LOCAL TRANSFER FOR QUASI-SPLIT CLASSICAL GROUPS AND CONGRUENCES MOD ℓ

by

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Abstract. — Let G be the group of rational points of a quasi-split p-adic special orthogonal, symplectic or unitary group for some odd prime number p. Following Arthur and Mok, there are an integer $N \geqslant 1$, a p-adic field E and a local functorial transfer from isomorphism classes of irreducible smooth complex representations of G to those of $\mathrm{GL}_N(E)$. By fixing a prime number ℓ different from p and an isomorphism between the field of complex numbers and an algebraic closure of the field of ℓ-adic numbers, we obtain a transfer map between representations with ℓ-adic coefficients. Now consider a cuspidal irreducible ℓ-adic representation π of G: we can define its reduction mod ℓ, which is a semi-simple smooth representation of G of finite length, with coefficients in a field of characteristic ℓ. Let π' be a cuspidal irreducible ℓ-adic representation of G whose reduction mod ℓ is isomorphic to that of π. We prove that the transfers of π and π' have reductions mod ℓ which may not be isomorphic, but which have isomorphic supercuspidal supports. When G is not the split special orthogonal group SO_2, we further prove that the reductions mod ℓ of the transfers of π and π' share a unique common generic component.

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1. Introduction

1.1.

Let F be a p-adic field for some odd prime number p and G be the group of rational points of a quasi-split special orthogonal, unitary or symplectic group defined over F. In the case where G is unitary, let E be the quadratic extension of F with respect to which G is defined ; otherwise, let E be equal to F. According to Arthur [**2**] for special orthogonal and symplectic groups, and to Mok [**46**] for unitary groups, there is a positive integer $N = N(G)$ and a map from isomorphism classes of irreducible (smooth) complex representations of G to those of the general linear group $\mathrm{GL}_N(E)$, called the local *transfer* or *base change*, which we will denote by $\mathbf{t}$.

1.2.

Let us fix a prime number ℓ different from p and an isomorphism of fields ι between $\mathbb{C}$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers. Replacing formally $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$ thanks to ι, we get a local transfer between isomorphism classes of irreducible smooth $\overline{\mathbb{Q}}_\ell$-representations, denoted $\mathbf{t}_\ell$. (We describe the dependency of $\mathbf{t}_\ell$ in the choice of ι – or equivalently the behavior of $\mathbf{t}$ with respect to automorphisms of $\mathbb{C}$: see Paragraph 6.4 for unramified representations, and Paragraph 9.2 for discrete series representations, of G.)

We can now consider irreducible $\overline{\mathbb{Q}}_\ell$-representations which are *integral* – that is, which carry a stable $\overline{\mathbb{Z}}_\ell$-lattice, where $\overline{\mathbb{Z}}_\ell$ denotes the ring of integers of $\overline{\mathbb{Q}}_\ell$. Given such a representation π, one can define its reduction mod ℓ: this is the semi-simplification of the reduction of any of its stable $\overline{\mathbb{Z}}_\ell$-lattices modulo the maximal ideal of $\overline{\mathbb{Z}}_\ell$. This is a smooth representation of finite length with coefficients in $\overline{\mathbb{F}}_\ell$, the residue field of $\overline{\mathbb{Z}}_\ell$, denoted $\mathbf{r}_\ell(\pi)$. One then can ask whether the map $\mathbf{t}_\ell$ preserves the fact of being integral, and how it behaves with respect to congruences mod ℓ.

Similar questions have already been answered for other local correspondences: see [**64, 16, 12**] for the local Langlands correspondence for GL_n, as well as [**17, 40**] for the local Jacquet-Langlands correspondence between inner forms of GL_n, for $n \geqslant 1$. (See also Paragraph 1.7 below and Appendix A, where we discuss the case of the cyclic local base change for GL_n.) In this paper, we prove the following theorem.

