

A BINARY OPERATION ON IRREDUCIBLE COMPONENTS OF LUSZTIG'S NILPOTENT VARIETIES II: APPLICATIONS AND CONJECTURES FOR REPRESENTATIONS OF GL_n OVER A NON-ARCHIMEDEAN LOCAL FIELD

EREZ LAPID AND ALBERTO MÍNGUEZ

ABSTRACT. In the first part of the paper we defined and studied a binary operation on the set of irreducible components of Lusztig's nilpotent varieties of a quiver. For type A we conjecture, following Geiss and Schröer, that this operation is compatible with taking the socle of parabolic induction of representations of general linear groups over a local non-archimedean field, at least when one of the irreducible components is rigid. We verify this conjecture in special cases.

CONTENTS

1.	Introduction	1
2.	Type A_n	3
3.	Relation to representation theory of GL	12
4.	\square -irreducible representations	15
5.	A new conjecture	17
6.	Regular multisegments	20
7.	Basic representations and irreducible components	24
8.	Proof of Theorem 6.5	29
9.	Odds and ends	32
	References	34

1. INTRODUCTION

Let F be a non-archimedean local field. The representation theory of the locally compact, totally disconnected groups $GL_n(F)$, $n \geq 0$ plays a particularly important role in the general theory of representations of p -adic groups. (All representations considered here are tacitly complex and smooth.) It was studied by Bernstein and Zelevinsky in their fundamental work in the 1970s, where they highlight the advantage of working with all n 's together [BZ76, BZ77, Zel80]. (For simplicity, we refer to it collectively as representation theory of GL .) Normalized parabolic induction, which following the usual convention is written as $\pi_1 \times \pi_2$, endows the category of finite length representations of $GL_n(F)$, $n \geq 0$ with the structure of a ring category over \mathbb{C} (i.e., a locally finite \mathbb{C} -linear abelian monoidal category, where the tensor functor is bilinear and biexact, and $\text{End}(\mathbf{1}) = \mathbb{C}$). The associativity

Date: March 21, 2022.

constraints are given by transitivity of parabolic induction and the unit object is given by the irreducible (one-dimensional) representation of $\mathrm{GL}_0 = 1$.

Compared to a general reductive group, GL_n is simpler in that L -packets are singletons and roughly speaking, “all supercuspidal representations look the same”.

It is well known that the representation theory of GL is closely related to the representation theory of quantum groups of type A [LNT03]. In particular, via Lusztig’s canonical basis, the irreducible objects are classified by the set Comp of irreducible components of nilpotent varieties of type A , which are in turn indexed by multisegments [Lus90, Lus91]. We denote this correspondence by $C \mapsto \pi(C)$. The nilpotent varieties classify modules of a given graded dimension of the preprojective algebra. (For a general quiver, one adds a nilpotency condition.)

A special role is played by the \square -irreducible representations, namely those (irreducible) π ’s such that $\pi \times \pi$ is irreducible. For instance, every unitarizable irreducible representation is \square -irreducible. The \square -irreducible representations and their analogs (“real” in Leclerc’s terminology) were studied by many people including Geiss, Hernandez, Kang, Kashiwara, Kim, Leclerc, Oh, Schröer and others in the context of monoidal categorification of cluster algebras [HL10, GLS06, KKKO18]. We refer the reader to [Kas18] and [HL21] for recent surveys on this topic. It is conjectured that $\pi(C)$ is \square -irreducible if and only if C is rigid, i.e., C contains an open orbit. This was proved in the so-called regular case in [LM18], together with a purely combinatorial characterization of \square -irreducibility in this case.

One of the nice properties of \square -irreducible representations is that if π_1 or π_2 is \square -irreducible, then the socle $\mathrm{soc}(\pi_1 \times \pi_2)$ is irreducible and occurs with multiplicity one in the Jordan–Hölder sequence of $\pi_1 \times \pi_2$. This raises the question of describing this socle explicitly. This is our motivating question. It naturally leads to look for a similar operation on Comp . Such an operation was conceived in the first part of the paper [AL] (for any quiver). Namely, given any two irreducible components $C_1, C_2 \in \mathrm{Comp}$, a third one, denoted $C_1 * C_2$, was defined to be the closure of the constructible set formed by all possible extensions of x_1 by x_2 for all (x_1, x_2) in a suitable open subset of $C_1 \times C_2$.

It is natural to expect that $\pi(C_1 * C_2)$ is always a subquotient, or perhaps even a subrepresentation, of $\pi(C_1) \times \pi(C_2)$. We formulate this as our main conjecture (Conjecture 5.1). It is closely related to a conjecture made by Geiss and Schröer in [GS05] in the context of dual canonical bases of simply laced Dynkin diagrams. A special case of Conjecture 5.1 was proposed in [LM20]. (We were unaware of [GS05] at the time.) In the case where C_1 or C_2 is rigid, one gets an especially nice necessary and sufficient (conjectural) irreducibility criterion for $\pi(C_1) \times \pi(C_2)$, namely $C_1 * C_2 = C_2 * C_1$. As our best evidence so far for the main conjecture, we show that it holds if C_1 or C_2 is regular and rigid (namely, the case analyzed in [LM18]). This strengthens the main result of [LM20] which proves the weaker conjecture of [ibid.] in a more restricted case.

We now describe the contents of the paper in more detail.

We first review and explicate the pertinent material of [AL] for $Q = A_n$ (§2). The computations of $C_1 * C_2$, $\mathrm{hom}(C_1, C_2)$ and $\mathrm{ext}^1(C_1, C_2)$ (the generic dimensions of Hom and Ext^1) become concrete and efficient, as long as we are content with randomized algorithms. We give examples of rigid and non-rigid irreducible components.

In §3 we recall the bijection between Comp and (a representative class of) irreducible representations of GL . We also discuss a variant and special cases of the conjectures of Geiss and Schröer.

The notion of \square -irreducible representations and their basic properties, following Kang–Kashiwara–Kim–Oh [KKKO15, KKKO18], are discussed in §4.

The main new conjecture, relating the binary operation $*$ to the socle of the parabolic induction, at least in the case where one of the irreducible components is rigid, is stated in §5. We give a few sanity checks and prove it in the special case where C_1 or C_2 corresponds to a single segment.

The results of [LM18], characterizing \square -irreducibility in the regular case, are reviewed in §6. We then state our best evidence so far for Conjecture 5.1, to wit, that it holds if C_1 or C_2 is regular and rigid (Theorem 6.5). As an aside, we give a simple necessary and sufficient condition for a regular representation to be a prime element in the Bernstein–Zelevinsky ring. We also formulate an intriguing, purely geometric, duality conjecture of Anton Mellit for Schubert varieties of type A .

In §7 we consider a special case of the Baumann–Kamnitzer–Tingley decompositions of Comp [BKT14] and its analogue to representation theory of GL . This is used in the proof of Theorem 6.5 in §8, which also builds on ideas from [LM16, LM18, LM20]. Clearly, more ideas are needed to go beyond the regular case.

We end the paper with a highly speculative conjectural characterization of the irreducible subquotients of any product $\pi_1 \times \cdots \times \pi_k$ (§9).

Acknowledgement. We are grateful to Bernard Leclerc for useful correspondence and to Max Gurevich for helpful discussions. The first-named author would like to thank the Hausdorff Research Institute for Mathematics in Bonn for its generous hospitality in July 2021. Conversations with Jan Schröer during that time were particularly helpful and we are very grateful to him. We also thank Anton Mellit for useful discussions and for allowing us to include Conjecture 6.2 here. Special thanks are due to Avraham Aizenbud for many conversations on the subject. Last but not least, we are greatly indebted to the referee for reading the paper carefully and making many useful suggestions.

2. TYPE A_n

We consider the quiver Q of type A_n with the standard orientation

$$\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$$

Thus, a representation of Q is a graded vector space $V = \bigoplus_{i=1}^n V_i$, together with a degree 1 linear transformation $T_+ : V \rightarrow V$.¹ In particular, T_+ is nilpotent of degree at most n . We will freely use the terminology and the results of [AL]. (However, we no longer use I to denote the index set of the vertices.)

We identify the set Ψ of positive roots with *segments* $[i, j]$, $1 \leq i \leq j \leq n$. (If $\alpha_1, \dots, \alpha_n$ are the simple roots in the usual ordering, then $[i, j]$ corresponds to $\alpha_i + \cdots + \alpha_j$.)

¹As in [AL] all vector spaces and varieties in this section are over a fixed algebraically closed field.

2.1. Recall that Ψ classifies the indecomposable modules. In the case at hand, to each segment $\Delta = [i, j]$ the corresponding representation $M_Q(\Delta)$ is a Jordan block whose graded dimension $\text{grdim } \Delta$ is the indicator function of $[i, j]$ (with $T_+^{j-i} \neq 0$). The simple objects (corresponding to the simple roots) are $M_Q([i, i])$, $i = 1, \dots, n$. The projective ones are $M_Q([i, n])$, $i = 1, \dots, n$.

The abelian monoid $\mathfrak{M} = \mathfrak{M}^n = \mathbb{N}\Psi$ freely generated by Ψ classifies all Q -representations up to isomorphism. The elements of \mathfrak{M} are called multisegments. A typical element $\mathfrak{m} \in \mathfrak{M}$ is written as

$$\mathfrak{m} = \Delta_1 + \dots + \Delta_k$$

where $\Delta_1, \dots, \Delta_k$ are segments. (Their order does not matter.) We write $M_Q(\mathfrak{m})$ for the corresponding representation. Extending grdim additively from Ψ to \mathfrak{M} , we have a disjoint union decomposition

$$\mathfrak{M} = \coprod_{\mathbf{d} \in \mathbb{N}^n} \mathfrak{M}(\mathbf{d}) \text{ where } \mathfrak{M}(\mathbf{d}) = \{\mathfrak{m} \in \mathfrak{M} \mid \text{grdim } \mathfrak{m} = \mathbf{d}\}.$$

Fix $V = \bigoplus_{i=1}^n V_i$ of graded dimension $\mathbf{d} = (\mathbf{d}(1), \dots, \mathbf{d}(n)) \in \mathbb{N}^n$ and set $V_i = 0$ and $\mathbf{d}(i) = 0$ if $i \notin \{1, \dots, n\}$. Let $G_V = \prod_{i=1}^n \text{GL}(V_i)$. The G_V -orbits on

$$R_Q(V) = \{T_+ : V \rightarrow V \mid T_+(V_i) \subseteq V_{i+1} \text{ for all } i\}$$

under conjugation are parameterized by $\mathfrak{M}(\mathbf{d})$. It is easy to determine the multisegment corresponding to the G_V -orbit of T_+ from the ranks of $T_+^j|_{V_i}$, $1 \leq i \leq i+j \leq n$ (see e.g., [KZ96, §1]).

Let $\Delta = [a, b]$ and $\Gamma = [c, d]$ be two segments. Following Zelevinsky, we say that Δ precedes Γ if $a+1 \leq c \leq b+1 \leq d$ and denote it by $\Delta \prec \Gamma$. We have

$$\dim \text{Hom}_Q(M_Q(\Gamma), M_Q(\Delta)) = \begin{cases} 1 & \text{if } \Delta \prec \vec{\Gamma}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\vec{\Gamma} = [c+1, d+1]$, and

$$\dim \text{Ext}_Q^1(M_Q(\Delta), M_Q(\Gamma)) = \begin{cases} 1 & \text{if } \Delta \prec \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for any multisegments $\mathfrak{m}, \mathfrak{n}$ we have

$$(2.1a) \quad \begin{aligned} \text{Ext}_Q^1(M_Q(\mathfrak{m}), M_Q(\mathfrak{n})) = 0 &\iff \\ \Delta \not\prec \Gamma &\text{ for every segment } \Delta \text{ of } \mathfrak{m} \text{ and } \Gamma \text{ of } \mathfrak{n} \end{aligned}$$

and

$$(2.1b) \quad \begin{aligned} \text{Hom}_Q(M_Q(\mathfrak{n}), M_Q(\mathfrak{m})) = 0 &\iff \\ \Delta \not\prec \vec{\Gamma} &\text{ for every segment } \Delta \text{ of } \mathfrak{m} \text{ and } \Gamma \text{ of } \mathfrak{n}. \end{aligned}$$

For any graded vector space $V = \bigoplus_{i=1}^n V_i$, the open G_V -orbit in $R_Q(V)$ corresponds to the multisegment $\mathfrak{m} \in \mathfrak{M}(\mathbf{d})$ such that $\Delta_i \not\prec \Delta_j$ for any two segments in \mathfrak{m} .

Recall that we can identify

$$R_Q^\circ(V) = \{T_- : V \rightarrow V \mid T_-(V_i) \subseteq V_{i-1} \text{ for all } i\}$$

with the dual space of $R_Q(V)$ by the pairing $(T_+, T_-) = \text{tr } T_+ T_- = \text{tr } T_- T_+$. It follows from [AL, (2.11)] that, for any two segments Δ, Γ we have

$$\dim \text{Hom}_{Q^\circ}(M_{Q^\circ}(\Delta), M_{Q^\circ}(\Gamma)) = \begin{cases} 1 & \text{if } \Delta \prec \vec{\Gamma}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim \text{Ext}_{Q^\circ}^1(M_{Q^\circ}(\Gamma), M_{Q^\circ}(\Delta)) = \begin{cases} 1 & \text{if } \Delta \prec \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for every two multisegments \mathbf{m} and \mathbf{n} we have

$$\text{Ext}_{Q^\circ}^1(M_{Q^\circ}(\mathbf{n}), M_{Q^\circ}(\mathbf{m})) = 0 \iff \Delta \not\prec \Gamma \text{ for every segment } \Delta \text{ of } \mathbf{m} \text{ and } \Gamma \text{ of } \mathbf{n}$$

and

$$(2.2) \quad \text{Hom}_{Q^\circ}(M_{Q^\circ}(\mathbf{m}), M_{Q^\circ}(\mathbf{n})) = 0 \iff \Delta \not\prec \vec{\Gamma} \text{ for every segment } \Delta \text{ of } \mathbf{m} \text{ and } \Gamma \text{ of } \mathbf{n}.$$

A special feature of Q is that it is isomorphic to Q° via $i \mapsto n+1-i$. This yields an (equivariant) equivalence of the categories of representations of Q and Q° . Composing it with the standard duality $M \mapsto M^*$ between finite-dimensional representations of Q and Q° we get a self-duality on the category of finite-dimensional representations of Q (or of Q°), which we denote by $^\vee$. For any segment $\Delta = [a, b]$ we have

$$M_Q(\Delta)^\vee = M_Q(\Delta^\vee), \quad M_{Q^\circ}(\Delta)^\vee = M_{Q^\circ}(\Delta^\vee)$$

where $\Delta^\vee = [n+1-b, n+1-a]$. Extending this involution to the set of multisegments by additivity we have

$$M_Q(\mathbf{m})^\vee = M_Q(\mathbf{m}^\vee), \quad M_{Q^\circ}(\mathbf{m})^\vee = M_{Q^\circ}(\mathbf{m}^\vee).$$

2.2. Recall that \overline{Q} is the double quiver (i.e., all orientations simultaneously)

$$\bullet \leftrightarrow \bullet \leftrightarrow \cdots \leftrightarrow \bullet$$

and the preprojective algebra Π is the finite-dimensional quotient of the path algebra of \overline{Q} by the relations

$$e_{i+1,i} e_{i,i+1} - e_{i-1,i} e_{i,i-1}, \quad i = 1, \dots, n.$$

Here $e_{r,s}$, $s = r \pm 1$ is the arrow from r to s , interpreted as 0 unless $1 \leq r, s \leq n$.

The nilpotent variety $\Lambda(V) = R_\Pi(V)$ is given by

$$\Lambda(V) = \{(T_+, T_-) \in R_Q(V) \times R_{Q^\circ}(V) \mid T_+ T_- = T_- T_+\}$$

and the maps

$$\pi_Q : \Lambda(V) \rightarrow R_Q(V), \quad \pi_{Q^\circ} : \Lambda(V) \rightarrow R_{Q^\circ}(V)$$

are the canonical projections.

Recall that we denote the set of irreducible components of $\Lambda(V)$ by $\text{Comp}(V)$, or simply by $\text{Comp}(\mathbf{d})$ if $\mathbf{d} = \text{grdim } V$, and let

$$\text{Comp} = \bigcup_{\mathbf{d} \in \mathbb{N}^n} \text{Comp}(\mathbf{d}).$$

We have a bijection

$$\lambda_Q : \mathfrak{M}(\mathbf{d}) \rightarrow \text{Comp}(\mathbf{d}), \quad \mathbf{m} \mapsto \overline{\mu_Q(\mathbf{m})}$$

where $\mu_Q(\mathbf{m})$ is the inverse image under π_Q of the G_V -orbit of $M_Q(\mathbf{m})$. Similarly, we have a bijection

$$\lambda_{Q^\circ} : \mathfrak{M}(\mathbf{d}) \rightarrow \text{Comp}(\mathbf{d}).$$

In fact,

$$(2.3) \quad \lambda_Q(\mathbf{m})^* = \lambda_{Q^\circ}(\mathbf{m}) \quad \forall \mathbf{m} \in \mathfrak{M}.$$

We also have

$$\lambda_Q(\mathbf{m}) = \lambda_{Q^\circ}(\mathbf{m}^t)$$

where \mathbf{m}^t is the Mœglin–Waldspurger involution [MgW86, Proposition II.6] – see also [KZ96, Rin99].

We recall that by definition, a Π -module x is *rigid* if $\text{Ext}_\Pi^1(x, x) = 0$. Also, an irreducible component $C \in \text{Comp}(V)$ is rigid, if it contains a rigid element, in which case the rigid elements form an open orbit in C . See [AL, §4] for more details.