Theorem 1.1. — *Let π_1, π_2 be integral cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representations of G, and assume that*

$$\mathbf{r}_\ell(\pi_1) \leqslant \mathbf{r}_\ell(\pi_2) \tag{1.1}$$

that is, $\mathbf{r}_\ell(\pi_1)$ is contained in $\mathbf{r}_\ell(\pi_2)$ as semi-simple $\overline{\mathbb{F}}_\ell$-representations of G. Then

(1) *The local transfer $\mathbf{t}_\ell(\pi_i)$ is an integral $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_N(E)$ for each $i = 1, 2$.*

(2) *The irreducible components of the semi-simple $\overline{\mathbb{F}}_\ell$-representation $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_1)) \oplus \mathbf{r}_\ell(\mathbf{t}_\ell(\pi_2))$ all have the same supercuspidal support (see below for a definition).*

(3) *Assume that G is not isomorphic to the split special orthogonal group $\mathrm{SO}_2(F) \simeq F^\times$. The semi-simple $\overline{\mathbb{F}}_\ell$-representations $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_1))$ and $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_2))$ have a unique generic irreducible component in common.*

As in the case of complex coefficients, an irreducible representation of $\mathrm{GL}_N(E)$ on an $\overline{\mathbb{F}}_\ell$-vector space V is said to be *generic* if V carries a non-zero $\overline{\mathbb{F}}_\ell$-linear form Λ such that $\Lambda(\pi(u)v) = \theta(u)v$ for all $v \in V$ and all upper triangular matrices u of $\mathrm{GL}_N(E)$, where θ is the $\overline{\mathbb{F}}_\ell$-character

$$u \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n})$$

and ψ is a non-zero $\overline{\mathbb{F}}_\ell$-character of F.

An irreducible $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_N(E)$ is *supercuspidal* if it does not occur as a subquotient of any representation parabolically induced from a proper Levi subgroup. The *supercuspidal support* of an irreducible $\overline{\mathbb{F}}_\ell$-representation π of $\mathrm{GL}_N(E)$ is a pair (M, ρ) made of a Levi subgroup of $\mathrm{GL}_N(E)$ and a supercuspidal representation ρ of M such that π occurs as a subquotient of the normalized parabolic induction of ρ. It is uniquely determined up to conjugacy ([**65**] V.4, [**38**] Théorème 8.16).

Note that, unless G is the split special orthogonal group $\mathrm{SO}_2(F) \simeq F^\times$, the centre of G is compact. When this is the case, any cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representation of G is integral. We will discuss the case of the split $\mathrm{SO}_2(F)$ in detail in Paragraph 9.3.

Also note that, if G is not isomorphic to $\mathrm{SO}_2(F) \simeq F^\times$, then (3) implies (2), since all irreducible components of the reduction mod ℓ of an integral irreducible $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_N(E)$ have the same supercuspidal support.

Before discussing the other assumptions of Theorem 1.1 (in Paragraph 1.6), let us explain how we prove it. The general strategy goes back to Khare [**31**] and Vignéras [**64**] who study the congruence properties of the local Langlands correspondence for $\mathrm{GL}_n(F)$ with $n \geqslant 1$.

1.3.

The first step is to pass from our given local situation to the following global situation (which is the purpose of Sections 2 to 4).

First, k is a totally real number field, l is either k or a totally imaginary quadratic extension of k and w is a finite place of k above p, inert in l, such that $k_w = F$ and $l_w = E$.

Next, $\mathbf{G}$ is a connected reductive group defined over k such that

(1) the group $\mathbf{G}(F)$ naturally identifies with G,

(2) the group $\mathbf{G}(k_v)$ is compact for any real place v and quasi-split for any finite place v,

(3) the k-group $\mathbf{G}$ is an inner form of a quasi-split special orthogonal, unitary or symplectic group $\mathbf{G}^*$.

Finally, Π_1 and Π_2 are irreducible automorphic representations of $\mathbf{G}(\mathbb{A}_k)$, where $\mathbb{A}_k$ denotes the ring of adèles of k, such that

(1) $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_1 and $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_2,

(2) the representations $\Pi_{1,v}$ and $\Pi_{2,v}$ are trivial for any real place v,

(3) there is a finite place $u \neq w$ of k such that $\Pi_{1,u}$ and $\Pi_{2,u}$ are both isomorphic to some cuspidal irreducible unitary representation ρ of $\mathbf{G}(k_u)$ which is compactly induced from a compact mod centre, open subgroup of $\mathbf{G}(k_u)$,

(4) there is a finite set S of places of k, containing all real places, such that for all $v \notin \mathrm{S}$:

(a) the group $\mathbf{G}$ is unramified over k_v,

(b) the representations $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified with respect to some hyperspecial maximal special compact subgroup of $\mathbf{G}(k_v)$,

(c) their Satake parameters (in the sense of Paragraph 3.4) are integral and congruent mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$,

where all tensor products are taken with respect to ι.

1.4.