Observe that by (2.1b) and (2.2), for any two multisegments \mathbf{m}, \mathbf{n}

$$(2.4) \quad \begin{aligned} &\text{if } \Delta \not\stackrel{\rightarrow}{\sim} \Gamma \text{ for every segment } \Delta \text{ of } \mathbf{m} \text{ and } \Gamma \text{ of } \mathbf{n}, \text{ then} \\ &\text{hom}_\Pi(\lambda_Q(\mathbf{n}), \lambda_Q(\mathbf{m})) = \text{hom}_\Pi(\lambda_{Q^\circ}(\mathbf{m}), \lambda_{Q^\circ}(\mathbf{n})) = 0. \end{aligned}$$

(Recall that by definition,

$$\text{hom}_\Pi(C_1, C_2) = \min_{(x_1, x_2) \in C_1 \times C_2} \dim \text{Hom}_\Pi(x_1, x_2)$$

for any irreducible components C_1, C_2 . Similarly, we will also use the notation $\text{ext}_\Pi^1(C_1, C_2)$ defined analogously.)

The bijection $\Lambda(V) \rightarrow \Lambda(V^*)$ takes (T_+, T_-) to (T_+^*, T_-^*) . On the other hand, the involution $i \mapsto n + 1 - i$ of \overline{Q} gives rise to an involution of Π and hence to an autoequivalence of the module category of Π . Composing it with $M \mapsto M^*$ we get another self-duality $M \mapsto M^\vee$ (which commutes with $*$). At the level of irreducible components, we have

$$\lambda_Q(\mathbf{m})^\vee = \lambda_Q(\mathbf{m}^\vee), \quad \lambda_{Q^\circ}(\mathbf{m})^\vee = \lambda_{Q^\circ}(\mathbf{m}^\vee).$$

Finally, we recall the binary operation $*$ on Comp defined in [AL, §3]. Given $C_i \in \text{Comp}(V^i)$, $i = 1, 2$ and $V = V^1 \oplus V^2$, $C_1 * C_2$ is the Zariski closure of the set of $x \in \Lambda(V)$ that fit in a short exact sequence

$$0 \rightarrow x_2 \rightarrow x \rightarrow x_1 \rightarrow 0$$

where $(x_1, x_2) \in C_1 \times C_2$ and $\dim \text{Ext}_\Pi^1(x_1, x_2)$ is minimal.

Recall that by definition, C_1 and C_2 *strongly commute* if $\text{Ext}_\Pi^1(x_1, x_2) = 0$ for some $(x_1, x_2) \in C_1 \times C_2$. Equivalently, the Zariski closure of the set of modules isomorphic to $x_1 \oplus x_2$ for some $(x_1, x_2) \in C_1 \times C_2$ is an irreducible component (necessarily equal to $C_1 * C_2 = C_2 * C_1$) – see [AL, Corollary 3.3].

If C_1 or C_2 is rigid, strong commutativity is equivalent to the a priori weaker condition $C_1 * C_2 = C_2 * C_1$ [AL, Proposition 6.1]. In this case, we sometimes omit the adjective “strong”.

By [AL, (3.2)] we have

$$(2.5) \quad (C_1 * C_2)^\vee = C_2^\vee * C_1^\vee$$

for any $C_1, C_2 \in \text{Comp}$.

2.3. Next, we expound [AL, §9] in the case at hand, and in particular the maps $\mathcal{T}_{M;N} : \text{Hom}_Q(M, N) \rightarrow \text{Hom}_Q(M, \tau N)$ of [AL, (9.8)] defined for any Π -modules M and N . The functor τ (and its left adjoint τ^-) are essentially the Coxeter functors with respect to Q . Concretely, τ (resp., τ^-) takes a pair $(\oplus_{i=1}^n V_i, T_+)$ to the pair $(\oplus_{i=1}^n V'_i, T'_+)$ where (letting $V_0 = V_{n+1} = 0$)

$$V'_i = \text{Ker}(T_+^{n-i+1}|_{V_{i-1}}) \quad (\text{resp., } \text{Coker}(T_+^i|_{V_i})) \quad i = 1, \dots, n$$

and T'_+ is the map induced by T_+ . In particular, $\tau M_Q(\Delta) = M_Q(\vec{\Delta})$ where by convention $M_Q([i, n+1]) = 0$ for all i , while $\tau^-(M_Q([a, b])) = M_Q([a-1, b-1])$ again, interpreted as 0 if $a = 1$.

Let $M \in \mu_Q(\mathfrak{m})$, for a multisegment

$$\mathfrak{m} = \sum_{i \in I} \Delta_i.$$

Then, there exists a graded basis $\mathfrak{f}_{i,r}$, $i \in I$, $r \in \Delta_i$ (of degree r) for M (as a graded vector space) such that

$$e_{r,r+1} \mathfrak{f}_{i,r} = \begin{cases} \mathfrak{f}_{i,r+1} & \text{if } r+1 \in \Delta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$U_{\mathfrak{m}} = \{(i, j) \in I \times I \mid \Delta_j \prec \Delta_i\},$$

so that $\#U_{\mathfrak{m}} = \dim \text{Hom}_Q(M_Q(\mathfrak{m}), \tau(M_Q(\mathfrak{m})))$. By [MgW86, Lemme II.4] there exist scalars $x_{i,j}$, $(i, j) \in U_{\mathfrak{m}}$ such that

$$e_{r+1,r} \mathfrak{f}_{i,r+1} = \sum_{j \in I \mid r \in \Delta_j \text{ and } (i,j) \in U_{\mathfrak{m}}} x_{i,j} \mathfrak{f}_{j,r}$$

for all $i \in I$ and r such that $r+1 \in \Delta_i$. We call $x_{i,j}$ the coordinates of M . They encode the data of M as a Π -module (cf. [AL, §9]). Of course, the coordinates depend on the choice of the basis $\mathfrak{f}_{i,r}$.

Suppose that in addition $N \in \mu_Q(\mathfrak{n})$ where

$$\mathfrak{n} = \sum_{j \in J} \Gamma_j.$$

Denote by $\mathfrak{g}_{j,r}$, $j \in J$, $r \in \Gamma_j$ a graded basis for N such that

$$e_{r,r+1} \mathfrak{g}_{j,r} = \begin{cases} \mathfrak{g}_{j,r+1} & \text{if } r+1 \in \Gamma_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $y_{i,j}$, $(i, j) \in U_{\mathfrak{n}}$ be the coordinates of N . Let

$$(2.6) \quad U_{\mathfrak{m};\mathfrak{n}} = \{(i, j) \in I \times J \mid \Gamma_j \prec \Delta_i\}, \quad V_{\mathfrak{m};\mathfrak{n}} = \{(i, j) \in I \times J \mid \Gamma_j \prec \vec{\Delta}_i\}.$$

In particular, $U_{\mathfrak{m}} = U_{\mathfrak{m};\mathfrak{m}}$. We will also write $V_{\mathfrak{m}} = V_{\mathfrak{m};\mathfrak{m}}$ for simplicity. Note that

$$\#U_{\mathfrak{m};\mathfrak{n}} = \dim \text{Hom}_Q(M_Q(\mathfrak{m}), \tau(M_Q(\mathfrak{n})))$$

and

$$\#V_{\mathfrak{m};\mathfrak{n}} = \dim \text{Hom}_Q(M_Q(\mathfrak{m}), M_Q(\mathfrak{n})).$$

Let $\mathbf{b}_{i,j}$, $(i, j) \in V_{\mathbf{m};\mathbf{n}}$ be the basis of $\text{Hom}_Q(\pi_Q(M), \pi_Q(N))$ given by

$$\mathbf{b}_{i,j}(\mathbf{f}_{k,r}) = \begin{cases} \mathbf{g}_{j,r} & \text{if } k = i \text{ and } r \in \Gamma_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{c}_{i,j}$, $(i, j) \in U_{\mathbf{m};\mathbf{n}}$ be the basis of $\text{Hom}_Q(\pi_Q(M), \tau(\pi_Q(N)))$ defined analogously. Then, for any $(i, j) \in V_{\mathbf{m};\mathbf{n}}$, $\mathcal{T}_{M;N}(\mathbf{b}_{i,j})$ is equal to

$$(2.7a) \quad \sum_{k \in I|(k,i) \in U_{\mathbf{m}} \text{ and } (k,j) \in U_{\mathbf{m};\mathbf{n}}} x_{k,i} \mathbf{c}_{k,j} - \sum_{k \in J|(j,k) \in U_{\mathbf{n}} \text{ and } (i,k) \in U_{\mathbf{m};\mathbf{n}}} y_{j,k} \mathbf{c}_{i,k}.$$

Consequently, for the dual map, for any $(i, j) \in U_{\mathbf{m};\mathbf{n}}$, $\mathcal{T}_{M;N}^*(\hat{\mathbf{c}}_{i,j})$ is equal to

$$(2.7b) \quad \sum_{k \in I|(i,k) \in U_{\mathbf{m}} \text{ and } (k,j) \in V_{\mathbf{m};\mathbf{n}}} x_{i,k} \hat{\mathbf{b}}_{k,j} - \sum_{k \in J|(k,j) \in U_{\mathbf{n}} \text{ and } (i,k) \in V_{\mathbf{m};\mathbf{n}}} y_{k,j} \hat{\mathbf{b}}_{i,k}$$

where $\hat{\mathbf{b}}_{i,j}$, $(i, j) \in V_{\mathbf{m};\mathbf{n}}$ and $\hat{\mathbf{c}}_{i,j}$, $(i, j) \in U_{\mathbf{m};\mathbf{n}}$ denote the dual bases.

2.4. Computational aspects. Given two irreducible components $C_1 = \lambda_Q(\mathbf{m})$, $C_2 = \lambda_Q(\mathbf{n}) \in \text{Comp}$ the computation of $\text{hom}_{\Pi}(C_1, C_2)$, (and consequently, $\text{ext}_{\Pi}^1(C_1, C_2)$) and the determination of $C_1 * C_2$ reduce to the following type of problem.

Let $A = A(x_1, \dots, x_k)$ be a matrix of size $n_1 \times n_2$ whose entries are polynomials in k variables (say, with integer coefficients) of degree $\leq d$. We would like to determine the generic rank r of A efficiently (i.e., in polynomial time). This problem is closely related to polynomial identity testing. Unfortunately, there is no known subexponential deterministic algorithm for this problem. However, the “obvious” Monte Carlo randomized algorithm works well. More precisely, let $\varepsilon > 0$ and choose integers y_1, \dots, y_k independently and uniformly randomly from a fixed set of size $s \geq d \min(n_1, n_2) \varepsilon^{-1}$. Then, the rank of $A(y_1, \dots, y_k)$ is equal to r with probability at least $1 - \varepsilon$. Indeed, if $p(x_1, \dots, x_k)$ is an $r \times r$ -minor of A that is not identically zero, then by the Schwartz–Zippel lemma [Sch80], the probability that $p(y_1, \dots, y_k) \neq 0$ (and in particular, $\text{rk } A(y_1, \dots, y_k) = r$) is at least $1 - \frac{\text{deg } p}{s} \geq 1 - \frac{dr}{s}$.

Recall that as in §2.3 above we can represent an element of $\mu_Q(\mathbf{m})$ by coordinates $x_{i,j}$, $(i, j) \in U_{\mathbf{m}}$. Therefore, a “random” element of $\mu_Q(\mathbf{m})$ (and hence of $\lambda_Q(\mathbf{m})$) is represented by random coordinates.

In order to compute $\text{hom}_{\Pi}(C_1, C_2)$ we use the relation [AL, (9.3)] and (2.7a) to reduce it to the problem above where $k = \#U_{\mathbf{m}} + \#U_{\mathbf{n}}$, $n_1 = \#U_{\mathbf{m};\mathbf{n}}$, $n_2 = \#V_{\mathbf{m};\mathbf{n}}$ and $d = 1$. In fact, the non-zero entries of $A(x_1, \dots, x_k)$ are of the form $\pm x_i$ for some i .

In special cases there is an efficient deterministic algorithm for the computation of $\text{hom}_{\Pi}(C_1, C_2)$ in terms of \mathbf{m} , \mathbf{n} . (See Remark 2.2 below.) However, we do not know how to do it in general.

By the same token, using [AL, Corollary 9.3], the rigidity condition for an irreducible component $C = \lambda_Q(\mathbf{m})$ can be checked efficiently by a randomized algorithm (cf. [LM18, LM20]). Once again, it would be interesting to find a simple combinatorial criterion or at least an efficient deterministic algorithm for the rigidity of an irreducible component in general. (See Theorem 6.1 below for a special case.) A closely related problem is to effectively characterize (as a subset) the open G_V -orbit in a given rigid irreducible component.

The binary operation $*$ gives rise to a binary operation on multisegments, also denoted by $*$, so that $\lambda_Q(\mathbf{m}) * \lambda_Q(\mathbf{n}) = \lambda_Q(\mathbf{m} * \mathbf{n})$. It would be interesting to give a purely combinatorial description of this operation on multisegments. As part of our results we will give a recipe for this in a special case – see Remark 8.3.

In general, we can determine $C_1 * C_2$ (or equivalently, $\mathbf{m} * \mathbf{n}$) by a randomized algorithm. To that end, we choose $M \in \mu_Q(\mathbf{m})$, and $N \in \mu_Q(\mathbf{n})$ with random coordinates. Using [AL, Proposition 9.1], the image under π_Q of a random extension of M by N is an extension E of $M_Q(\mathbf{m})$ by $M_Q(\mathbf{n})$ which is determined by a random element of $\text{Ker } \mathcal{T}_{N;M}^* \subseteq \text{Hom}_Q(M_Q(\mathbf{n}), \tau M_Q(\mathbf{m}))^* \simeq \text{Ext}_Q^1(M_Q(\mathbf{m}), M_Q(\mathbf{n}))$. The map $\mathcal{T}_{N;M}^*$ was explicated in (2.7b). We can therefore recover the multisegment pertaining to E , as explained in §2.1. Alternatively, choosing random $x_i = (T_i^+, T_i^-) \in C_i$, $i = 1, 2$, with $T_i^\pm : V^i \rightarrow V^i$ of degree ± 1 , an extension of x_1 by x_2 is of the form

$$x = (T^+, T^-) = ((T_1^+ + T_{1,2}^+) \oplus T_2^+, (T_1^- + T_{1,2}^-) \oplus T_2^-)$$

for a pair of linear transformations $T_{1,2}^\pm : V^1 \rightarrow V^2$ of degrees ± 1 satisfying

$$T_2^+ T_{1,2}^- + T_{1,2}^+ T_1^- = T_2^- T_{1,2}^+ + T_{1,2}^- T_1^+$$

Choosing a random element in this vector space, the G_V -orbit of T^+ gives the parameter of $C_1 * C_2$ with high probability.

Similarly, given $C, C_1 \in \text{Comp}$ with C_1 rigid, [AL, Proposition 5.1] gives rise to an efficient randomized algorithm for finding the unique $C_2 \in \text{Comp}$ such that $C = C_1 * C_2$, if exists. Namely, choose random points $x \in C$ and $x_1 \in C_1$ and a random element $\varphi \in \text{Hom}_\Pi(x, x_1)$ (using [AL, (9.3)]). If φ is surjective, then with high probability $\pi_Q(\text{Ker } \varphi)$ gives rise to the multisegment defining C_2 . Otherwise, C_2 does not exist with high probability.

2.5. Examples. We have already mentioned some simple examples of rigid modules and irreducible components in [AL, §4]. For instance, if every segment in \mathbf{m} is of the form $[i, i]$ (corresponding to simple roots), then $\lambda_Q(\mathbf{m})$ and $\lambda_{Q^\circ}(\mathbf{m})$ are rigid. The same conclusion holds if $\Delta_i \not\prec \Delta_j$ for every segments Δ_i, Δ_j of \mathbf{m} .

For $n \leq 4$, all irreducible components are rigid (and Π is representation-finite).

2.5.1. Laminae and quasi-laminae. Let $S(r)$, $r = 1, \dots, n$ be the simple Π -module whose graded dimension is 1 in the r -th coordinate and 0 elsewhere. Let $P(r) = \Pi e_r$ be the indecomposable projective Π -module whose cosocle is $S(r)$. By [AL, (9.9)] we have

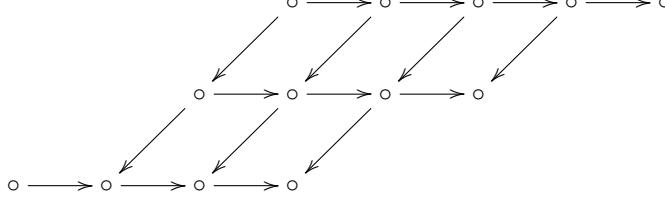
$$\pi_Q(P(r)) = M_Q\left(\sum_{i=1}^r [i, i + n - r]\right), \quad \pi_{Q^\circ}(P(r)) = M_{Q^\circ}\left(\sum_{j=1}^{n-r+1} [j, j + r - 1]\right).$$

The submodule lattice of $P(r)$ is isomorphic to the sublattice of the Young lattice consisting of Young diagrams contained in the rectangle of size $r \times (n - r + 1)$ (cf. [Rin99]). In particular, it is distributive.

Suppose that $1 \leq a_1 < \dots < a_r$ and $b_1 < \dots < b_r \leq n$ are two sequences such that $a_i \leq b_i + 1$ for all i . Consider a graded basis e_j^i , $a_i \leq j \leq b_i$, $i = 1, \dots, r$ (graded by j) and homogenous maps T_\pm of degree ± 1 given by

$$T_+ e_j^i = \begin{cases} e_{j+1}^i & j < b_i, \\ 0 & \text{otherwise;} \end{cases} \quad T_- e_j^i = \begin{cases} e_{j-1}^{i-1} & \text{if } i > 1 \text{ and } j \leq b_{i-1} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

See diagram below for the sequences $1 < 3 < 4$ and $4 < 6 < 8$; the dots represent the basis elements and T_{\pm} are represented by the horizontal and diagonal arrows, respectively.



The resulting Π -module M is a subquotient of $P(r)$. Conversely, all subquotients of $P(r)$ are obtained this way. (The subrepresentations of $P(r)$ correspond to the case $b_i = i + n - r$ for all $i = 1, \dots, r$.) We have

$$\pi_Q(M) = M_Q\left(\sum_{i=1}^r [a_i, b_i]\right)$$

where by convention we discard empty segments. (Such multisegments were called ladders in [LM14].)

Note that M is indecomposable if and only if $a_{i+1} \leq b_i + 1$ for all $i = 1, \dots, r-1$. Following Ringel's terminology, an indecomposable subquotient of any $P(r)$ is called a *lamina*.² Direct sums of laminae are termed *laminated modules*. In particular, any subquotient of $P(r)$ is laminated. (We will call them *quasi-laminae*.)

Every quasi-lamina is rigid. This follows from [AL, Lemma 8.1] by an easy induction. More precisely, if $x = \lambda_Q(\mathbf{m})$ with $\mathbf{m} = [a_1, b_1] + \dots + [a_r, b_r]$ with $a_1 < \dots < a_r$ and $b_1 < \dots < b_r$, then we may take $x_1 = \lambda_Q([a_r, b_r])$ and $x_2 = \lambda_Q([a_1, b_1] + \dots + [a_{r-1}, b_{r-1}])$ and use [AL, Corollary 9.3]. Indeed, the map $\mathcal{T}_{x;x_1}$ is trivially surjective since its codomain is 0. The map $\mathcal{T}_{x_1;x}$ is surjective since by (2.7a), on the complement of its kernel (which is at most one-dimensional) it is represented by a square, upper triangular matrix whose diagonal entries are coordinates of x (which are free variables).

Example 2.1. Consider an irreducible component $C = \lambda_Q(\mathbf{m})$ of a lamina with

$$\mathbf{m} = \sum_{i=1}^r [a_i, b_i] \text{ with } a_1 < \dots < a_r \text{ and } b_1 < \dots < b_r.$$

For each i let t_i be such that $a_i - 1 \leq t_i \leq b_i$ and $t_1 < \dots < t_r$. Let $C_i = \lambda_Q(\mathbf{m}_i)$, $i = 1, 2$ where

$$\mathbf{m}_1 = \sum_{i=1}^r [a_i, t_i], \quad \mathbf{m}_2 = \sum_{i=1}^r [t_i + 1, b_i]$$

where we ignore any empty segments in the sums. Then, $C = C_1 * C_2$.

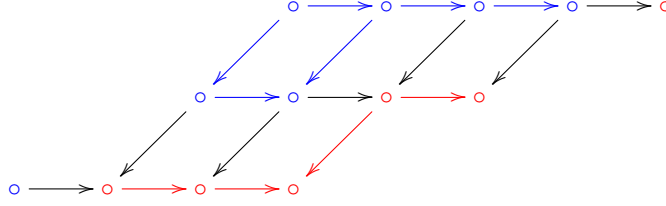
For instance, in the case

$$\mathbf{m} = [1, 4] + [3, 6] + [4, 8], \quad \mathbf{m}_1 = [1, 1] + [3, 4] + [4, 7], \quad \mathbf{m}_2 = [2, 4] + [5, 6] + [8, 8],$$

a rigid element of C is represented by the following diagram (cf. §2.5.1). It is an extension of a rigid element of C_1 by a rigid element of C_2 where the former (resp.,

²This terminology is specific for type A .

latter) is represented by the blue (resp., red) dots and arrows.



Remark 2.2. Let $C_1 = \lambda_Q(\mathbf{m})$ and $C_2 = \lambda_Q(\mathbf{n})$ with $\mathbf{m} = \sum_{i \in I} \Delta_i$ and $\mathbf{n} = \sum_{j \in J} \Gamma_j$. As in [LM16, §6.3], let

$$G_{\mathbf{m};\mathbf{n}} = (U_{\mathbf{m};\mathbf{n}}, V_{\mathbf{m};\mathbf{n}}, E_{\mathbf{m};\mathbf{n}})$$

be the bipartite graph where $U_{\mathbf{m};\mathbf{n}}$ and $V_{\mathbf{m};\mathbf{n}}$ are as in (2.6) and

$$E_{\mathbf{m};\mathbf{n}} = \{((i, k), (i, j)) \in U_{\mathbf{m};\mathbf{n}} \times V_{\mathbf{m};\mathbf{n}} \mid (j, k) \in U_{\mathbf{n}}\} \cup \\ \{((i, k), (j, k)) \in U_{\mathbf{m};\mathbf{n}} \times V_{\mathbf{m};\mathbf{n}} \mid (i, j) \in U_{\mathbf{m}}\}.$$

Denote by $\mathcal{M}(\mathbf{m}, \mathbf{n})$ the condition that $G_{\mathbf{m};\mathbf{n}}$ admits a matching that covers all vertices of $U_{\mathbf{m};\mathbf{n}}$.³ As noted in [LM20], $\mathcal{M}(\mathbf{n}, \mathbf{m})$ is a necessary condition for $\lambda_Q(\mathbf{m}) * \lambda_Q(\mathbf{n}) = \lambda_Q(\mathbf{m} + \mathbf{n})$ (cf. [AL, (9.4)] and (2.7a)). This condition is not sufficient in general. However, suppose that C_1 or C_2 is a quasi-lamina. Then, the conditions $\lambda_Q(\mathbf{m}) * \lambda_Q(\mathbf{n}) = \lambda_Q(\mathbf{m} + \mathbf{n})$ and $\mathcal{M}(\mathbf{n}, \mathbf{m})$ are equivalent by [LM16, Proposition 6.20]. In particular, C_1 and C_2 commute if and only if both $\mathcal{M}(\mathbf{m}, \mathbf{n})$ and $\mathcal{M}(\mathbf{n}, \mathbf{m})$ are satisfied. This criterion is very useful in practice. We will use it in §8 where we prove our main result. (See Lemma 8.2.)

More generally, let $\mathfrak{R}_{\mathbf{m};\mathbf{n}}$ be a matching of $G_{\mathbf{m};\mathbf{n}}$ of maximal size. Then, using the argument of [LM16, §6.4] and [LM20, §8.3] one can show that if C_1 or C_2 is a quasi-lamina, then $\min_{(M,N) \in \mu_Q(\mathbf{m}) \times \mu_Q(\mathbf{n})} \dim \text{Coker } \mathcal{T}_{N;M}$ is equal to the number of vertices in $U_{\mathbf{m};\mathbf{n}}$ that are not covered by $\mathfrak{R}_{\mathbf{m};\mathbf{n}}$. Equivalently, $\text{hom}_{\Pi}(C_2, C_1)$ is equal to the number of vertices of $V_{\mathbf{m};\mathbf{n}}$ that are not covered by $\mathfrak{R}_{\mathbf{m};\mathbf{n}}$. In particular, since the size of $\mathfrak{R}_{\mathbf{m};\mathbf{n}}$ can be computed efficiently by a deterministic algorithm, the same is true for $\text{hom}_{\Pi}(C_1, C_2)$ and $\text{ext}_{\Pi}^1(C_1, C_2)$ in this case. We omit the details.

2.5.2. Non-rigid components.

Example 2.3. An example of a non-rigid irreducible component for $n = 5$ was analyzed in detail in [GS05], following [Lec03]. Consider $\mathbf{d} = (1, 2, 2, 2, 1)$, so that $\dim \Lambda(V) = 12$ and $\dim G_V = 14$. Let

$$C = \lambda_Q(\mathbf{m}) = \lambda_{Q^\circ}(\mathbf{m}) \text{ where } \mathbf{m} = [4, 5] + [2, 4] + [3, 3] + [1, 2].$$

Then, C is the closure of a one-parameter family \mathcal{F} of 11-dimensional orbits. For any $x \in \mathcal{F}$ we have $\dim \text{End}_{\Pi}(x) = 3$ and $\dim \text{Ext}_{\Pi}^1(x, x) = 2$, while for any two distinct $x, y \in \mathcal{F}$ we have $\dim \text{Hom}_{\Pi}(x, y) = 2$ and $\text{Ext}_{\Pi}^1(x, y) = 0$.

Thus, by [AL, Corollary 3.3], $C \oplus C$ is an irreducible component even though for any $x \in \mathcal{F}$ there is a short exact sequence

$$0 \rightarrow x \rightarrow P(2) \oplus P(4) \rightarrow x \rightarrow 0.$$

³Recall that a matching in a graph is a set \mathfrak{R} of edges, no two of which have a vertex in common. The vertices covered by \mathfrak{R} are those which belong to one of the edges in \mathfrak{R} .

Example 2.4. As was pointed out in [AL, Remark 4.3], there exist non-rigid components C for which $C * C$ is rigid. An example for $n = 7$ is

$$C = \lambda_Q([4, 7] + [5, 6] + [2, 5] + [3, 4] + [1, 3])$$

for which

$$C * C = \lambda_Q([4, 7] + [2, 7] + [5, 6] + [3, 6] + [4, 5] + [1, 5] + [2, 4] + [3, 3] + [1, 3]).$$

2.5.3. Finally, we explicate a special case of [AL, Lemma 9.5] (cf. [AL, Remark 8.5]).

Lemma 2.5. *Suppose that $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$ where the Δ_i 's are distinct. Assume that $C = \lambda_Q(\mathfrak{m})$ is rigid. Then, we can write non-trivially $C = C_1 * C_2$ where C_1 and C_2 are rigid and $\text{ext}_{\mathbb{H}}^1(C, C_1) = 0$.*

Proof. Indeed, without loss of generality we may assume that $\Delta_i \not\prec \Delta_k$ and $\Delta_i \not\prec \overrightarrow{\Delta}_k$ for all $i < k$. (For instance, this holds if $b(\Delta_k) + e(\Delta_k) \leq b(\Delta_i) + e(\Delta_i)$ for all $i < k$.) Let $x \in C$ be rigid. Then, $\pi_Q(x) \simeq M_Q(\mathfrak{m})$. Write $\mathfrak{m} = \mathfrak{m}' + \mathfrak{m}''$ where $\mathfrak{m}' = \Delta_1 + \cdots + \Delta_{k-1}$ and $\mathfrak{m}'' = \Delta_k$. Then, the conditions of all parts of [AL, Lemma 9.5] are satisfied for the decomposition $M_Q(\mathfrak{m}) = M_Q(\mathfrak{m}') \oplus M_Q(\mathfrak{m}'')$. Taking $C_1 = \lambda_Q(\mathfrak{m}')$ and $C_2 = \lambda_Q(\mathfrak{m}'')$, the result follows. \square

3. RELATION TO REPRESENTATION THEORY OF GL

From now on, let F be a local non-archimedean field with normalized absolute value $|\cdot|$. Let

$$\mathcal{C} = \bigoplus_{m \geq 0} \mathcal{C}(\text{GL}_m(F))$$

be the category of complex, smooth representations of finite length of $\text{GL}_m(F)$, $m \geq 0$. This is a ring category with tensor functor given by (normalized) parabolic induction $\pi_1 \times \pi_2$ with respect to the parabolic subgroup of block upper triangular matrices. The unit object $\mathbf{1}$ is the one-dimensional representation of $\text{GL}_0(F) = 1$.

By Zelevinsky's classification [Zel80], there is a bijection

$$\mathfrak{m} \rightarrow Z(\mathfrak{m})$$

between \mathfrak{M}^n and the set $\text{Irr} = \text{Irr}^n$ of irreducible subquotients (up to isomorphism) of

$$|\cdot|^{a_1} \times \cdots \times |\cdot|^{a_k}$$

(a representation of $\text{GL}_k(F)$) where a_1, \dots, a_k range over all finite sequences of integers between 1 and n . Under this bijection, for any $\mathbf{d} \in \mathbb{N}^n$, $\mathfrak{M}(\mathbf{d})$ corresponds to the set $\text{Irr}(\mathbf{d})$ of irreducible subquotients of

$$\overbrace{|\cdot| \times \cdots \times |\cdot|}^{\mathbf{d}(1)} \times \cdots \times \overbrace{|\cdot|^n \times \cdots \times |\cdot|^n}^{\mathbf{d}(n)}.$$

More precisely, given a segment $\Delta = [a, b]$ define

$$Z(\Delta) = \text{soc}(|\cdot|^a \times \cdots \times |\cdot|^b), \quad L(\Delta) = \text{soc}(|\cdot|^b \times \cdots \times |\cdot|^a).$$

Thus, $Z(\Delta)$ is the one-dimensional character $|\det \cdot|^{(a+b)/2}$ of $\text{GL}_{b-a+1}(F)$ and $L(\Delta)$ is the twist of the Steinberg representation of $\text{GL}_{b-a+1}(F)$ by $|\det \cdot|^{(a+b)/2}$. Given a multisegment \mathfrak{m} we write it as $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ in such a way that $\Delta_i \not\prec \Delta_j$ for all $i < j$. Then, up to isomorphism, the standard module representation

$$(3.1) \quad \zeta(\mathfrak{m}) = Z(\Delta_1) \times \cdots \times Z(\Delta_r)$$

depends only on \mathfrak{m} and not on the choice of permissible ordering of the Δ_i 's and

$$Z(\mathfrak{m}) := \text{soc}(\zeta(\mathfrak{m})).$$

We also have a bijection

$$\mathfrak{m} \mapsto L(\mathfrak{m}) := \text{soc}(L(\Delta_r) \times \cdots \times L(\Delta_1))$$

where $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$, ordered as above.

Let \mathcal{R} be the Bernstein–Zelevinsky Hopf algebra over \mathbb{Z} , namely, the Grothendieck group of \mathcal{C} with product and coproduct induced from the tensor functor (i.e., parabolic induction) and Jacquet functor, respectively. Then, \mathcal{R} is a (commutative) polynomial ring over \mathbb{Z} freely generated by the so-called segment representations. The latter are obtained from Zelevinsky's more general notion of segments [Zel80, §3]. If we limit ourselves to our restricted notion of segments (indexed by pairs of integers $i \leq j$ in $\{1, \dots, n\}$), then we get a Hopf subalgebra \mathcal{R}^n (a polynomial ring over \mathbb{Z} freely generated by $Z([i, j])$, $1 \leq i \leq j \leq n$). The irreducible representations $Z(\mathfrak{m})$, $\mathfrak{m} \in \mathfrak{M}^n$ form a \mathbb{Z} -basis for \mathcal{R}^n . We caution that \mathcal{R}^n should not be confused with the graded part of \mathcal{R} corresponding to the Grothendieck group of $\text{GL}_n(F)$. If π is an irreducible representation of $\text{GL}_k(F)$ we will write $\deg \pi = k$.

Define a bijection

$$\text{Comp} \rightarrow \text{Irr} : C \mapsto \pi(C)$$

by

$$(3.2) \quad Z(\mathfrak{m}) = \pi(\lambda_Q(\mathfrak{m})),$$

for every $\mathfrak{m} \in \mathfrak{M}$. By (2.3) and [MgW86] we also have

$$L(\mathfrak{m}) = \pi(\lambda_{Q^\circ}(\mathfrak{m})) \quad \forall \mathfrak{m} \in \mathfrak{M}.$$

Clearly, $\pi(\text{Comp}(\mathbf{d})) = \text{Irr}(\mathbf{d})$ for every $\mathbf{d} \in \mathbb{N}^n$.

Remark 3.1. Although in this paper we will only use the bijection $C \mapsto \pi(C)$ as a bookkeeping device, it lies much deeper. One can identify $\mathcal{R}^n \otimes \mathbb{C}$ with the Hopf algebra $\mathbb{C}[N]$ of regular functions on the maximal unipotent subgroup N of GL_{n+1} of upper unitriangular matrices. Under this identification, the class of $Z([i, j])$ becomes the (i, j) -coordinate function. On the other hand, $\mathbb{C}[N]$ is also the dual of the universal enveloping algebra $U(\text{Lie } N)$ (a cocommutative Hopf algebra), i.e., the positive part of $U(\mathfrak{sl}_{n+1})$. In turn, $U(\text{Lie } N)$ is endowed with Lusztig's canonical basis [Lus90]. Under the identification above, the dual canonical basis in $\mathbb{C}[N]$ coincides with the classes of irreducible representations in \mathcal{R}^n . The bijection $C \mapsto \pi(C)$ realizes this fact, taking into account that the canonical basis is parameterized by the set

$$\text{Comp} = \text{Comp}^n = \coprod_{\mathbf{d} \in \mathbb{N}^n} \text{Comp}(\mathbf{d})$$

(see [Lus91]) – this parameterization is compatible with the \mathbb{N}^n -grading of $U(\text{Lie } N)$. See [LNT03, Gur21a] and the references therein for more details.

For any irreducible component C we have

$$(3.3) \quad \pi(C)^\vee \otimes |\cdot|^{n+1} = \pi(C^\vee)$$

where on the left-hand side, $^\vee$ denotes the contragredient and we twist by the character $|\det|^{n+1}$.

We also remark that for any $\pi_1, \pi_2, \sigma \in \text{Irr}$ we have (cf. [MgW86, p. 173])

$$(3.4) \quad \sigma \hookrightarrow \pi_1 \times \pi_2 \iff \sigma^\vee \hookrightarrow \pi_2^\vee \times \pi_1^\vee \iff \pi_2 \times \pi_1 \twoheadrightarrow \sigma.$$

Remark 3.2. For any $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{M}$, $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$ occurs in the Jordan–Hölder sequence of $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ (in fact, with multiplicity one). In particular, if $Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2)$ is irreducible, then it is equal to $Z(\mathfrak{m}_1 + \mathfrak{m}_2)$. Thus, by [[(9.5)]2103.12027], if $C_1, C_2 \in \text{Comp}$ strongly commute and $\pi(C_1) \times \pi(C_2)$ is irreducible, then it is equal to $\pi(C_1 * C_2) = \pi(C_1 \oplus C_2)$.

Example 3.3. Going back to Example 2.3, let $C = \lambda_Q(\mathfrak{m})$, $C_1 = \lambda_Q(\mathfrak{m}_1)$, $C_2 = \lambda_Q(\mathfrak{m}_2)$ where

$$\begin{aligned} \mathfrak{m} &= [4, 5] + [2, 4] + [3, 3] + [1, 2], \\ \mathfrak{m}_1 &= [1, 4] + [2, 5], \quad \mathfrak{m}_2 = [1, 2] + [2, 3] + [3, 4] + [4, 5]. \end{aligned}$$

Then, by [Lec03],

$$\pi(C) \times \pi(C) = \pi(C \oplus C) + \pi(C_1) \times \pi(C_2) = \pi(C \oplus C) + \pi(C_1 \oplus C_2).$$

Note that $C \oplus C = \lambda_Q(\mathfrak{m} + \mathfrak{m})$ and $C_1 \oplus C_2 = \lambda_Q(\mathfrak{m}_1 + \mathfrak{m}_2)$.