The next step – which is the purpose of Section 5 – is to associate to Π_1 and Π_2 two cuspidal irreducible automorphic representations $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ of $\mathrm{GL}_N(\mathbb{A}_l)$ such that, for any finite place v, the local transfer of $\Pi_{i,v}$ is isomorphic to $\widetilde{\Pi}_{i,v}$, for $i = 1, 2$. For this, we use the results of Taïbi [**61**] if $\mathbf{G}^*$ is symplectic or special orthogonal, and Labesse [**35**] if $\mathbf{G}^*$ is unitary. Namely, let $\widetilde{\Pi}_i$ be

– the Arthur parameter associated with Π_i if $\mathbf{G}^*$ is symplectic or special othogonal ([**61**]),
– the stable base change of Π_i to $\mathrm{GL}_N(\mathbb{A}_l)$ if $\mathbf{G}^*$ is unitary ([**35**]).

In both cases, $\widetilde{\Pi}_i$ is algebraic regular and [**61, 35**] provide $\widetilde{\Pi}_i$ with certain local-global compatibilities at all finite places. In order to ensure that these local-global compatibilities are what we want, namely, that the local transfer of $\Pi_{i,v}$ is isomorphic to $\widetilde{\Pi}_{i,v}$ at all finite v, and in prevision of the next step, we need $\widetilde{\Pi}_i$ to be cuspidal.

In order to choose Π_1, Π_2 so that $\widetilde{\Pi}_1$, $\widetilde{\Pi}_2$ are cuspidal, we use the auxiliary cuspidal representation ρ of Paragraph 1.3. More precisely, we prove the following result (see Lemma 9.1).

Proposition 1.2. — *Given k, w and $\mathbf{G}$ as in Paragraph 1.3, the finite place u of k and the representation ρ of $\mathbf{G}(k_u)$ can be chosen so that the local transfer of ρ is cuspidal.*

If $\mathbf{G}^*$ is unitary, it suffices to choose u so that $\mathbf{G}$ is split over k_u. In the symplectic and special orthogonal cases, this is the purpose of Appendices B and C. (In particular, the place u has to divide 2 in the symplectic case.)

1.5.

We now have two algebraic regular, cuspidal irreducible automorphic representations $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ of $\mathrm{GL}_N(\mathbb{A}_l)$ such that, for $i = 1, 2$ and all finite places v, the transfer of $\Pi_{i,v}$ is isomorphic to $\widetilde{\Pi}_{i,v}$. Besides, it follows from the properties of the transfer from $\mathbf{G}(\mathbb{A}_k)$ to $\mathrm{GL}_N(\mathbb{A}_l)$ that the conjugate of the contragredient of $\widetilde{\Pi}_i$ by the generator c of $\mathrm{Gal}(l/k)$ is isomorphic to $\widetilde{\Pi}_i$.

From the properties of Π_1 and Π_2 at all places $v \notin \mathrm{S}$, and from the congruence properties of the unramified local transfer that we establish in Section 6, it follows that, for all $v \notin \mathrm{S}$:

(1) the local components $\widetilde{\Pi}_{1,v}$ and $\widetilde{\Pi}_{2,v}$ are unramified,

(2) the Satake parameters of $\widetilde{\Pi}_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\widetilde{\Pi}_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral and congruent mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

We now apply the results of [**5**], which give us two continuous ℓ-adic Galois representations

$$\Sigma_i : \mathrm{Gal}(\overline{\mathbb{Q}}/l) \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell), \quad i = 1, 2,$$

such that, for any finite place v of l not dividing ℓ, the (ℓ-adic) Weil-Deligne representation associated with $\widetilde{\Pi}_{i,v}|\det|_v^{(1-N)/2}$ by the local Langlands correspondence is isomorphic to the Frobenius-semisimplification of the Weil-Deligne representation associated with $\Sigma_{i,v}$, the restriction of Σ_i to a decomposition subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/l)$ at v. (Here $|\cdot|_v$ denotes the absolute value of l_v normalized so that the absolute value of any uniformizer of l_v is the inverse of the cardinality of the residue field of l_v.)

Thanks to our local conditions at all $v \notin \mathrm{S}$, the representations $\Sigma_{1,v}$, $\Sigma_{2,v}$ are congruent mod ℓ. A density argument then implies that Σ_1 and Σ_2 are congruent mod ℓ. In particular, $\Sigma_{1,w}$, $\Sigma_{2,w}$ are congruent mod ℓ.

Associated with $\Sigma_{i,w}$, there is a Frobenius-semisimple Weil-Deligne representation (ρ_i, N_i). We show in Section 7 that the fact that $\Sigma_{1,w}$ and $\Sigma_{2,w}$ are congruent mod ℓ implies that the smooth semi-simple representations ρ_1 and ρ_2 are integral and congruent mod ℓ. Since ρ_i corresponds to the cuspidal support of $\widetilde{\Pi}_{i,w}|\det|_w^{(1-N)/2}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ (thanks to the local-global compatibility at w given by [**5**]), it follows from the mod ℓ local Langlands correspondence of Vignéras [**64**] that

$$\widetilde{\Pi}_{1,w}|\det|_w^{(1-N)/2}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell, \quad \widetilde{\Pi}_{2,w}|\det|_w^{(1-N)/2}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$$

are integral and have the same *mod ℓ supercuspidal support* (which is the supercuspidal support of any irreducible component of the reduction mod ℓ): it follows that the supercuspidal support of the generic irreducible component of the reduction mod ℓ of $\widetilde{\Pi}_{i,w}|\det|_w^{(1-N)/2}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$, denoted δ_i, is independent of $i\in\{1,2\}$. Since a generic irreducible $\overline{\mathbb{F}}_\ell$-representation is uniquely determined by its supercuspidal support, we deduce that δ_1, δ_2 are isomorphic. The Main Theorem 1.1 now follows from the fact that $\widetilde{\Pi}_{i,w}|\det|_w^{(1-N)/2}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ is isomorphic to $\mathbf{t}_\ell(\pi_i)$. We refer to Sections 8 and 9 for more details.

1.6.

Now let us discuss the assumptions of the main theorem.

First, the construction of Π_1 does not require π_1 to be cuspidal: it would be enough to assume that $\pi_1\otimes_{\overline{\mathbb{Q}}_\ell}\mathbb{C}$ is a discrete series representation of G (for one, or equivalently any, choice of the field isomorphism ι: see Remark 9.3).

However, in order to construct the representation Π_2 satisfying our local conditions at all places $v\notin \mathrm{S}$ by the method of Khare–Vignéras, we need π_2 to be cuspidal – even more precisely, we need π_2 to be compactly induced from some open, compact mod centre subgroup of G, which is true of any cuspidal representation of G, thanks to the work of Stevens [**58**] and since p is odd. Consequently, both the cuspidality of π_2 and (1.1) imply that π_1 should be cuspidal, as the parabolic restriction functors commute with reduction mod ℓ.

For the same reason, we want the auxiliary representation ρ of $\mathbf{G}(k_u)$ to be compactly induced from an open, compact mod centre subgroup.

Moreover, as has been explained in Paragraph 1.4, we also need ρ to have a cuspidal transfer to $\mathrm{GL}_N(k_u)$. This is why the symplectic group requires a special treatment (see Appendix C), since no cuspidal representation of a p-adic symplectic group has a cuspidal transfer when p is odd, and the work of Stevens [**58**] is not available when $p=2$.

On the other hand, we show that part (3) of Theorem 1.1 does not hold in general for non-cuspidal representations: Remark 6.4 gives an example of integral unramified irreducible $\overline{\mathbb{Q}}_\ell$-representations π_1 and π_2 of $\mathrm{SO}_5(F)$ such that $\mathbf{r}_\ell(\pi_1)=\mathbf{r}_\ell(\pi_2)$ but $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_1))$ and $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_2))$ have no irreducible component in common.

Finally, our Assumption (1.1) is inspired from Vignéras [**64**] 3.5. It is tempting to conjecturate that the conclusion of Theorem 1.1 still holds when (1.1) is replaced by the weaker condition "$\mathbf{r}_\ell(\pi_1)$ *and* $\mathbf{r}_\ell(\pi_2)$ *have a component in common*", but we have no evidence that such a conjecture should be true.

1.7.

In Appendix A, we discuss the case of the local base change from $\mathrm{GL}_n(F)$ to $\mathrm{GL}_n(K)$ for a cyclic extension K of F, denoted $\mathbf{b}_{K/F}$.

As in Paragraph 1.2, choosing a field isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ gives an ℓ-adic local base change map $\mathbf{b}_{K/F,\ell}$. By using the properties of the local Langlands correspondence for GL_n with respect to conjugacy by an automorphism of $\mathbb{C}$, we prove that $\mathbf{b}_{K/F,\ell}$ does not depend on the choice of ι (see Proposition A.1). We also use certain results of Zou [**69**] 1.10 to prove an analogue of Theorem 1.1 for $\mathbf{b}_{K/F,\ell}$ (see Paragraph A.4), and give an example of integral cuspidal $\overline{\mathbb{Q}}_\ell$-representations π_1, π_2 of $\mathrm{GL}_2(F)$ such that $\mathbf{r}_\ell(\pi_1) = \mathbf{r}_\ell(\pi_2)$ but $\mathbf{r}_\ell(\mathbf{b}_\ell(\pi_1)) \neq \mathbf{r}_\ell(\mathbf{b}_\ell(\pi_2))$ (Paragraph A.5).