Conjectures of Geiss and Schröer. The following is for the most part, a special case of a conjecture of Geiss and Schröer for type A .

Conjecture 3.4.A (cf. Conjecture 5.3 of [GS05]). *Let $C_i \in \text{Comp}(\mathfrak{d}_i)$, $i = 1, 2$. Then, $\pi(C_1) \times \pi(C_2)$ is irreducible if and only if*

$$(3.5) \quad \begin{aligned} &\text{there exist nonempty open subsets } U_i \subseteq C_i, i = 1, 2 \\ &\text{such that } \text{Ext}_{\mathbb{H}}^1(x_1, x_2) = 0 \text{ for all } x_1 \in U_1, x_2 \in U_2. \end{aligned}$$

In particular (by [AL, Corollary 3.3]) if $\pi(C_1) \times \pi(C_2)$ is irreducible, then C_1 and C_2 strongly commute (and hence $\pi(C_1) \times \pi(C_2) = \pi(C_1 \oplus C_2)$ by Remark 3.2).

In fact, the original formulation in [ibid.] (which is attributed in part to Marsh and Reineke) is pertaining to the product of two elements in the dual canonical basis of the negative part of the quantized enveloping algebra of the Lie algebra corresponding to a simply laced Dynkin diagram. For type A , [GS05, Conjecture 5.3] amounts to the “if” part of Conjecture 3.4.A as well as a related, though not equivalent, variant of the “only if” part.

In general, the strong commutativity of C_1 and C_2 does not imply (3.5) even if $C_1 = C_2$ (see Example 2.3). However, if C_1 or C_2 is rigid, then Conjecture 3.4.A specializes as follows.

Conjecture 3.4.B. *Let $C_i \in \text{Comp}(\mathfrak{d}_i)$, $i = 1, 2$. Suppose that C_1 or C_2 is rigid. Then, $\pi(C_1) \times \pi(C_2)$ is irreducible if and only if C_1 and C_2 commute (see [AL, Proposition 6.1]).*

Recall that in general, strong (as well as weak) commutativity can be checked efficiently by a randomized algorithm.

Some form of Conjecture 3.4.B appears as Conjecture 1 in [LM20] – we were unaware of [GS05, Conjecture 5.3] at the time.

Finally, we also single out the case $C_1 = C_2$, in which Conjecture 3.4.A specializes as follows. We say that π is \square -irreducible if $\pi \times \pi$ is irreducible.

Conjecture 3.4.C. *For any $C \in \text{Comp}(\mathfrak{d})$*

$$\pi(C) \text{ is } \square\text{-irreducible} \iff C \text{ is rigid.}$$

This conjecture was stated in [LM18], albeit with an inaccurate attribution to a later reference.

At any rate, as far as we know, Conjectures 3.4.B and 3.4.C (let alone Conjecture 3.4.A) are wide open in general, even in one direction. We also remark that if neither C_i is rigid and $C_1 \neq C_2$, then it is unclear how to check condition (3.5) of Conjecture 3.4.A algorithmically.

4. \square -IRREDUCIBLE REPRESENTATIONS

For any two representations of finite length π and σ of GL we denote by

$$R_{\pi;\sigma} : \pi \times \sigma \rightarrow \sigma \times \pi$$

the normalized intertwining operator obtained by taking the leading term in the Laurent expansion of the unnormalized intertwining operator at $s = 0$. (See [LM18] for more details.)

We recall the following results, which are adaptations of basic results of Kang, Kashiwara, Kim and Oh.

Theorem 4.1.A ([LM18], after [KKKO15]). *The following conditions are equivalent for a representation π of finite length of $\mathrm{GL}_r(F)$.*

- (1) π is \square -irreducible.
- (2) $\mathrm{End}_{\mathrm{GL}_{2r}(F)}(\pi \times \pi) = \mathbb{C}$.
- (3) $R_{\pi;\pi}$ is a scalar.

Let us say that a representation τ of finite length is SI if its socle is irreducible and occurs with multiplicity one in the Jordan–Hölder sequence of τ .

Theorem 4.1.B ([LM18], after [KKKO15]). *Let π_1 and π_2 be two irreducible representations of GL. Suppose that at least one of them is \square -irreducible. Then,*

- (1) $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ are SI.
- (2) $\mathrm{soc}(\pi_1 \times \pi_2)$ is the image of the normalized intertwining operator $\pi_2 \times \pi_1 \rightarrow \pi_1 \times \pi_2$.
- (3) $\pi_1 \times \pi_2$ is irreducible if and only if $\mathrm{soc}(\pi_1 \times \pi_2) \simeq \mathrm{soc}(\pi_2 \times \pi_1)$.
- (4) If $\pi_1 \times \pi_2$ is irreducible and both π_1 and π_2 are \square -irreducible, then $\pi_1 \times \pi_2$ is \square -irreducible.

Note that part 3 was not stated explicitly in [LM18] but it follows easily for the other parts. Indeed, if $\sigma := \mathrm{soc}(\pi_1 \times \pi_2) \simeq \mathrm{soc}(\pi_2 \times \pi_1)$, then by (3.4) σ occurs both as a subrepresentation and as a quotient of $\tau := \pi_1 \times \pi_2$. Since the multiplicity of σ in the Jordan–Hölder sequence of τ is one, this means that σ is a direct summand of τ . However, since $\sigma = \mathrm{soc}(\tau)$, this implies that $\tau = \sigma$.

Corollary 4.1.C. *Let π_1, \dots, π_k be irreducible representations such that at most one of them is not \square -irreducible. Assume that $\pi_i \times \pi_j$ is irreducible for all $i \neq j$. Then, $\pi_1 \times \dots \times \pi_k$ is irreducible.*

This follows by induction on k , by analyzing the normalized intertwining operator. See [GM21] for a more general statement.

Denote by cos the cosocle of a representation.

Theorem 4.1.D (cf. Corollary 3.7 of [KKKO15]). *Let π be a \square -irreducible representation of GL . Then, the two functions*

$$\begin{aligned}\sigma &\mapsto \mathrm{soc}(\sigma \times \pi) = \mathrm{cos}(\pi \times \sigma) \\ \sigma &\mapsto \mathrm{soc}(\pi \times \sigma) = \mathrm{cos}(\sigma \times \pi)\end{aligned}$$

are injective on the set of irreducible representations of GL .

For convenience, we adapt the proof of Proposition 3.6 and Corollary 3.7 of [KKKO15] to the setting of representation theory of GL .

Proof. We first note that it is enough to prove the injectivity of the first map, since the injectivity of the second one would then follow by passing to the contragredient.

Let $r = \deg \pi$ and let J_π be the left-adjoint of the functor $\sigma \mapsto \sigma \times \pi$. Thus, for any representation τ of $\mathrm{GL}_s(F)$, $J_\pi(\tau)$ is the $(\mathrm{GL}_r(F), \pi)$ -coinvariants (in the second factor) of the (normalized) Jacquet module $J(\tau)$ of τ with respect to the parabolic subgroup of type $(s - r, r)$ (interpreted as 0 if $s < r$).

Suppose now that τ is irreducible and $\omega := J_\pi(\tau) \neq 0$. Let

$$\iota : \tau \rightarrow \omega \times \pi$$

be the canonical morphism corresponding to the identity morphism of ω . Since the latter does not factor through any proper submodule of ω ,

$$(4.1) \quad \iota(\tau) \text{ is not contained in } \kappa \times \pi \text{ for any proper subrepresentation } \kappa \subsetneq \omega.$$

Moreover, since τ is irreducible, ι is a monomorphism.

Next, we claim that

$$(4.2) \quad \text{the image of the intertwining operator } R_{\pi; \omega} : \pi \times \omega \rightarrow \omega \times \pi \text{ is } \iota(\tau).$$

Recall that the restriction of

$$R_{\pi; \omega \times \pi} : \pi \times \omega \times \pi \rightarrow \omega \times \pi \times \pi$$

to $\pi \times \tau$ (via ι) is either 0 or $R_{\pi; \tau}$, and in any case its image is contained in $\tau \times \pi$ (see [LM18, Lemma 2.3(3)]). On the other hand,

$$R_{\pi; \omega \times \pi} = (\mathrm{id}_\omega \times R_{\pi; \pi}) \circ (R_{\pi; \omega} \times \mathrm{id}_\pi)$$

which by assumption, is a non-zero scalar multiple of $R_{\pi; \omega} \times \mathrm{id}_\pi$. Thus, we get a commutative diagram

$$\begin{array}{ccc} \pi \times \tau & \longrightarrow & \tau \times \pi \\ \downarrow \mathrm{id}_\pi \times \iota & & \downarrow \iota \times \mathrm{id}_\pi \\ \pi \times \omega \times \pi & \xrightarrow{R_{\pi; \omega} \times \mathrm{id}_\pi} & \omega \times \pi \times \pi \end{array}$$

This means that

$$\pi \times \iota(\tau) \subseteq R_{\pi; \omega}^{-1}(\iota(\tau)) \times \pi.$$

Hence, by [LM18, Corollary 2.2], there exists a subrepresentation κ of ω such that $R_{\pi; \omega}(\pi \times \kappa) \subseteq \iota(\tau)$ and $\iota(\tau) \subseteq \kappa \times \pi$. By (4.1), $\kappa = \omega$, which implies (4.2).

It follows that we can factorize $R_{\pi; \omega}$ as $\iota \circ \eta$ where

$$\eta : \pi \times \omega \rightarrow \tau.$$

We claim that for any proper subrepresentation κ of ω we have $\eta|_{\pi \times \kappa} \equiv 0$. Indeed, otherwise, the restriction of $R_{\pi; \omega}$ to $\pi \times \kappa$ would be non-zero, and hence equal to $R_{\pi; \kappa}$. This would contradict (4.1) and (4.2).

We infer that ω has a unique maximal (proper) subrepresentation, i.e., $\cos(\omega)$ is irreducible.

Finally, if σ is an irreducible representation of GL and $\tau = \text{soc}(\sigma \times \pi)$, then the relation

$$\text{Hom}(\omega, \sigma) = \text{Hom}(\tau, \sigma \times \pi)$$

implies that $\sigma = \cos(\omega)$. In particular, σ is determined by τ . \square

5. A NEW CONJECTURE

We make the following conjecture which is inspired by [GS05, Conjecture 5.3].

Conjecture 5.1. $\pi(C_1 * C_2)$ is a subrepresentation of $\pi(C_1) \times \pi(C_2)$. In particular, if $\pi(C_1)$ or $\pi(C_2)$ is \square -irreducible, then $\pi(C_1 * C_2) = \text{soc}(\pi(C_1) \times \pi(C_2))$.

Roughly speaking, Conjecture 5.1 says that a generic extension of a generic $x_1 \in C_1$ by a generic $x_2 \in C_2$ determines an irreducible subrepresentation of $\pi(C_1) \times \pi(C_2)$.

A good example to bear in mind is the following.

Example 5.2. As in Example 2.1 let $C = \lambda_Q(\mathbf{m})$ be a lamina with

$$\mathbf{m} = \sum_{i=1}^r [a_i, b_i] \text{ with } a_1 < \cdots < a_r \text{ and } b_1 < \cdots < b_r.$$

For each i let t_i be such that $a_i - 1 \leq t_i \leq b_i$ and $t_1 < \cdots < t_r$. Then, $C = C_1 * C_2$ where

$$C_1 = \lambda_Q\left(\sum_{i=1}^r [a_i, t_i]\right) \text{ and } C_2 = \lambda_Q\left(\sum_{i=1}^r [t_i + 1, b_i]\right).$$

(As usual, we ignore any empty segments in the sums.) By [KL12], we have $\pi(C) = \text{soc}(\pi(C_1) \times \pi(C_2))$.

A weak form of Conjecture 5.1 is that $\pi(C_1 * C_2)$ occurs as a subquotient of $\pi(C_1) \times \pi(C_2)$. In fact, this would follow from the first part of [GS05, Conjecture 5.3] (for type A) together with [AL, Theorem 3.1]. However, even this weaker conjecture is open as far as we know (even assuming that C_1 or C_2 is rigid).

The decomposition of $\pi(C) \times \pi(C)$ in the Grothendieck group was computed in [Lap21] using computer calculations for C of the form (6.1) below, for $n \leq 6$. In these cases,

- (1) The multiplicity of $\pi(C * C)$ in the Jordan–Hölder sequence of $\pi(C) \times \pi(C)$ is always one or two.
- (2) This multiplicity is one if and only if C strongly commutes with itself if and only if $\pi(C) \times \pi(C)$ is of length at most two.

Some consequences of Conjecture 5.1 and its interrelation with Conjectures 3.4.A–3.4.C are stated in the following

Lemma 5.3. *Assume that Conjecture 5.1 holds. Then, for any $\mathbf{m}, \mathbf{n} \in \mathfrak{M}$*

- (1) *If $\lambda_Q(\mathbf{m} + \mathbf{n}) = \lambda_Q(\mathbf{m}) * \lambda_Q(\mathbf{n})$ (cf. [AL, (9.4)]), then $Z(\mathbf{m} + \mathbf{n}) \hookrightarrow Z(\mathbf{m}) \times Z(\mathbf{n})$.*
- (2) *If $\lambda_Q(\mathbf{m})$ and $\lambda_Q(\mathbf{n})$ strongly commute, then $Z(\mathbf{m} + \mathbf{n})$ is a direct summand of $Z(\mathbf{m}) \times Z(\mathbf{n})$. In particular, if $Z(\mathbf{m})$ or $Z(\mathbf{n})$ is \square -irreducible, then $Z(\mathbf{m}) \times Z(\mathbf{n})$ is irreducible.*
- (3) *If $\lambda_Q(\mathbf{m})$ and $\lambda_Q(\mathbf{n})$ do not strongly commute, then $Z(\mathbf{m}) \times Z(\mathbf{n})$ is reducible.*

Indeed, the first part follows from (3.2). The last two parts follow from [[(9.5)]2103.12027, (3.4) and Remark 3.2. (For the last part, the weaker form of Conjecture 5.1 suffices.)

Note that in the case where $Z(\mathfrak{m})$ or $Z(\mathfrak{n})$ is \square -irreducible, the first part is precisely Conjecture 2 of [LM20]. Thus, Conjecture 5.1 is a strengthening of [LM20, Conjecture 2].

If we assume Conjecture 3.4.C, then Conjecture 5.1 also gives a randomized algorithm to determine the multisegment \mathfrak{m}_2 (if exists) such that $Z(\mathfrak{m}) = \text{soc}(Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2))$ given two multisegments $\mathfrak{m}, \mathfrak{m}_1$ such $Z(\mathfrak{m}_1)$ is \square -irreducible. (See Theorem 4.1.D and §2.4.)

An easy case of Conjecture 5.1 is the following.

Lemma 5.4. *Let $\mathfrak{m}_1, \mathfrak{m}_2$ be two multisegments and let $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$. Assume that $\text{Ext}_Q^1(M_Q(\mathfrak{m}_1), M_Q(\mathfrak{m}_2)) = 0$ (cf. (2.1a)). Let $C_i = \lambda_Q(\mathfrak{m}_i)$, $i = 1, 2$ and $C = \lambda_Q(\mathfrak{m})$. Then, $C_1 * C_2 = C$ and $\text{soc}(\pi(C_1) \times \pi(C_2)) = \pi(C)$. The same holds if we replace Q by Q° throughout.*

Proof. If $\text{Ext}_Q^1(M_Q(\mathfrak{m}_1), M_Q(\mathfrak{m}_2)) = 0$, then any extension of $x_1 \in \mu_Q(\mathfrak{m}_1)$ by $x_2 \in \mu_Q(\mathfrak{m}_2)$ lies in $\mu_Q(\mathfrak{m})$. Hence, $C = C_1 * C_2$ by [AL, (3.3)]. The relation

$$Z(\mathfrak{m}) = \text{soc}(Z(\mathfrak{m}_1) \times Z(\mathfrak{m}_2))$$

(under the above condition) is well known and follows from the fact that

$$\zeta(\mathfrak{m}) = \zeta(\mathfrak{m}_1) \times \zeta(\mathfrak{m}_2).$$

The second part is similar. □

The following analogue of [AL, Lemma 8.1] gives another piece of evidence for Conjecture 5.1.

Lemma 5.5. *Let π_1, π_2 be irreducible representations of GL with π_2 \square -irreducible. Let $\pi = \text{soc}(\pi_1 \times \pi_2)$. Assume that $\pi \times \pi_1$ is irreducible. Then, π_1 and π are \square -irreducible.*

Proof. The \square -irreducibility of π was proved in [LM18, Lemma 2.10]. To show that π_1 is also \square -irreducible, consider the morphisms

$$\pi_2 \times \pi_1 \times \pi_1 \xrightarrow{f} \pi_1 \times \pi_2 \times \pi_1 \xrightarrow{g} \pi_1 \times \pi_1 \times \pi_2$$

where $f = R_{\pi_2; \pi_1} \times \text{id}_{\pi_1}$ and $g = \text{id}_{\pi_1} \times R_{\pi_2; \pi_1}$. The image of f is $\pi \times \pi_1$. The composition $h = g \circ f$ is non-zero since $\pi \times \pi_1 \not\subseteq \pi_1 \times \omega$ where $\omega = \text{Ker } R_{\pi_2; \pi_1}$ by [LM18, Corollary 2.2]. Therefore, by [LM18, Lemma 2.3], h is equal to a nonzero scalar multiple of $R_{\pi_2; \pi_1 \times \pi_1}$.

The image of g is $\pi_1 \times \pi$, which is irreducible by assumption. Therefore, the image of h coincides with $\pi_1 \times \pi$.

Let τ be an irreducible subrepresentation of $\pi_1 \times \pi_1$. The restriction f' of f to $\pi_2 \times \tau$ is nonzero since $\pi_2 \times \tau \not\subseteq \omega \times \pi_1$, again by [LM18, Corollary 2.2]. Since the image of f is irreducible, it coincides with the image of f' . Let $h' = g \circ f'$ be the restriction of h to $\pi_2 \times \tau$. Then, the image of h' is equal to the image of h . In particular, h' is non-zero. Therefore, by [LM18, Lemma 2.3], h' is a non-zero scalar multiple of $R_{\pi_2; \tau}$. It follows that the image of h' is contained in $\tau \times \pi_2$. We obtain $\pi_1 \times \pi \subseteq \tau \times \pi_2$, and therefore $\tau = \pi_1 \times \pi_1$, as before. We conclude that π_1 is \square -irreducible. □

Definition 5.6. Let $C \in \text{Comp}$. We say that C is good if for every $D \in \text{Comp}$ we have

$$\pi(C * D) \hookrightarrow \pi(C) \times \pi(D) \text{ and } \pi(D * C) \hookrightarrow \pi(D) \times \pi(C).$$

In this case, if C is rigid, we also say that any rigid element of C is good.

Conjecture 5.1 states that every $C \in \text{Comp}$ is good.

We hasten to say that at the moment, we do not know how to prove that a non-rigid irreducible component is good even in a single example.

Remark 5.7. Suppose that \mathfrak{C} is a set of irreducible components that is invariant under $C \mapsto C^\vee$. Then, in order to prove that every $C \in \mathfrak{C}$ is good, it suffices to show that $\pi(C * D) \hookrightarrow \pi(C) \times \pi(D)$ for $C \in \mathfrak{C}$ and $D \in \text{Comp}$. Indeed, by (3.4), (3.3) and (2.5)

$$\begin{aligned} \pi(D * C) \hookrightarrow \pi(D) \times \pi(C) &\iff \pi(D * C)^\vee \hookrightarrow \pi(C)^\vee \times \pi(D)^\vee \\ &\iff \pi(C^\vee * D^\vee) \hookrightarrow \pi(C^\vee) \times \pi(D^\vee). \end{aligned}$$

Another sanity check for Conjecture 5.1 is the following

Lemma 5.8. Suppose that $C_1, \dots, C_k \in \text{Comp}$ are good and pairwise strongly commute and that $\pi(C_i)$ is \square -irreducible for all i . Then, $C_1 * \dots * C_k$ is good.⁴

Proof. By induction, it is enough to consider the case $k = 2$. Let $C_i \in \text{Comp}$, $i = 1, 2, 3$ and suppose that C_1 and C_2 are good and strongly commute. Let $\pi_i = \pi(C_i)$, $i = 1, 2, 3$ and suppose that π_1 and π_2 are \square -irreducible. By assumption and Theorem 4.1.B, $\pi_1 \times \pi_2$ is \square -irreducible and $\pi_1 \times \pi_2 = \pi(C_1 * C_2)$. In particular, $\pi_1 \times \pi_2 \times \pi_3$ is SI. Thus,

$$\begin{aligned} \text{soc}(\pi_1 \times \pi_2 \times \pi_3) &= \text{soc}(\pi_1 \times \text{soc}(\pi_2 \times \pi_3)) = \text{soc}(\pi_1 \times \pi(C_2 * C_3)) \\ &= \pi(C_1 * (C_2 * C_3)) = \pi((C_1 * C_2) * C_3). \end{aligned}$$

Here we used the fact that C_2 and C_1 are good in the second and third equalities, respectively, as well as [AL, Corollary 7.4] for the last one. In a similar vein we show that

$$\text{soc}(\pi_3 \times \pi_1 \times \pi_2) = \pi(C_3 * (C_1 * C_2)).$$

It follows that $C_1 * C_2$ is good as required. \square

For convenience, for $C \in \text{Comp}(\mathbf{d})$ we set $\max C = \max\{i \mid \mathbf{d}(i) \neq 0\}$ with the convention that $\max C = 0$ if $C = 0$. Similarly define $\min C$.

To go further, we prove the following general reduction step.

Lemma 5.9. Let $C \in \text{Comp}$. Assume that $\pi(C * D)$ is a subrepresentation of $\pi(C) \times \pi(D)$ for every $D \in \text{Comp}$ such that $\max D \leq \max C$, or alternatively, for every $D \in \text{Comp}$ such that $\min D \leq \min C$. Then, $\pi(C * D)$ is a subrepresentation of $\pi(C) \times \pi(D)$ for every $D \in \text{Comp}$.

Proof. We consider the first case. We argue by induction on $t := \max D$. The statement is trivial if $D = 0$. For the induction step, we may assume of course that $t > \max C$, otherwise there is nothing to prove. Let $D = \lambda_{Q^\circ}(\mathbf{n})$ be the Q° -parameterization of D by a multisegment. Decompose \mathbf{n} as $\mathbf{n}_1 + \mathbf{n}_2$ (possibly

⁴When there is no danger of confusion we will write $C_1 * \dots * C_k$ for the ‘‘product’’ in arbitrary order.

with $\mathbf{n}_1 = 0$) where $\mathbf{n}_2 \neq 0$ consists of the segments Δ in \mathbf{n} (counted with their multiplicities) such that $e(\Delta) = t$. Let $D_i = \lambda_{Q^\circ}(\mathbf{n}_i)$, $i = 1, 2$. Note that $\max D_1 < t$. Then, D_2 is rigid and by Lemma 5.4 and (2.4), $D = D_1 * D_2$ and $\text{hom}_\Pi(D_2, C) = 0$. Therefore, by [AL, Corollary 7.4]

$$C * D = (C * D_1) * D_2.$$

Thus,

$$\pi(C * D) = \pi((C * D_1) * D_2).$$

Since $\max(C * D_1) < t$ we may apply Lemma 5.4 to $C * D_1$ and D_2 to obtain

$$\pi((C * D_1) * D_2) = \text{soc}(\pi(C * D_1) \times \pi(D_2)).$$

By induction hypothesis,

$$\pi(C * D_1) \hookrightarrow \pi(C) \times \pi(D_1).$$

Hence,

$$\text{soc}(\pi(C * D_1) \times \pi(D_2)) \hookrightarrow \text{soc}(\pi(C) \times \pi(D_1) \times \pi(D_2)).$$

However, by [LM20, Lemma 9]

$$\text{soc}(\pi(C) \times \pi(D_1) \times \pi(D_2)) = \text{soc}(\pi(C) \times \pi(D)).$$

All in all, we obtain

$$\pi(C * D) \hookrightarrow \text{soc}(\pi(C) \times \pi(D))$$

as required.

A similar argument (using λ_Q) applies to the case where $\min D \leq \min C$. \square

From Lemmas 5.9 and 5.4 and Remark 5.7 we conclude

Corollary 5.10. *Let Δ be a segment. Then, $\lambda_Q(\Delta)$ and $\lambda_{Q^\circ}(\Delta)$ are good.*

Remark 5.11. The proof of Lemma 5.9, together with Lemma 5.4, gives a combinatorial recipe for the computation of $\mathbf{m} * \mathbf{n}$ in the case where \mathbf{m} or \mathbf{n} is a single segment. Note that the recipe involves the Mœglin-Waldspurger involution since we switch between λ_Q and λ_{Q° .

6. REGULAR MULTISEGMENTS

In this section we recall the results of [LM18] for regular multisegments and state our main theoretical result towards Conjecture 5.1.

We start with a special case. Consider the quiver $Q = A_{2n-1}$ and graded dimension $\mathbf{d} = (1, 2, \dots, n, n-1, \dots, 1)$

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \dots \xrightarrow{n} \bullet \xrightarrow{n-1} \dots \xrightarrow{1} \bullet$$

Fix $V = \bigoplus_{i=1}^{2n-1} V_i$ with $\dim V_i = \mathbf{d}(i) = \min(i, 2n-i)$ (so that $\dim V = n^2$). As in [KS97, §8], consider the following open, G_V -stable subset of $R_Q(V)$

$$R_Q^b(V) = \{T_+ \in R_Q(V) \mid T_+|_{V_i} \text{ is injective } \forall i < n \text{ and surjective } \forall i \geq n\}.$$

For any $T_+ \in R_Q^b(V)$ we assign the following two complete flags of V_n

$$\begin{aligned} 0 \subsetneq T_+^{n-1}(V_1) \subsetneq T_+^{n-2}(V_2) \subsetneq \dots \subsetneq T_+(V_{n-1}) \subsetneq V_n, \\ 0 \subsetneq \text{Ker}(T_+|_{V_n}) \subsetneq \text{Ker}(T_+^2|_{V_n}) \subsetneq \dots \subsetneq \text{Ker}(T_+^{n-1}|_{V_n}) \subsetneq V_n. \end{aligned}$$

The resulting map $R_Q^b(V) \rightarrow X \times X$, where X is the (complete) flag variety of GL_n , is a principal $\prod_{i \neq n} \mathrm{GL}(V_i)$ -bundle. Hence, we get an isomorphism of GL_n -varieties

$$R_Q^b(V) / \prod_{i \neq n} \mathrm{GL}(V_i) \simeq X \times X.$$

Thus, the G_V -orbits in $R_Q^b(V)$ correspond to the GL_n -orbits in $X \times X$, which are parameterized by the symmetric group S_n .

If $Y_w, w \in S_n$ is a GL_n -orbit in $X \times X$ (i.e., a Bruhat cell), then

$$(6.1) \quad C_w = \lambda_Q([1, w(1) + n - 1] + \cdots + [n, w(n) + n - 1]) \in \mathrm{Comp}(\mathbf{d})$$

is the closure of the conormal bundle of the G_V -orbit in $R_Q^b(V)$ corresponding to Y_w .

Denote by X_w the closure of Y_w (i.e., the Schubert variety).

For example, $X_e = Y_e = \Delta X$ (diagonal embedding) for the identity permutation, while for the longest element, Y_{w_0} is open and $X_{w_0} = X \times X$.

The following result affirms a special case of Conjecture 3.4.C.

Theorem 6.1 ([LM18]). *The following conditions on $w \in S_n$ are equivalent.*⁵

- (1) C_w is rigid.
- (2) The conormal bundle of $Y_w \subseteq X$ admits an open GL_n -orbit.
- (3) $X_{w_0 w}$ is smooth.
- (4) $\pi(C_w) \times \pi(C_w)$ is irreducible.

We also recall that $X_{w_0 w}$ is smooth $\iff X_{w_0 w}$ is rationally smooth $\iff w$ is 1324 and 2143 avoiding [Deo85, LS90].

Note that Example 2.3 is not covered by Theorem 6.1, but we will shortly discuss the more general results of [LM18], which cover this example.

In passing, we mention that the equivalence of the purely geometric conditions 2 and 3 (in the more general context) led Anton Mellit to make the following conjecture, for which there is overwhelming computational support.

Conjecture 6.2 (Mellit). *Let $x, w \in S_n$ with $Y_w \subseteq X_x$ (i.e., $w \leq x$ in the Bruhat order). Suppose that X_x is smooth. Then, the following conditions are equivalent.*

- (1) The conormal bundle of $Y_w \subseteq X_x$ admits an open GL_n -orbit.
- (2) The smooth locus of $X_{w_0 w}$ contains $Y_{w_0 x}$.

In the case where x is 231-avoiding (which implies that X_x is smooth), this conjecture (along with a representation-theoretic criterion) was proved in [LM18], albeit indirectly.

Remark 6.3. In general, for any two parabolic subgroups P and Q of GL_n , one can realize in a similar way the GL_n -action on $P \backslash \mathrm{GL}_n \times Q \backslash \mathrm{GL}_n$ in the geometry of a G_V -action on an open subset of $R_Q(V)$ for a suitable graded vector space V . Unfortunately, the naive generalization of Theorem 6.1 to this context is not true – nor do we have a conjectural replacement for the smoothness condition.

For any segment $\Delta = [a, b]$ we write $b(\Delta) = a$ and $e(\Delta) = b$.

The multisegments considered above satisfy the following property (cf. (6.1)): writing $\mathbf{m} = \Delta_1 + \cdots + \Delta_k$, we have $b(\Delta_i) \neq b(\Delta_j)$ and $e(\Delta_i) \neq e(\Delta_j)$ for all $i \neq j$. Such multisegments are called regular.

⁵The equivalence of the first two conditions is easy.

Recall the setup of [LM18, §6]. We say that a (necessarily regular) multisegment is of type 4231 if it can be written as $\mathbf{m} = \Delta_1 + \cdots + \Delta_k$ with $k \geq 4$ such that the following conditions are satisfied.⁶

- (1) $\Delta_i \prec \Delta_{i-1}$, $i = 3, \dots, k$.
- (2) $b(\Delta_k) < b(\Delta_1) < b(\Delta_{k-1})$.
- (3) $e(\Delta_3) < e(\Delta_1) < e(\Delta_2)$.

Similarly, a (necessarily regular) multisegment is of type 3412 if it can be written as $\mathbf{m} = \Delta_1 + \cdots + \Delta_k$ with $k \geq 4$ such that the following conditions are satisfied.

- (1) $\Delta_i \prec \Delta_{i-1}$, $i = 4, \dots, k$ and $\Delta_2 \prec \Delta_1$.
- (2) $b(\Delta_2) < b(\Delta_k) < b(\Delta_1) < b(\Delta_{k-1})$.
- (3) $e(\Delta_4) < e(\Delta_2) < e(\Delta_3) < e(\Delta_1)$.

We say that a regular multisegment is balanced if it does not admit any multisegment of type 4231 or type 3412 as a submultisegment. The main result of [LM18], for which Theorem 6.1 is a special case, is the following.

Theorem 6.4 ([LM18]). *Suppose that \mathbf{m} is a regular multisegment. Then, the following conditions are equivalent.*

- (1) \mathbf{m} is balanced.
- (2) $Z(\mathbf{m})$ is \square -irreducible.
- (3) $\lambda_Q(\mathbf{m})$ is rigid.

We can now state our main result towards Conjecture 5.1.

Theorem 6.5. *Suppose that $C = \lambda_Q(\mathbf{m})$ where \mathbf{m} is a balanced (regular) multisegment. Then, C is good. In particular, by Lemma 5.8, every rigid laminated module (see §2.5.1) is good.*

Theorem 6.5 considerably extends Theorem 1 of [LM20]. (The latter applies only to rigid laminated modules and proves a weak version of Conjecture 5.1.) We will prove Theorem 6.5 in §8 below by combining ideas from [LM20] and [LM18] and pushing them further.

Addendum: primality for regular multisegments⁷. As a complement, we give a (surprisingly?) simple necessary and sufficient condition for the property that $Z(\mathbf{m})$ is a prime (i.e., irreducible) element in \mathcal{R} , in the case where \mathbf{m} is regular.

Note that if $Z(\mathbf{m})$ is prime, then we cannot write non-trivially $Z(\mathbf{m}) = Z(\mathbf{m}_1) \times Z(\mathbf{m}_2)$ (where necessarily $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$). In general, it is not clear whether the converse holds.

We first need an elementary lemma. For this section only, let A be an integral domain. Let $R = A[x_i, i \in I]$ be the (commutative) ring of polynomials over A in the variables x_i , $i \in I$ for some finite nonempty set I . For any subset $J \subseteq I$ let $x_J \in R$ be the monomial $\prod_{i \in J} x_i$.

Lemma 6.6. *Consider a polynomial $f \in R$ of the form $f = \sum_{J \subseteq I} c_J x_J$, $c_J \in A$.*

Assume that for any $i \in I$, there exists $J \subseteq I$ such that $i \in J$ and $c_J \neq 0$.

Consider the graph Γ whose vertex set is I and we connect i and j if there is no $J \subseteq I$ such that $i, j \in J$ and $c_J \neq 0$. Assume that Γ is connected.

⁶Note that in [LM18, Definition 6.10] a different ordering of the Δ_i 's was used, justifying the terminology.

⁷The results of this subsection will not be used in the rest of the paper.

Finally, assume that $\sum_{J \subseteq I} Ac_J = A$.
Then, f is irreducible in R .

Proof. By assumption, the degree of x_i in f is 1 for every $i \in I$.

Suppose that $f = gh$. Then, for each i , precisely one of g and h is independent of x_i (i.e., belongs to $A[x_j, j \neq i]$).

We claim that if g is independent of x_i , then g is also independent of x_j for any edge (i, j) in Γ .

Indeed, write $h = a + bx_i$ where $a, b \in A[x_k, k \neq i]$ and $b \neq 0$. Assume on the contrary that g is dependent on x_j . Then, h is independent of x_j and therefore $a, b \in A[x_k, k \neq i, j]$. Write $g = c + dx_j$ with $c, d \in A[x_k, k \neq i, j]$ and $d \neq 0$. Then, $f = ac + bcx_i + adx_j + bdx_ix_j$ and therefore x_ix_j appears non-trivially in f , in contradiction to the assumption that (i, j) is an edge.

By our assumption on Γ , it follows that either g or h is a constant, which must be a unit by our last assumption. \square

Let now \mathbf{m} be a multisegment. We say that \mathbf{m} is *split* if we can write $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ nontrivially, where for every two segments Δ_1 in \mathbf{m}_1 and Δ_2 in \mathbf{m}_2 we have $\Delta_1 \not\prec \Delta_2$ and $\Delta_2 \not\prec \Delta_1$ (i.e., $Z(\Delta_1) \times Z(\Delta_2)$ is irreducible).

Proposition 6.7. *The following conditions are equivalent for a regular multisegment \mathbf{m} .*

- (1) $Z(\mathbf{m})$ is not a prime element of \mathcal{R} .
- (2) $Z(\mathbf{m}) = Z(\mathbf{m}_1) \times Z(\mathbf{m}_2)$ for some nontrivial decomposition $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$.
- (3) \mathbf{m} is split.

Proof. Clearly, $3 \implies 2 \implies 1$. (These implications hold without the assumption that \mathbf{m} is regular.)

It remains to prove that if \mathbf{m} is not split (but regular), then $Z(\mathbf{m})$ is prime in \mathcal{R} .

Write $\mathbf{m} = \Delta_1 + \dots + \Delta_k$ and

$$(6.2) \quad \zeta(\mathbf{m}) = \sum_{\mathbf{n} \in S} c'_n Z(\mathbf{n}),$$

(see (3.1)) where c'_n are positive integers for all \mathbf{n} . We have $c'_m = 1$, and the set S was described explicitly by Zelevinsky [Zel80, §7]. Denote by I' the set of all segments that occur in a multisegment in S . In particular, $I' \supseteq \{\Delta_1, \dots, \Delta_k\}$. The following facts follow easily from Zelevinsky's description.