1.8.

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Notation

Throughout the paper, let p be a prime number, let $\mathbb{Q}_p$ be the field of p-adic numbers and let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. By a p-adic field, we mean a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$.

2. Globalizing quadratic and Hermitian forms

The purpose of this section is to prove the following result.

Theorem 2.1. — *Let F be a p-adic field and let G be a quasi-split special orthogonal, unitary or symplectic group over F. There exist a totally real number field k and a connected reductive group $\mathbf{G}$ over k such that*

(1) *$\mathbf{G}$ is an inner form of a quasi-split special orthogonal, unitary or symplectic k-group,*
(2) *there is a finite place w of k above p such that $k_w = F$ and $\mathbf{G}(F)$ is isomorphic to G,*
(3) *the group $\mathbf{G}(k_v)$ is compact for any real place v, and quasi-split for any finite place v.*

This theorem will be used in Section 4 where we prove the existence of automorphic representations of $\mathbf{G}(\mathbb{A})$ with prescribed conditions on their local components, where $\mathbb{A}$ denotes the ring of adèles of k.

In Section 9, we will need a stronger version of Theorem 2.1: in order to transfer automorphic representations of $\mathbf{G}(\mathbb{A})$ to a general linear group, we will need to realize $\mathbf{G}$ as a pure inner form in the orthogonal and unitary cases, and a rigid inner form in the symplectic case. This is why, rather than Theorem 2.1, we will prove the stronger Theorems 2.8 and 2.11 below. For the symplectic case, see Paragraph 2.8.

We emphasize that p may be equal to 2 in this section.

2.1. Quadratic forms

In this paragraph, k denotes either a p-adic field for some prime number p, or a real Archimedean local field, or a totally real number field, and q is a (non-degenerate) quadratic form of dimension $d \geqslant 2$ over k. There exist non-zero scalars $\lambda_1, \dots, \lambda_d \in k^\times$ such that q is equivalent to $\lambda_1 x_1^2 + \cdots + \lambda_d x_d^2$. The quantity

$$\delta = \delta(q) = \lambda_1 \dots \lambda_d \bmod k^{\times 2} \in k^\times / k^{\times 2}$$

does not depend on this choice. It is called the discriminant of q. In the sequel, we assume that the discriminant δ is fixed. All quadratic forms are assumed to be non-degenerate.

If k is a p-adic field, then q is, up to equivalence, uniquely determined by its Hasse invariant

$$\varepsilon(q) = \prod_{i<j} (\lambda_i, \lambda_j) \in \{-1, 1\}$$

where $(\cdot, \cdot)$ is the Hilbert symbol over k (see [**53**] IV.2.3 Theorem 7 or [**28**] Theorem 9.24).

If k is isomorphic to the field of real numbers, q is, up to equivalence, entirely determined by its signature (a, b) with $a + b = d$ and $(-1)^b = \delta$. Its Hasse invariant is equal to $(-1)^{b(b-1)/2}$. If $\delta > 0$, then $b = 2c$ for some $c \in \{0, \dots, \lfloor d/2 \rfloor\}$ and the Hasse invariant is $(-1)^c$.

Now suppose that k is a totally real number field, and $\delta_v > 0$ for all real places v. The Hasse principle (see [**52**] Theorem 6.6.6) ensures that q is uniquely determined, up to equivalence, by all its localizations $q_v = q \otimes_k k_v$, where v ranges over all places of k. In other words, it is determined by the Hasse invariants $\varepsilon(q_v)$ for all finite v and the signatures $(d - 2c(q_v), 2c(q_v))$ for all real v. Conversely, a family

$$((\varepsilon_v)_{v \text{ finite}}, (c_v)_{v \text{ real}}), \quad \varepsilon_v \in \{-1, 1\}, \quad c_v \in \{0, \dots, \lfloor d/2 \rfloor\},$$

corresponds to a (unique) quadratic form of dimension d over k and discriminant δ if and only if one has $\varepsilon_v = 1$ for almost all finite places v and

$$\prod_{v \text{ finite}} \varepsilon_v \cdot \prod_{v \text{ real}} (-1)^{c_v} = 1 \tag{2.1}$$

(see [**52**] Theorem 6.6.10). We give more details in §2.2 and §2.3, depending on the parity of d.