- (1) Every $\mathbf{n} \in S$ is regular.
- (2) For every $\Delta \in I'$ we have $b(\Delta) \in \{b(\Delta_1), \dots, b(\Delta_k)\}$ and $e(\Delta) \in \{e(\Delta_1), \dots, e(\Delta_k)\}$.
- (3) Suppose that $\Delta_r \prec \Delta_s$, but there does not exist t such that $\Delta_r \prec \Delta_t$ and $\Delta_t \prec \Delta_s$. (For brevity, we will write $\Delta_r \prec\prec \Delta_s$ in this case.) Let \mathbf{n} be the multisegment obtained from \mathbf{m} by replacing Δ_r and Δ_s by $\Delta_r \cup \Delta_s$ and $\Delta_r \cap \Delta_s$ (or just $\Delta_r \cup \Delta_s$ if $\Delta_r \cap \Delta_s = \emptyset$). We will say that \mathbf{n} is a neighbor of \mathbf{m} . Then, $\mathbf{n} \in S$, and in particular $\Delta_r \cup \Delta_s \in I'$. Moreover, for any $\mathbf{m}' \in S \setminus \{\mathbf{m}, \mathbf{n}\}$, the representation $Z(\mathbf{n})$ does not occur in $\zeta(\mathbf{m}')$.

Inverting the relation (6.2) (and its analogues for $\zeta(\mathbf{n})$, for any $\mathbf{n} \in S$) we get

$$(6.3) \quad Z(\mathbf{m}) = \sum_{\mathbf{n} \in S} c_n \zeta(\mathbf{n})$$

where c_n are integers with $c_m = 1$. The only other pertinent piece of information for us is that $c_n = -c'_n \neq 0$ for every neighbor n of m . (In fact, one can show that $c'_n = 1$ in this case.)⁸

Let I be the set of segments that occur in a multisegment $n \in S$ for which $c_n \neq 0$. Thus, $\{\Delta_1, \dots, \Delta_k\} \subseteq I \subseteq I'$ and if $\Delta_r \prec \Delta_s$, then $\Delta_r \cup \Delta_s \in I$. (In reality, $I = I'$.) Consider the free variables $Z(\Delta)$, $\Delta \in I$. The relation (6.3) expresses $f = Z(m) \in \mathcal{R}$ as a sum of monomials in $Z(\Delta)$, $\Delta \in I$ (corresponding to $\zeta(n)$) with coefficients c_n . Consider the graph Γ defined in Lemma 6.6.

Let $\Delta \in I$. If r is such that $b(\Delta) = b(\Delta_r)$, then (Δ, Δ_r) is an edge in Γ (since every $n \in S$ is regular). Similarly, if s is such that $e(\Delta) = e(\Delta_s)$, then (Δ, Δ_s) is an edge in Γ .

Suppose that $\Delta_r \prec \Delta_s$. Then, $\Delta_r \cup \Delta_s \in I$ and both $(\Delta_r, \Delta_r \cup \Delta_s)$ and $(\Delta_r \cup \Delta_s, \Delta_s)$ are edges in Γ (by the previous paragraph).

On the other hand, the fact that m is not split means that the graph whose vertices are $\{1, \dots, k\}$ and whose edges are given by $\Delta_r \prec \Delta_s$, is connected. It follows that Γ is connected. The primality of $Z(m)$ in the subring of \mathcal{R} generated by $Z(\Delta)$, $\Delta \in I$ (and hence, the primality of $Z(m)$ in \mathcal{R} itself), follows from Lemma 6.6 (for $A = \mathbb{Z}$). \square

Remark 6.8. Proposition 6.7 ceases to hold if we only assume that all the segments in m occur with multiplicity one. For instance, consider $m = m_1 + m_2$ where

$$m_1 = [1, 2] + [2, 3] = \Delta_1 + \Delta_2, \quad m_2 = [3, 3] = \Delta_3.$$

Then, $Z(m) = Z(m_1) \times Z(m_2)$ even though m is not split since $\Delta_1 \prec \Delta_2$ and $\Delta_1 \prec \Delta_3$.

7. BASIC REPRESENTATIONS AND IRREDUCIBLE COMPONENTS

In this section we prove an analogue of a special case of the Baumann–Kamnitzer–Tingley decomposition [BKT14] for the representation theory of GL.

7.1. By definition, a *basic* representation is one of the form $\sigma = Z(\Delta)$ or $\sigma = L(\Delta)$ for some segment Δ .⁹

Let σ be a basic representation. Depending on whether $\sigma = Z(\Delta)$ or $\sigma = L(\Delta)$, we say that a basic representation is σ -saturated if it is of the form $Z(\Delta')$ (resp., $L(\Delta')$) where $\Delta' \subseteq \Delta$ and $e(\Delta') = e(\Delta)$ (resp., $b(\Delta') = b(\Delta)$). We denote by $\text{Irr}_{\sigma\text{-basic}}$ the set of basic σ -saturated representations.

Note that in both cases, if $\sigma_1, \dots, \sigma_k \in \text{Irr}_{\sigma\text{-basic}}$, then $\sigma_1 \times \dots \times \sigma_k$ is \square -irreducible. Such a product (possibly with $k = 0$) will be called a σ -saturated representation. We denote by $\text{Irr}_{\sigma\text{-sat}}$ the set of σ -saturated representations. Any $\sigma' \in \text{Irr}_{\sigma\text{-sat}}$ can be written uniquely, up to reordering of the factors, as $\sigma_1 \times \dots \times \sigma_k$ where $\sigma_i \in \text{Irr}_{\sigma\text{-basic}}$. We will call k the σ -index of σ' .

More generally, let $\pi \in \text{Irr}$. Depending on whether $\sigma = Z(\Delta)$ or $\sigma = L(\Delta)$, we define the σ -index of π , denoted by $\text{ind}_\sigma(\pi)$, to be the multiplicity of $e(\Delta)$ (resp., $b(\Delta)$) in the cuspidal support of π if the latter is contained (as a set) in Δ , and $-\infty$ otherwise. This definition is consistent with the case of σ -saturated representations.

⁸In general, the c'_n 's (and hence also the c_n 's, up to explicit signs) are given by the value at 1 of certain Kazhdan–Lusztig polynomials with respect to the symmetric group S_k – see e.g. [Hen07]. In particular, $c_n \neq 0$ for all $n \in S$. However, we will not use this fact.

⁹The discussion of this subsection and the next one is valid for any segment in the sense of Zelevinsky.

For instance, if $\pi = Z(\Delta_1 + \cdots + \Delta_k)$ with $\Delta_i \subseteq \Delta$ for all i , then the $Z(\Delta)$ -index of π is the number of i 's such that $e(\Delta_i) = e(\Delta)$. Similarly, if $\pi = L(\Delta_1 + \cdots + \Delta_k)$ with $\Delta_i \subseteq \Delta$ for all i , then the $L(\Delta)$ -index of π is the number of i 's such that $b(\Delta_i) = b(\Delta)$.

In particular, if $\text{ind}_\sigma(\pi) > 0$, then there exist $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$ and $\pi' \in \text{Irr}$ such that

$$(7.1) \quad \pi = \text{soc}(\sigma' \times \pi').$$

(Namely, if $\sigma = Z(\Delta)$ and $\pi = Z(\Delta_1 + \cdots + \Delta_k)$, then we may take $\sigma' = Z(\Delta_i)$ for any i such that $e(\Delta_i) = e(\Delta)$; if $\sigma = L(\Delta)$ and $\pi = L(\Delta_1 + \cdots + \Delta_k)$, then we may take $\sigma' = L(\Delta_i)$ for any i such that $b(\Delta_i) = b(\Delta)$.)

It is also clear that if $\pi_1, \pi_2 \in \text{Irr}$, then for any irreducible subquotient τ of $\pi_1 \times \pi_2$ we have

$$(7.2) \quad \text{ind}_\sigma(\tau) = \text{ind}_\sigma(\pi_1) + \text{ind}_\sigma(\pi_2).$$

Denote by¹⁰

$$J : \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}$$

the ‘‘total’’ Jacquet module, i.e., J is the direct sum over $m \geq 0$ of the functors

$$\begin{aligned} \mathcal{C}(\text{GL}_m(F)) &\rightarrow \bigoplus_{m_1, m_2 \geq 0: m_1 + m_2 = m} \mathcal{C}(\text{GL}_{m_1}(F) \times \text{GL}_{m_2}(F)) \\ &= \bigoplus_{m_1, m_2 \geq 0: m_1 + m_2 = m} \mathcal{C}(\text{GL}_{m_1}(F)) \boxtimes \mathcal{C}(\text{GL}_{m_2}(F)) \end{aligned}$$

where for each summand we take the Jacquet module with respect to the standard parabolic subgroup of type (m_1, m_2) of $\text{GL}_m(F)$.

Note that if $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$, then for every irreducible subquotient $\pi_1 \otimes \pi_2$ of $J(\sigma')$ other than $\sigma' \otimes \mathbf{1}$ we have $\text{ind}_\sigma(\pi_1) = 0$. More generally, it follows from the geometric lemma of Bernstein and Zelevinsky that if $\sigma' \in \text{Irr}_{\sigma\text{-sat}}$, then for every irreducible subquotient $\pi_1 \otimes \pi_2$ of $J(\sigma')$ other than $\sigma' \otimes \mathbf{1}$ we have

$$(7.3) \quad \text{ind}_\sigma(\pi_1) < \text{ind}_\sigma(\sigma').$$

7.2. For any $\pi \in \mathcal{C}$ (not necessarily irreducible) we define the hereditary σ -index by

$$\text{h-ind}_\sigma(\pi) = \max\{\text{ind}_\sigma(\pi_1) \mid \pi_1 \otimes \pi_2 \text{ is an irreducible subquotient of } J(\pi)\},$$

interpreted as 0 if $\pi = 0$. We say that π is σ -reduced if its hereditary σ -index is 0. We denote by $\mathcal{C}_{\sigma\text{-red}} \subseteq \mathcal{C}$ the subcategory of σ -reduced representations and by $\text{Irr}_{\sigma\text{-red}} \subseteq \text{Irr}$ its irreducible objects.

Clearly, $\text{h-ind}_\sigma(\pi) \geq \max(0, \text{ind}_\sigma(\pi))$ for any $\pi \in \text{Irr}$, and $\text{h-ind}_\sigma(\pi_1) \leq \text{h-ind}_\sigma(\pi_2)$ if π_1 is a subquotient of π_2 .

It follows from (7.2) and the geometric lemma that for any $\pi_1, \pi_2 \in \mathcal{C}$ we have

$$(7.4) \quad \text{h-ind}_\sigma(\pi_1 \times \pi_2) = \text{h-ind}_\sigma(\pi_1) + \text{h-ind}_\sigma(\pi_2).$$

In particular,

$$(7.5) \quad \text{if } \pi_1 \text{ and } \pi_2 \text{ are } \sigma\text{-reduced, then } \pi_1 \times \pi_2 \text{ is also } \sigma\text{-reduced.}$$

Also, it follows from (7.3) that if $\sigma' \in \text{Irr}_{\sigma\text{-sat}}$, then for every irreducible subquotient $\pi_1 \otimes \pi_2$ of $J(\sigma')$ other than $\sigma' \otimes \mathbf{1}$ we have

$$(7.6) \quad \text{h-ind}_\sigma(\pi_1) < \text{ind}_\sigma(\sigma').$$

¹⁰Here \boxtimes denotes Deligne’s tensor product of categories – see e.g. [EGNO15, §1.11]

In particular,

$$(7.7) \quad \text{h-ind}_\sigma(\sigma') = \text{ind}_\sigma(\sigma').$$

We say that two representations $\pi_1, \pi_2 \in \mathcal{C}$ have *disjoint supports* if for every irreducible subquotient τ_i of π_i , $i = 1, 2$ the cuspidal support of τ_1 and τ_2 are distinct as multisets. We use similar terminology for representations of $\text{GL} \times \text{GL}$.

For any $\pi_1, \pi_2 \in \mathcal{C}$ we denote by $\mathcal{K}_{\pi_1, \pi_2}$ the kernel of the canonical surjection

$$J(\pi_1 \times \pi_2) \twoheadrightarrow \pi_1 \otimes \pi_2.$$

Following [LM20, §7.2] we write $\pi_1 \not\cong \pi_2$ if $\mathcal{K}_{\pi_1, \pi_2}$ and $\pi_1 \otimes \pi_2$ have disjoint supports (as representations of $\text{GL} \times \text{GL}$).

By the geometric lemma, (7.6) and [LM20, Corollary 5] we get

Lemma 7.1. *Let $\pi_1 \in \text{Irr}_{\sigma\text{-sat}}$ and $\pi_2 \in \mathcal{C}_{\sigma\text{-red}}$. Then, for any irreducible subquotient $\tau_1 \otimes \tau_2$ of $\mathcal{K}_{\pi_1, \pi_2}$ we have*

$$\text{h-ind}_\sigma(\tau_1) < \text{ind}_\sigma(\pi_1) \text{ or } \deg \tau_1 > \deg \pi_1.$$

In particular, $\pi_1 \not\cong \pi_2$. Hence, if π_2 is SI, then $\pi_1 \times \pi_2$ is SI.

Fix a basic representation σ .

Lemma 7.2. *The following conditions are equivalent for $\pi \in \mathcal{C}$.*

- (1) $\text{h-ind}_\sigma(\pi) > 0$.
- (2) $J(\pi)$ admits an irreducible subquotient of the form $\sigma' \otimes \pi'$ with $\text{ind}_\sigma(\sigma') > 0$.
- (3) $J(\pi)$ admits an irreducible subquotient of the form $\sigma' \otimes \pi'$ where $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$.
- (4) $J(\pi)$ admits an irreducible quotient of the form $\sigma' \otimes \pi'$ where $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$.
- (5) There exists a non-zero homomorphism $\pi \rightarrow \sigma' \times \pi'$ for some $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$ and $\pi' \in \text{Irr}$.

In particular, if $\pi \in \text{Irr}$, then π is σ -reduced if and only if

$$\pi \not\hookrightarrow \sigma' \times \pi'$$

for every $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$ and $\pi' \in \text{Irr}$.

Proof. 1 \iff 2 by definition of h-ind.

4 \implies 3 \implies 2 is obvious.

Conditions 4 and 5 are equivalent by Frobenius reciprocity.

Finally, suppose that $\tau \otimes \omega$ is an irreducible subquotient of $J(\pi)$ such that $\text{ind}_\sigma(\tau) > 0$. Since $\text{ind}_\sigma(\tau)$ depends only on the cuspidal support of τ , we may assume without loss of generality that $\tau \otimes \omega$ is a quotient of $J(\pi)$. Thus, by Frobenius reciprocity there exists a non-zero homomorphism $\pi \rightarrow \tau \times \omega$. Hence, by (7.1), there exists a non-zero homomorphism $\pi \rightarrow \sigma' \times \tau' \times \omega$ for some $\sigma' \in \text{Irr}_{\sigma\text{-basic}}$ and $\tau' \in \text{Irr}$. Hence, there exists a non-zero homomorphism $\pi \rightarrow \sigma' \times \pi'$ for some irreducible subquotient π' of $\tau' \times \omega$. Thus, 2 \implies 5. \square

Proposition 7.3. *Let π be an irreducible representation of GL . Then, there exist $\pi_1 \in \text{Irr}_{\sigma\text{-sat}}$ and $\pi_2 \in \text{Irr}_{\sigma\text{-red}}$ such that*

$$(7.8) \quad \pi = \text{soc}(\pi_1 \times \pi_2).$$

Moreover, π_1 and π_2 are uniquely determined by these conditions. More precisely,

- (1) $\text{h-ind}_\sigma(\pi) = \text{h-ind}_\sigma(\pi_1) = \text{ind}_\sigma(\pi_1)$.
- (2) $\pi_1 \otimes \pi_2$ occurs with multiplicity one in the Jordan–Hölder sequence of $J(\pi)$.

(3) If $\pi'_1 \otimes \pi'_2$ is an irreducible subquotient of $J(\pi)$ such that $\text{h-ind}_\sigma(\pi'_1) = \text{h-ind}_\sigma(\pi)$, then either $\pi'_1 \otimes \pi'_2 = \pi_1 \otimes \pi_2$ or $\deg \pi'_1 > \deg \pi_1$.

Thus, $\pi_1 \otimes \pi_2$ is the unique irreducible subquotient of $J(\pi)$ with $\deg \pi_1$ minimal such that $\text{h-ind}_\sigma(\pi_1) = \text{h-ind}_\sigma(\pi)$.

We will refer to (7.8) as the σ -decomposition of π .

Proof. The existence follows from Lemma 7.2 by induction on the degree of π . Suppose that (7.8) holds with $\pi_1 \in \text{Irr}_{\sigma\text{-sat}}$ and $\pi_2 \in \text{Irr}_{\sigma\text{-red}}$. Then, $\pi_1 \otimes \pi_2$ occurs as a quotient of $J(\pi)$ and hence,

$$\begin{aligned} \text{ind}_\sigma(\pi_1) &\leq \text{h-ind}_\sigma(\pi) \leq \text{h-ind}_\sigma(\pi_1 \times \pi_2) \stackrel{(7.4)}{=} \text{h-ind}_\sigma(\pi_1) + \text{h-ind}_\sigma(\pi_2) \\ &= \text{h-ind}_\sigma(\pi_1) \stackrel{(7.7)}{=} \text{ind}_\sigma(\pi_1), \end{aligned}$$

which implies the first part. The other parts follow from Lemma 7.1. \square

Lemma 7.4. *Let π_1, π_2 be irreducible representations with σ -decompositions*

$$\pi_i = \text{soc}(\sigma_i \times \tau_i), \quad i = 1, 2.$$

If $\pi_1 \times \pi_2$ is irreducible, then its σ -decomposition is

$$\pi_1 \times \pi_2 = \text{soc}(\sigma' \times \tau') \quad \text{where } \sigma' = \sigma_1 \times \sigma_2 \text{ and } \tau' = \tau_1 \times \tau_2.$$

In particular, $\tau_1 \times \tau_2$ is irreducible.

Hence, if π is \square -irreducible with σ -decomposition $\pi = \text{soc}(\pi_1 \times \pi_2)$, then π_2 is \square -irreducible.

Proof. Clearly, $\sigma' \in \text{Irr}_{\sigma\text{-sat}}$, $\tau' \in \mathcal{C}_{\sigma\text{-red}}$, $\sigma' \otimes \tau'$ is a subquotient of $J(\pi_1 \times \pi_2)$ and $\text{ind}_\sigma(\sigma') = \text{h-ind}_\sigma(\pi_1 \times \pi_2)$. Moreover, by Lemma 7.1, $J(\sigma' \times \tau')$, and hence also $J(\pi_1 \times \pi_2)$, does not admit an irreducible subquotient $\omega_1 \otimes \omega_2$ such that $\text{h-ind}_\sigma(\omega_1) = \text{h-ind}_\sigma(\pi_1 \times \pi_2)$ and $\deg \omega_1 < \deg \sigma'$. The Lemma therefore follows from Proposition 7.3. \square

7.3. We turn to the analogous concepts for irreducible components.

First, we recall a general fact. Let R be any ring and z a module over R . We say that an R -module x is z -saturated (resp., z -reduced) if $\text{Hom}_R(x', z) \neq 0$ for every non-trivial submodule x' of x (resp., $\text{Hom}_R(x, z) = 0$). The class of z -saturated modules contains z itself, and is closed under arbitrary direct products, passing to submodules, and taking extensions. The class of z -reduced modules is closed under quotients, arbitrary direct sums, and extensions. Note that if x_1 is z -saturated and x_2 is z -reduced, then $\text{Hom}_R(x_2, x_1) = 0$.

Any module x admits a canonical short exact sequence

$$(7.9) \quad 0 \rightarrow x[z] \rightarrow x \rightarrow x\{z\} \rightarrow 0$$

where $x[z]$ is z -reduced and $x\{z\}$ is z -saturated. Namely, $x[z]$ is the maximal z -reduced submodule of x . Moreover, if y is any z -saturated module, then any morphism $x \rightarrow y$ factors through $x \rightarrow x\{z\}$.

In other words, the (full) subcategories of z -reduced and z -saturated objects form a torsion pair in the category of R -modules in the sense of [BKT14, §3.1].

We will apply this discussion to the case of (finite-dimensional) modules over the preprojective algebra Π .

Fix a basic representation σ and let C_σ be the corresponding irreducible component. (We will call C_σ a *basic irreducible component*.) Note that C_σ is rigid. Let

$x_\sigma \in C_\sigma$ be rigid. Note that the submodules of x_σ are rigid and correspond to the basic σ -saturated representations of GL (see §7.1). Moreover, $\text{Ext}_\Pi^1(x_1, x_2) = 0$ for any submodules x_1, x_2 of x_σ . An irreducible component containing a submodule of x_σ will be called a *subcomponent* of C_σ .

Lemma 7.5. *The x_σ -saturated modules are the direct sums of submodules of x_σ . They are all rigid.*

Proof. Clearly, direct sums of submodules of x_σ are x_σ -saturated, and by the above, they are all rigid. Conversely, suppose that x is x_σ -saturated, say of graded dimension $\mathbf{d} = (d_1, \dots, d_n)$. Assume for concreteness that $\sigma = Z(\Delta)$. It is clear that $d_i = 0$ for all $i \notin \Delta$, otherwise we would have $\text{Hom}_\Pi(x', x_\sigma) = 0$ for the submodule x' generated by a homogeneous vector of degree i . Suppose that $\pi_Q(x) = M_Q(\Delta_1 + \dots + \Delta_k)$. By the above, $\Delta_i \subseteq \Delta$ for all i . We show that $e(\Delta_i) = e(\Delta)$ for all i . This will finish the proof because then, $x = x_{Z(\Delta_1)} \oplus \dots \oplus x_{Z(\Delta_k)}$.

Assume on the contrary that $e(\Delta_i) < e(\Delta)$ for some i , and choose i for which $e(\Delta_i)$ is minimal. Then, $\Delta_j \not\prec \Delta_i$ for all j and therefore, x admits $x_{Z(\Delta_i)}$ as a submodule, whereas $\text{Hom}_\Pi(x_{Z(\Delta_i)}, x_\sigma) = 0$. We get a contradiction.

The argument for the case $\sigma = L(\Delta)$ is similar. \square

We say that an irreducible component is σ -saturated if it contains an x_σ -saturated module. Thus, C is σ -saturated if and only if $C = C_1 * \dots * C_k = C_1 \oplus \dots \oplus C_k$ for some subcomponents C_1, \dots, C_k of C_σ . The σ -saturated irreducible components are rigid and correspond to the σ -saturated representations of GL. By Corollary 5.10 and Lemma 5.8, all σ -saturated irreducible components are good.

Finally, we say that an irreducible component C is σ -reduced if it contains an x_σ -reduced module, i.e., if $\text{hom}_\Pi(C, C_\sigma) = 0$.

Lemma 7.6. *Every irreducible component C can be written uniquely as $C_1 * C_2$ where C_1 is σ -saturated and C_2 is σ -reduced.*

We call this the σ -decomposition of C .

Proof. Suppose that $C \subseteq \Lambda_{\mathbf{d}}$ and let d be the total dimension of \mathbf{d} . For any module y , the set

$$\Lambda_{\mathbf{d}}(y) := \{x \in \Lambda_{\mathbf{d}} \mid \exists \text{ an epimorphism } x \rightarrow y\}$$

is constructible. Note that for any σ -saturated module x_1 of dimension d_1 say, we have

$$\{x \in \Lambda_{\mathbf{d}} \mid x\{x_\sigma\} = x_1\} = \Lambda_{\mathbf{d}}(x_1) \setminus \cup_y \Lambda_{\mathbf{d}}(y)$$

where y ranges over all x_σ -saturated modules with $d_1 < \dim y \leq d$ for which there exists an epimorphism $y \rightarrow x_1$. Since up to isomorphism, there are only finitely many x_σ -saturated modules of a given dimension, it follows that there exists an x_σ -saturated module x_1 such that $x\{x_\sigma\} = x_1$ for all x in an open, nonempty subset C' of C .

Let C_1 be the irreducible component containing x_1 . Since every morphism $x \rightarrow x\{x_\sigma\}$ factors through the canonical projection $x \rightarrow x\{x_\sigma\}$, we infer from [AL, Proposition 5.1] that $C = C_1 * C_2$ where $C_2 \in \text{Comp}$ is the closure of $\{x[x_\sigma] \mid x \in S\}$ for a suitable open subset $S \neq \emptyset$ of C' . It is clear that C_2 is σ -reduced.

Conversely, suppose that $C = C'_1 * C'_2$ where C'_1 is σ -saturated and C'_2 is σ -reduced. Let x'_1 be a rigid element of C'_1 and

$$C''_2 = \{x \in C'_2 \mid \text{Hom}_\Pi(x, x_\sigma) = 0\},$$

an open nonempty subset of C'_2 . Then, any short exact sequence

$$0 \rightarrow x_2 \rightarrow x \rightarrow x'_1 \rightarrow 0$$

with $x_2 \in C''_2$ is isomorphic to (7.9) (with $z = x_\sigma$). In particular, if also $x \in C'$, then necessarily $x'_1 = x_1$ and $x_2 \in C_2$. The uniqueness follows. \square

Remark 7.7. In fact, Lemma 7.6 is a very special case of the results of [BKT14]. (Cf. [ibid., §5.5].) More precisely, in the notation of [ibid.] the set of σ -saturated (resp., σ -reduced) irreducible components is \mathfrak{F}^w (resp., \mathfrak{T}^w) where w is the product of the simple reflections pertaining to the elements of Δ , ordered by Q or Q° , depending on whether $\sigma = Z(\Delta)$ or $L(\Delta)$.

The σ -decomposition is compatible with that of Proposition 7.3. More precisely,

Corollary 7.8. *Suppose that $C = C_1 * C_2$ is the σ -decomposition of C . Then,*

$$(7.10) \quad \pi(C) = \text{soc}(\pi(C_1) \times \pi(C_2))$$

is the σ -decomposition of $\pi(C)$. In particular, C is σ -reduced if and only if $\pi(C)$ is σ -reduced.

Proof. Indeed, since C_1 is good, (7.10) holds. Clearly, $\pi(C_1)$ is σ -saturated. It remains to show that if C is σ -reduced, then $\pi(C)$ is σ -reduced. Suppose on the contrary that $\pi(C)$ is not σ -reduced. Then, by Lemma 7.2, $\pi(C) = \text{soc}(\pi_1 \times \pi_2)$ where π_1 is a basic σ -saturated representation. Correspondingly, $C = C_1 * C_2$ with C_1 a subcomponent of C_σ , in contradiction to the assumption that C is σ -reduced. \square

8. PROOF OF THEOREM 6.5

Let σ be a basic representation and C an irreducible component with σ -decomposition $C = C_1 * C_2$. It follows from [AL, Corollary 8.7] that if C is rigid, then C_2 is rigid. The converse is not true in general. However, we have the following. (As usual, we denote by C_σ the irreducible component corresponding to σ .)

Proposition 8.1. *Let σ be a basic representation and C an irreducible component with σ -decomposition $C = C_1 * C_2$. Assume that*

- (1) C commutes with every subcomponent of C_σ .
- (2) C_2 is rigid.
- (3) $\pi(C_2)$ is \square -irreducible.
- (4) $\pi(C_2 * D) = \text{soc}(\pi(C_2) \times \pi(D))$ for every $D \in \text{Comp}$.

Then,

- (1) C is rigid.
- (2) $\pi(C)$ is \square -irreducible.
- (3) $\pi(C * D) = \text{soc}(\pi(C) \times \pi(D))$ for every $D \in \text{Comp}$.

Proof. The first part follows from [AL, Corollary 8.2] since C_1 and C_2 are rigid and C commutes with C_1 . The second part follows from Lemma 5.5 since $\pi(C) \times \pi(C_1)$ is irreducible.

It remains to show that

$$\pi(C * D) = \text{soc}(\pi(C) \times \pi(D))$$

for any irreducible component D .

Assume first that D is σ -reduced. Then, since C_1 is σ -saturated, we have $\text{hom}_\Pi(D, C_1) = 0$ and hence by [AL, Corollary 7.4]

$$C * D = (C_1 * C_2) * D = C_1 * (C_2 * D).$$

On the other hand, $\pi(C_2) \times \pi(D)$ is σ -reduced by (7.5) and SI since $\pi(C_2)$ is \square -irreducible. Hence, $\pi(C_1) \times \pi(C_2) \times \pi(D)$ is SI by Lemma 7.1. Therefore,

$$\begin{aligned} \text{soc}(\pi(C) \times \pi(D)) &= \text{soc}(\pi(C_1) \times \pi(C_2) \times \pi(D)) \\ &= \text{soc}(\pi(C_1) \times \text{soc}(\pi(C_2) \times \pi(D))). \end{aligned}$$

Since C_1 is good, by the assumption on C_2 we get

$$\text{soc}(\pi(C) \times \pi(D)) = \pi(C_1 * (C_2 * D)) = \pi(C * D).$$

In the general case, let $D = D_1 * D_2$ be the σ -decomposition of D . By assumption on C and Lemma 7.5, C and D_1 commute. Hence, by [AL, Corollary 7.4]

$$C * D = C * (D_1 * D_2) = (C * D_1) * D_2 = (D_1 * C) * D_2 = D_1 * (C * D_2).$$

Moreover, $\pi(C) \times \pi(D_1)$ is irreducible, hence \square -irreducible since $\pi(C)$ and $\pi(D_1)$ are both \square -irreducible. Hence $\pi(C) \times \pi(D_1) \times \pi(D_2)$ is SI, and therefore

$$\begin{aligned} \text{soc}(\pi(C) \times \pi(D)) &= \text{soc}(\pi(C) \times \pi(D_1) \times \pi(D_2)) \\ &= \text{soc}(\pi(D_1) \times \text{soc}(\pi(C) \times \pi(D_2))). \end{aligned}$$

We already proved that $\text{soc}(\pi(C) \times \pi(D_2)) = \pi(C * D_2)$. Therefore,

$$\text{soc}(\pi(C) \times \pi(D)) = \pi(D_1 * (C * D_2)) = \pi(C * D).$$

The proposition follows. \square

Recall that by Remark 5.7, in order to show Theorem 6.5, it is enough to show that if $C = \lambda_Q(\mathfrak{m})$ where \mathfrak{m} is (regular) balanced, then

$$\pi(C * D) = \text{soc}(\pi(C) \times \pi(D))$$

for every $D \in \text{Comp}$. We prove this, together with the rigidity of C and the \square -irreducibility of $Z(\mathfrak{m})$ by induction on the size of \mathfrak{m} . The induction step follows from Proposition 8.1 using the following combinatorial lemma which is similar to [LM18, Lemma 7.4], and proved along the same lines.

Lemma 8.2. *Suppose $C = \lambda_Q(\mathfrak{m})$ where $\mathfrak{m} \neq 0$ is balanced. Then, there exists a basic irreducible component C_σ and a σ -reduced irreducible component $C' = \lambda_Q(\mathfrak{m}')$ such that*

- (1) $C = C_\sigma * C'$.
- (2) \mathfrak{m}' is balanced.
- (3) C commutes with every subcomponent of C_σ .

Proof. Write $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$ where $b(\Delta_1) > \dots > b(\Delta_k)$. Let $1 \leq m \leq k$ be the largest index such that $e(\Delta_1) > \dots > e(\Delta_m)$. Let $1 \leq l \leq m$ be the smallest index such that $\Delta_{i+1} \prec \Delta_i$ for all $l \leq i < m$. (In other words, l is the largest index between 2 and m such that $b(\Delta_{l-1}) > e(\Delta_l) + 1$ if such an index exists; otherwise $l = 1$.)

The proof splits into two cases according to whether there exist indices i and j such that $l \leq i < m < j \leq k$ and $e(\Delta_{i+1}) < e(\Delta_j) < e(\Delta_i)$. Assume first that such indices do not exist. We will show that the required conditions are satisfied for $\sigma = Z(\Delta)$ and $\mathfrak{m}' = \mathfrak{m} - \Delta$ where we set $\Delta = \Delta_l$. Clearly, \mathfrak{m}' is balanced, since

it is a submultisegment of \mathbf{m} . It is also clear from the definition of l that $\Delta \not\prec \Delta_i$ for all i . Hence, $C = C_\sigma * C'$. Moreover, C' is σ -reduced since $e(\Delta_i) \neq e(\Delta)$ for all $i \neq l$. Suppose that $\sigma' = Z(\Delta')$ where $\Delta' \subseteq \Delta$ and $e(\Delta') = e(\Delta)$. In order to show that C commutes with $C_{\sigma'}$ we use Remark 2.2 and its notation. The condition $\mathcal{M}(\mathbf{m}, \Delta')$ is trivial since $U_{\mathbf{m}; \Delta'} = \emptyset$. In order to show $\mathcal{M}(\Delta', \mathbf{m})$ it suffices to show that if $X = \{j \mid \Delta_j \prec \Delta'\}$ and $Y = \{i \mid \Delta_i \prec \overset{\rightarrow}{\Delta'}\}$, then the function

$$f : X \rightarrow Y, \quad f(j) = \max\{i \in Y \mid \Delta_j \prec \Delta_i\}$$

is injective. Note that f is well-defined and $l \leq f(j) < j$ for all j .

Assume on the contrary that $f(j) = f(j')$ for some $j, j' \in X$ with $j' < j$. Clearly, $\Delta_j \not\prec \Delta_{j'}$, otherwise $f(j) \geq j' > f(j')$. On the other hand $b(\Delta_{j'}) < b(\Delta') \leq e(\Delta_j) + 1$. Since also $b(\Delta_j) < b(\Delta_{j'})$, we necessarily have $e(\Delta_j) > e(\Delta_{j'})$. In particular, $j > m$. Let i be the largest index between l and m such that $\Delta_{j'} \prec \Delta_i$ and $\Delta_j \prec \Delta_i$. If $i = m$, then $j' > m + 1$ (since $e(\Delta_{m+1}) > e(\Delta_m)$) and $\Delta_m + \Delta_{m+1} + \Delta_{j'} + \Delta_j$ is a submultisegment of type 3412, which is a contradiction to the assumption on \mathbf{m} . Thus, $i < m$ and therefore $e(\Delta_{i+1}) < e(\Delta_i)$. By the maximality of i , we necessarily have $e(\Delta_{i+1}) < e(\Delta_j)$, in contradiction to our assumption on \mathbf{m} .

It remains to consider the case where there exist indices r and s such that $l \leq r < m < s \leq k$ and $e(\Delta_{r+1}) < e(\Delta_s) < e(\Delta_r)$. Let $1 \leq l' \leq m$ be the smallest index such that $b(\Delta_i) = b(\Delta_{i+1}) + 1$ for all $l' \leq i < m$. (Clearly, $l' \geq l$.) Let $\sigma = L([b(\Delta_m), b(\Delta_{l'})])$ and $\mathbf{m}' = \Delta'_1 + \cdots + \Delta'_k$ where

$$\Delta'_i = \begin{cases} -\Delta_i & i = l', \dots, m \\ \Delta_i & \text{otherwise.} \end{cases}$$

(We omit indices for which Δ'_i is empty.) Since $b(\Delta'_i) \neq b(\Delta_m)$ for all i , C' is σ -reduced. Also, by the choice of l' , \mathbf{m}' is regular and it is balanced since any submultisegment of \mathbf{m}' of type 4231 or of type 3412 would give (upon replacing Δ'_i by Δ_i) a submultisegment of \mathbf{m} of the same type.