2.2. The odd orthogonal case

If k is a p-adic field, there are two equivalence classes of quadratic forms of dimension $2n + 1$ and discriminant δ, in bijection with $\{-1, 1\}$ through the Hasse invariant. The special orthogonal groups associated with these quadratic forms are non-isomorphic. The one with Hasse invariant

$$(-1, -1)^{n(n+1)/2} \cdot (-1, \delta)^n \tag{2.2}$$

(that is $x_1 x_2 + \cdots + x_{2n-1} x_{2n} + (-1)^n \delta x_{2n+1}^2$) is split. The other one is non-quasi-split.

If k is isomorphic to the field of real numbers, there are $n + 1$ equivalence classes of quadratic forms of dimension $2n + 1$ and discriminant δ. The special orthogonal groups associated with these quadratic forms are non-isomorphic. Exactly one of them is compact: this is the one with signature $(2n+1,0)$ if $\delta > 0$, and $(0,2n+1)$ if $\delta < 0$.

Proposition 2.2. — *Let k be a totally real number field of degree r, and $\delta \in k^\times/k^{\times 2}$. Suppose that $\delta_v > 0$ for all real places v. There is a quadratic form q of dimension $2n+1$ and discriminant δ such that* $\mathrm{SO}(q)$ *is compact at all real places and quasi-split at all finite places if and only if $rn(n+1)/2$ is even. When this is the case, q is unique up to equivalence.*

Proof. — A quadratic form q over k of dimension $2n+1$ and discriminant δ is entirely determined, up to equivalence, by the Hasse invariants $\varepsilon(q_v) \in \{-1,1\}$ for all finite places v and the signatures $(2n+1-2c(q_v), 2c(q_v))$ for all real places v of k. Non-equivalent quadratic forms define non-isomorphic special orthogonal groups.

For $\mathrm{SO}(q)$ to be compact at all real places and quasi-split at all finite places, q must have invariants $c_v = 0$ for all real v and $\varepsilon_v = (-1,-1)_v^{n(n+1)/2} \cdot (-1,\delta_v)_v^n$ for all finite v, where $(\cdot,\cdot)_v$ is the Hilbert symbol with respect to k_v. By (2.1), such a q exists if and only if

$$\prod_{v \text{ finite}} (-1,-1)_v^{n(n+1)/2} \times \prod_{v \text{ finite}} (-1,\delta_v)_v^n = 1.$$

Thanks to the Hilbert reciprocity law ([**48**] VII), the left hand side is equal to

$$\prod_{v \text{ real}} (-1,-1)_v^{n(n+1)/2} \times \prod_{v \text{ real}} (-1,\delta_v)_v^n = (-1)^{rn(n+1)/2}$$

(since $\delta_v > 0$ for all real v), which gives the expected result. □

Remark 2.3. — Given any k, let q be a quadratic form of dimension $2n+1$ and discriminant 1 over k. Then, for any $\delta \in k^\times$, the quadratic form δq has discriminant δ and $\mathrm{SO}(\delta q) = \mathrm{SO}(q)$.

2.3. The even orthogonal case

In this paragraph, we assume that the dimension of q is $2n$. It will be convenient to use the normalized discriminant $\alpha = (-1)^n\delta$.

Suppose first that k is a p-adic field.

- If $n = 1$, there is only one equivalence class of quadratic forms of dimension 2 and normalized discriminant $\alpha = 1$. Its Hasse invariant is 1. The special orthogonal group associated with it is isomorphic to $\mathrm{GL}_1(k)$.
- Suppose that $n \geqslant 2$ or $\alpha \neq 1$. There are two equivalence classes of quadratic forms of dimension $2n$ and discriminant δ, characterized by their Hasse invariant. The special orthogonal groups associated with them are non-isomorphic if and only if $\alpha = 1$. When this is the case, the one with Hasse invariant $(-1,-1)^{n(n-1)/2}$ (that is the quadratic form $x_1x_2 + \cdots + x_{2n-1}x_{2n}$) is split, and the other one is non-quasi-split. Otherwise, let l be the quadratic extension of k generated by a square root of α: if q is a quadratic form of dimension $2n$ and discriminant δ over k, then λq has same discriminant and opposite Hasse invariant for any scalar $\lambda \in k^\times$ which is not an l/k-norm, and $\mathrm{SO}(\lambda q) = \mathrm{SO}(q)$.

If k is isomorphic to the field of real numbers, there are $n+1$ equivalence classes of quadratic forms of dimension $2n$ and discriminant δ. Quadratic forms with signatures (a,b) and (a',b') define isomorphic special orthogonal groups if and only if one has $b' \in \{a,b\}$. If $\delta < 0$, there is no compact special orthogonal group. If $\delta > 0$, there is exactly one compact special orthogonal group: this is the one with $b \in \{0, 2n\}$.

Proposition 2.4. — *Let k be a totally real number field of degree r, and $\delta \in k^\times/k^{\times 2}$. Suppose that $\delta_v > 0$ for all real places v.*

(1) *There is a quadratic form q of dimension $2n$ and discriminant δ such that $\mathrm{SO}(q)$ is compact at all real places and quasi-split at all finite places if and only if either n is odd, or $\delta \neq (-1)^n$, or $rn(n-1)/2$ is even.*

(2) *Assume that $\delta \neq (-1)^n$. For any finite place w such that $\delta_w \neq (-1)^n$ and any $\varepsilon \in \{-1,1\}$, there is a quadratic form q as in (1) satisfying the extra condition $\varepsilon(q \otimes k_w) = \varepsilon$.*

Proof. — A quadratic form q over k of dimension $2n$ and discriminant δ is entirely determined, up to equivalence, by the Hasse invariants $\varepsilon(q_v) \in \{-1,1\}$ for all finite places v and the signatures $(2n - 2c(q_v), 2c(q_v))$ for all real places v. A quadratic form f with same dimension and discriminant as q defines a special orthogonal group isomorphic to $\mathrm{SO}(q)$ if and only if they have the same Hasse invariants for all finite v such that $\alpha_v = 1$, and $c(f_v) \in \{n - c(q_v), c(q_v)\}$ for all real places v.

For $\mathrm{SO}(q)$ to be compact at all real places and quasi-split at all finite places, q must have invariants $c_v \in \{0, n\}$ for all real places v and $\varepsilon_v = (-1,-1)_v^{n(n-1)/2}$ for all finite places v such that $\alpha_v = 1$. (Recall that $\alpha = (-1)^n\delta$.) By (2.1), such a q exists if and only if

$$\prod_{\substack{v \text{ finite} \\ \alpha_v \neq 1}} \varepsilon_v \times \prod_{\substack{v \text{ finite} \\ \alpha_v = 1}} (-1,-1)_v^{n(n-1)/2} \times (-1)^{ns} = 1$$

where s is the number of real places such that $c_v = n$. If n is odd, we may adjust $s \in \{0, \ldots, r\}$ so that this product is 1. If $\alpha \neq 1$, we may adjust the signs ε_v for the finite v such that $\alpha_v \neq 1$ so that this product is 1. (Since the number of such v is at least 2, we may even assume that ε_w is equal to a given sign ε for a given w as in (2).) If n is even and $\alpha = 1$, the condition is

$$\prod_{v \text{ finite}} (-1,-1)_v^{n(n-1)/2} = 1$$

and Hilbert's reciprocity law says that the left hand side is equal to

$$\prod_{v \text{ real}} (-1,-1)_v^{n(n-1)/2} = (-1)^{rn(n-1)/2},$$

which gives the expected result. ☐

2.4. Globalizing the base field

The following lemma will be useful in the remainder of this section.

Lemma 2.5. — *Let F be a p-adic field.*

(1) *There exists a totally real number field k of even degree such that $k_w = F$ for some finite place w of k dividing p.*

(2) *If $p \neq 2$, we may further assume that there exists a finite place u of k such that $k_u \simeq \mathbb{Q}_2$.*

Proof. — We follow the proof of [**2**] Lemma 6.2.1. Let us write $F = \mathbb{Q}_p(\beta)$ for some root $\beta \in F$ of a monic irreducible polynomial f of degree $r = [F : \mathbb{Q}_p]$ with coefficients in $\mathbb{Q}_p$. Given a field E, we identify the space of monic polynomials of degree r with coefficients in E with E^r.