Next, we show that $C = C_\sigma * C'$. Let

$$\begin{aligned} \mathbf{m}_1 &= \mathbf{m}'_1 = \Delta_1 + \cdots + \Delta_{l'-1}, \\ \mathbf{m}_2 &= \Delta_{l'} + \cdots + \Delta_m, \quad \mathbf{m}'_2 = \Delta'_{l'} + \cdots + \Delta'_m, \\ \mathbf{m}_3 &= \mathbf{m}'_3 = \Delta_{m+1} + \cdots + \Delta_k, \\ \mathbf{m}_4 &= \mathbf{m}_2 + \mathbf{m}_3, \quad \mathbf{m}'_4 = \mathbf{m}'_2 + \mathbf{m}'_3, \\ C_i &= \lambda_Q(\mathbf{m}_i), \quad C'_i = \lambda_Q(\mathbf{m}'_i), \quad i = 1, 2, 3, 4. \end{aligned}$$

By Lemma 5.4 we have

$$C_2 * C_3 = C_4, \quad C'_2 * C'_3 = C'_4, \quad C_1 * C_4 = C, \quad C'_1 * C'_4 = C'.$$

Moreover, by the choice of l' , C_σ and C'_1 commute. Therefore,

$$C_\sigma * C' = C_\sigma * (C'_1 * C'_4) = (C_\sigma * C'_1) * C'_4 = (C'_1 * C_\sigma) * C'_4 = C'_1 * (C_\sigma * C'_4).$$

Also, $\text{hom}_\Pi(C'_3, C_\sigma) = 0$ and therefore,

$$C_\sigma * C'_4 = C_\sigma * (C'_2 * C'_3) = (C_\sigma * C'_2) * C'_3.$$

Note that $C_\sigma * C'_2 = C_2$ (see Example 5.2). We therefore get

$$C_\sigma * C'_4 = C_2 * C'_3 = C_2 * C_3 = C_4.$$

Hence,

$$C_\sigma * C' = C'_1 * C_4 = C_1 * C_4 = C.$$

It remains to show that C commutes with every subcomponent $\lambda_Q(\mathbf{n}')$ of C_σ . Note that $\mathbf{n}' = [b(\Delta_m), b(\Delta_m)] + \cdots + [s, s]$ for some $s \in [b(\Delta_m), b(\Delta_{l'})]$. We use Remark 2.2. The condition $\mathcal{M}(\mathbf{m}, \mathbf{n}')$ follows from the fact that $\Delta_{i+1} \prec \Delta_i$ and $b(\Delta_{i+1}) = b(\Delta_i) - 1$ whenever $b(\Delta_i) \in [b(\Delta_m) + 1, b(\Delta_{l'}) + 1]$. In order to show $\mathcal{M}(\mathbf{n}', \mathbf{m})$, it suffices to check that $e(\Delta_i) \neq b(\Delta_m) - 1$ for all i .

Assume on the contrary that there exists i such that $e(\Delta_i) = b(\Delta_m) - 1$. Clearly, $i \neq s$ since $e(\Delta_s) > e(\Delta_m) > e(\Delta_i)$. Necessarily $i < s$, otherwise $\Delta_r + \cdots + \Delta_m + \Delta_s + \Delta_i$ is a submultisegment of type 4231. By definition of m , $e(\Delta_{m+1}) > e(\Delta_m)$. Suppose that $e(\Delta_{m+1}) < e(\Delta_l)$. Let $l \leq t < m$ be the index such that $e(\Delta_{t+1}) < e(\Delta_{m+1}) < e(\Delta_t)$. Then, $\Delta_t + \cdots + \Delta_{m+1} + \Delta_i$ is a submultisegment of type 4231, in contradiction to the assumption on \mathbf{m} . Otherwise, $e(\Delta_{m+1}) > e(\Delta_l)$ and hence, $\Delta_r + \cdots + \Delta_{m+1} + \Delta_i + \Delta_s$ is a submultisegment of type 3412. Once again we get a contradiction.

The proof of the lemma is complete. \square

As indicated before, Theorem 6.5 now follows from Remark 5.7, Proposition 8.1 and Lemma 8.2 by induction on the size of \mathbf{m} .

Remark 8.3. The proofs of Proposition 8.1 and Lemma 8.2, together with Remark 5.11, give a combinatorial recipe for the computation of $\mathbf{m} * \mathbf{n}$ in the case where \mathbf{m} or \mathbf{n} is balanced.

Remark 8.4. Proposition 8.1 can be used to provide additional examples of rigid, good irreducible components. It would be interesting to see what are the limits of this method. At any rate, there are examples of rigid irreducible components $C \neq 0$ that do not commute with any basic irreducible component C_σ for which $C = C_\sigma * C'$ for some C' .

Remark 8.5. It would be interesting to extend the concepts and results of §7.1–§7.3 to the more general framework of [BKT14].

9. ODDS AND ENDS

We end the paper with several questions and speculations.

Question 9.1. Given $C_1, C_2 \in \text{Comp}$, what is the representation-theoretic counterpart of the condition $\text{hom}_\Pi(C_1, C_2) = 0$?

Recall the condition \bowtie introduced in §7.2. One may wonder whether the conditions $\text{hom}_\Pi(C_2, C_1) = 0$ and $\pi(C_1) \bowtie \pi(C_2)$ are equivalent. At this stage we are unable to prove or disprove either direction.

Let $C_i \in \text{Comp}(\mathbf{d}_i)$, $i = 1, \dots, k$ and let $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_k$. For any $C \in \text{Comp}(\mathbf{d})$ we would like to give a criterion for the condition

$$(9.1) \quad \pi(C) \text{ occurs as a subquotient of } \pi(C_1) \times \cdots \times \pi(C_k).$$

Fix graded vector space V^i of graded dimension \mathbf{d}_i , $i = 1, \dots, k$ and let $V = V^1 \oplus \cdots \oplus V^k$. We identify $\Lambda(V^1) \times \cdots \times \Lambda(V^k)$ with a closed subvariety of $\Lambda(V)$ by the embedding

$$(x_1, \dots, x_k) \mapsto x_1 \oplus \cdots \oplus x_k.$$

Thus, by definition,

$$C_1 \oplus \cdots \oplus C_k \text{ is the closure of } G_V \cdot (C_1 \times \cdots \times C_k).$$

Let

$$p_i : \Lambda(V^1) \times \cdots \times \Lambda(V^k) \rightarrow \Lambda(V^i)$$

be the projection onto the i -th factor.

We make the following conjecture.¹¹

Conjecture 9.2. *For any $C \in \text{Comp}(V)$, (9.1) holds if and only if there exists an irreducible component D of $C \cap (C_1 \times \cdots \times C_k)$ such that for all $i = 1, \dots, k$ the restriction of p_i is a dominant map from D to C_i .*

In particular,

- (1) *If all but at most one of the C_i 's is rigid, then*

$$\text{JH}(\pi(C_1) \times \cdots \times \pi(C_k)) = \{\pi(C) \mid C \supseteq C_1 \oplus \cdots \oplus C_k\}.$$

- (2) *If all C_i 's are rigid, then*

$$\text{JH}(\pi(C_1) \times \cdots \times \pi(C_k)) = \{\pi(C) \mid x_1 \oplus \cdots \oplus x_k \in C\}.$$

where x_i is a rigid element of C_i , $i = 1, \dots, k$.

At the moment, we have very little evidence towards this conjecture. However, we can make a few consistency checks and remarks.

- (1) (Cf. [AL, (9.7)] and Remark 3.2). Suppose that $C_i = \lambda_Q(\mathfrak{m}_i)$, $i = 1, \dots, k$. Then,

$$\lambda_Q(\mathfrak{m}_1 + \cdots + \mathfrak{m}_k) \supseteq C_1 \oplus \cdots \oplus C_k$$

while

$Z(\mathfrak{m}_1 + \cdots + \mathfrak{m}_k)$ occurs as a subquotient of $Z(\mathfrak{m}_1) \times \cdots \times Z(\mathfrak{m}_k)$

(in fact, with multiplicity one).

- (2) Let $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$ and take $C_i = \lambda_Q(\Delta_i)$. Then, it is clear that $\lambda_Q(\mathfrak{m}') \supseteq C_1 \oplus \cdots \oplus C_k$ if and only if $M_Q(\mathfrak{m}')$ lies in the closure of the G_V -orbit of $M_Q(\mathfrak{m})$. On the other hand, by [Zel80, §7], this condition also characterizes the occurrence of $Z(\mathfrak{m}')$ as a subquotient in the standard module $\zeta(\mathfrak{m})$. Thus, Conjecture 9.2 is true in this case. The same holds if we take $C_i = \lambda_{Q^\circ}(\Delta_i)$.
- (3) In the case $k = 2$, by [AL, (3.7)], Conjecture 9.2 implies that $\pi(C_1 * C_2)$ is an irreducible subquotient of $\pi(C_1) \times \pi(C_2)$, which is a weak form of Conjecture 5.1. In particular, as in Lemma 5.3, Conjecture 9.2 implies that if C_1 and C_2 do not strongly commute, then $\pi(C_1) \times \pi(C_2)$ is reducible. Also, Conjecture 9.2 implies Conjecture 3.4.B.
- (4) Conjecture 9.2 implies that if C is rigid, then $\pi(C)$ is \square -irreducible. By the above, Conjecture 9.2 also implies that if C does not strongly commute with itself, then $\pi(C)$ is not \square -irreducible. We expect that as in Example 2.3, if C strongly commutes with itself but is not rigid, then there exists an open, nonempty subset $C' \subseteq C$ such that for every $x \in C'$ there exists a self-extension of x that is not contained in $C * C = C \oplus C$. This would imply that assuming Conjecture 9.2, if C is non-rigid, then $\pi(C)$ is not \square -irreducible. However, as things stand, we do not know whether Conjecture 9.2 implies Conjecture 3.4.C.

¹¹We thank Avraham Aizenbud and Jan Schröer for discussions related to this conjecture.

- (5) In general, by [CBS02], $C_1 \oplus \cdots \oplus C_k$ is an irreducible component if and only if C_i strongly commutes with C_j for all $i \neq j$. If this is the case, and in addition all but at most one of the C_i 's is rigid, then Conjecture 9.2 asserts that $\pi(C_1) \times \cdots \times \pi(C_k)$ is irreducible. This is consistent with Conjectures 3.4.B and 3.4.C, in view of Corollary 4.1.C.

Part of the subtlety with Conjecture 9.2, already in the case where all C_i 's are rigid, is to get a handle on the condition

$$C \supseteq C_1 \oplus \cdots \oplus C_k.$$

This condition clearly implies that for every irreducible component D

$$\begin{aligned} \mathrm{hom}_\Pi(D, C) &\leq \mathrm{hom}_\Pi(D, C_1) + \cdots + \mathrm{hom}_\Pi(D, C_k), \\ \mathrm{hom}_\Pi(C, D) &\leq \mathrm{hom}_\Pi(C_1, D) + \cdots + \mathrm{hom}_\Pi(C_k, D). \end{aligned}$$

We do not know whether the converse implication also holds, let alone whether it suffices to consider only certain D 's, which would make this condition feasible to check. (A related problem was studied by Riedtmann, Zwara and others, see e.g., [Rie86, Zwa00].)

Example 9.3. Let $\Delta_i = [i, i + r]$, $i = 1, \dots, r$ and $k \leq r$. For any $i = 1, \dots, k$ consider

$$C_i = \lambda_Q(\mathbf{m}_i) \text{ where } \mathbf{m}_i = \sum_{1 \leq j \leq r | j \equiv i \pmod k} \Delta_j.$$

It is known that any irreducible subquotient of $Z(\mathbf{m}_1) \times \cdots \times Z(\mathbf{m}_k)$ is of the form $Z(\mathbf{m})$ with $\mathbf{m} = \sum_{i=1}^r [w(i), i + r]$ where $w \in S_r$ is a permutation which does not admit a decreasing subsequence of size $k + 1$ [Gur20]. It is also easy to see that if $C \supseteq C_1 \oplus \cdots \oplus C_k$, then $C = \lambda_Q(\mathbf{m})$ for \mathbf{m} of this type. Computer calculations affirms that conversely, at least up to $r = 8$, for such \mathbf{m} , $Z(\mathbf{m})$ occurs in $Z(\mathbf{m}_1) \times \cdots \times Z(\mathbf{m}_k)$. We expect that this holds in general. (This is known for $k = 2$, in which case the multiplicities are one [Gur21b].) It would be already interesting to see that this expectation is consistent with Conjecture 9.2.

REFERENCES

- [AL] Avraham Aizenbud and Erez Lapid, *A binary operation on irreducible components of Lusztig's nilpotent varieties I: definition and properties*, arXiv:2103.12027.
- [BKT14] Pierre Baumann, Joel Kamnitzer, and Peter Tingley, *Affine Mirković-Vilonen polytopes*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 113–205. MR 3270589
- [BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70. MR 0425030
- [BZ77] ———, *Induced representations of reductive p -adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR 579172
- [CBS02] William Crawley-Boevey and Jan Schröer, *Irreducible components of varieties of modules*, J. Reine Angew. Math. **553** (2002), 201–220. MR 1944812
- [Deo85] Vinay V. Deodhar, *Local Poincaré duality and nonsingularity of Schubert varieties*, Comm. Algebra **13** (1985), no. 6, 1379–1388. MR 788771 (86i:14015)
- [EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. MR 3242743
- [GLS06] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), no. 3, 589–632. MR 2242628
- [GM21] Maxim Gurevich and Alberto Mínguez, *Cyclic representations of general linear p -adic groups*, J. Algebra **585** (2021), 25–35. MR 4272560

- [GS05] Christof Geiss and Jan Schröer, *Extension-orthogonal components of preprojective varieties*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 1953–1962. MR 2115084
- [Gur20] Maxim Gurevich, *Decomposition rules for the ring of representations of non-Archimedean GL_n* , Int. Math. Res. Not. IMRN (2020), no. 20, 6815–6855. MR 4172670
- [Gur21a] ———, *On the Hecke-algebraic approach for general linear groups over a p -adic field*, Interactions of Quantum Affine Algebras with Cluster Algebras, Current Algebras and Categorification, Progress in Mathematics, vol. 337, Springer, 2021, [In Honor of Vyjayanthi Chari on the Occasion of Her 60th Birthday].
- [Gur21b] ———, *Quantum invariants for decomposition problems in type A rings of representations*, J. Combin. Theory Ser. A **180** (2021), Paper No. 105431, 46. MR 4213657
- [Hen07] Anthony Henderson, *Nilpotent orbits of linear and cyclic quivers and Kazhdan-Lusztig polynomials of type A* , Represent. Theory **11** (2007), 95–121 (electronic). MR 2320806
- [HL10] David Hernandez and Bernard Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341. MR 2682185
- [HL21] ———, *Quantum affine algebras and cluster algebras*, Interactions of Quantum Affine Algebras with Cluster Algebras, Current Algebras and Categorification, Progress in Mathematics, vol. 337, Birkhäuser/Springer, Basel, 2021, [In Honor of Vyjayanthi Chari on the Occasion of Her 60th Birthday].
- [Kas18] Masaki Kashiwara, *Crystal bases and categorifications—Chern Medal lecture*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 249–258. MR 3966729
- [KKKO15] Seok-Jin Kang, Masaki Kashiwara, Myungho Kim, and Se-jin Oh, *Simplicity of heads and socles of tensor products*, Compos. Math. **151** (2015), no. 2, 377–396. MR 3314831
- [KKKO18] ———, *Monoidal categorification of cluster algebras*, J. Amer. Math. Soc. **31** (2018), no. 2, 349–426. MR 3758148
- [KL12] Arno Kret and Erez Lapid, *Jacquet modules of ladder representations*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 21–22, 937–940. MR 2996769
- [KS97] Masaki Kashiwara and Yoshihisa Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), no. 1, 9–36. MR 1458969 (99e:17025)
- [KZ96] Harold Knight and Andrei Zelevinsky, *Representations of quivers of type A and the multisegment duality*, Adv. Math. **117** (1996), no. 2, 273–293. MR 1371654 (97e:16029)
- [Lap21] Erez Lapid, *Explicit decomposition of certain induced representations of the general linear group*, Relative trace formulas, Simons Symp., Springer, Cham, 2021, pp. 321–327.
- [Lec03] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$* , Transform. Groups **8** (2003), no. 1, 95–104. MR 1959765
- [LM14] Erez Lapid and Alberto Minguez, *On a determinantal formula of Tadić*, Amer. J. Math. **136** (2014), no. 1, 111–142. MR 3163355
- [LM16] ———, *On parabolic induction on inner forms of the general linear group over a non-archimedean local field*, Selecta Math. (N.S.) **22** (2016), no. 4, 2347–2400. MR 3573961
- [LM18] ———, *Geometric conditions for \square -irreducibility of certain representations of the general linear group over a non-archimedean local field*, Adv. Math. **339** (2018), 113–190. MR 3866895
- [LM20] ———, *Conjectures and results about parabolic induction of representations of $GL_n(F)$* , Invent. Math. **222** (2020), no. 3, 695–747. MR 4169050
- [LNT03] Bernard Leclerc, Maxim Nazarov, and Jean-Yves Thibon, *Induced representations of affine Hecke algebras and canonical bases of quantum groups*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 115–153. MR 1985725
- [LS90] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $Sl(n)/B$* , Proc. Indian Acad. Sci. Math. Sci. **100** (1990), no. 1, 45–52. MR 1051089
- [Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498. MR 1035415
- [Lus91] ———, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365–421. MR 1088333

- [MgW86] C. Mœglin and J.-L. Waldspurger, *Sur l'involution de Zelevinski*, J. Reine Angew. Math. **372** (1986), 136–177. MR 863522
- [Rie86] Christine Riedtmann, *Degenerations for representations of quivers with relations*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 2, 275–301. MR 868301
- [Rin99] Claus Michael Ringel, *The multisegment duality and the preprojective algebras of type A*, AMA Algebra Montp. Announc. (1999), Paper 2, 6. MR 1734897
- [Sch80] J. T. Schwartz, *Fast probabilistic algorithms for verification of polynomial identities*, J. Assoc. Comput. Mach. **27** (1980), no. 4, 701–717. MR 594695
- [Zel80] A. V. Zelevinsky, *Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$* , Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR 584084 (83g:22012)
- [Zwa00] Grzegorz Zwara, *Degenerations of finite-dimensional modules are given by extensions*, Compositio Math. **121** (2000), no. 2, 205–218. MR 1757882

Email address: erez.m.lapid@gmail.com

DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 7610001 ISRAEL

Email address: minguez.alberto@gmail.com

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN AUSTRIA