By Krasner's lemma (see [**50**] 3.1.5), there is an open neighborhood U_p of f in $\mathbb{Q}_p^r$ such that any $g \in U_p$ has a root $\beta' \in \overline{\mathbb{Q}}_p$ such that $\mathbb{Q}_p(\beta') = F$. Let U_∞ be the open subset of $\mathbb{R}^r$ made of all monic polynomials with r distinct real roots. Since the diagonal image of $\mathbb{Q}^r$ in $\mathbb{R}^r \times \mathbb{Q}_p^r$ is dense, the intersection

$$\mathbb{Q}^r \cap (U_\infty \times U_p)$$

is non-empty. We may replace f by a polynomial in this intersection, which we still denote by f. The number field $k = \mathbb{Q}(\beta)$ is totally real, and $k_w = F$ for some finite place w of k dividing p. If the degree of k is even, we are done. Otherwise, we choose a monic irreducible polynomial g of degree 2 over $\mathbb{Q}$ which splits over $\mathbb{R}$ and $\mathbb{Q}_p$, whose existence can be proven in the same way as above. Then replace k by $k(\gamma)$ where γ is a root of g in $\mathbb{Q}_p$.

Suppose now that $p \neq 2$, and let U_2 be the open subset of $\mathbb{Q}_2^r$ made of all monic polynomials with r distinct roots in $\mathbb{Q}_2$. We may replace f by a polynomial in $\mathbb{Q}^r \cap (U_\infty \times U_p \times U_2)$, which we still denote by f. The number field $k = \mathbb{Q}(\beta)$ is totally real, $k_w = F$ for some finite place w of k dividing p, and 2 is totally split in k. If the degree of k is even, we are done. Otherwise, we choose a monic irreducible polynomial g of degree 2 over $\mathbb{Q}$ which splits over $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{Q}_2$, then replace k by $k(\gamma)$ where γ is a root of g in $\mathbb{Q}_p$. □

Remark 2.6. — With a similar argument, one can prove in addition to part (1) of Lemma 2.5 that, if E is a quadratic extension of F in $\overline{\mathbb{Q}}_p$, there is a totally imaginary quadratic extension l of k such that $l_w = E$.

Remark 2.7. — Part (2) of Lemma 2.5 will be needed in Section 9, *in the symplectic case*, in order to apply the results of Appendix C.

2.5. Proof of Theorem 2.1 in the special orthogonal case

We prove Theorem 2.1 in the case where G is special orthogonal, that is, there is a quadratic form Q over F such that G is isomorphic to $\mathrm{SO}(Q)$. We will prove the following stronger result.

Theorem 2.8. — *Let Q be a quadratic form over F such that $\mathrm{SO}(Q)$ is quasi-split. There exist a totally real number field k and a quadratic form q over k such that*

(1) *there is a finite place w of k dividing p such that*
   (a) *the field k_w is equal to F,*
   (b) *the quadratic forms $q \otimes F$ and Q are equivalent,*

(2) *the group $\mathrm{SO}(q \otimes k_v)$ is compact for all real v, and quasi-split for all finite v.*

Proof. — By Lemma 2.5, there exists a totally real number field k of even degree such that k_w and F are equal for some finite place w of k dividing p. Fix a $\gamma \in F^\times$ such that the discriminant of Q is $\gamma F^{\times 2}$, and fix a $\delta \in k^\times$ such that $\gamma^{-1}\delta_w \in F^{\times 2}$ and $\delta_v > 0$ for all real v.

By Proposition 2.2 when Q has odd dimension and Proposition 2.4 when Q has even dimension, there is a quadratic form q of discriminant δ satisfying (2). Moreover, the quadratic forms $q \otimes F$ and Q have the same discriminant and define quasi-split special orthogonal groups.

If Q has odd dimension, or if Q has dimension $2n$ and $\gamma = (-1)^n$, they are thus equivalent.

Otherwise, use Proposition 2.4(2) with $\varepsilon = \varepsilon(Q)$ to ensure that $q \otimes F$ and Q have the same Hasse invariant: they are thus equivalent. □

Remark 2.9. — In addition to Theorem 2.8, there is always a finite place $u \neq w$ of k such that the group $\mathrm{SO}(q \otimes k_u)$ is split: one can choose

(1) any finite place different from w in the odd orthogonal case,
(2) any finite place $u \neq w$ such that $(-1)^n\delta_u \in k_u^{\times 2}$ in the even orthogonal case.

2.6. Hermitian forms

In this paragraph, l is a separable quadratic k-algebra (where k is as in Paragraph 2.1) and h is a (non-degenerate) l/k-Hermitian form of dimension $n \geqslant 1$. There is a choice of non-zero scalars $\lambda_1, \dots, \lambda_n \in k^\times$ such that h is equivalent to $\lambda_1 \mathrm{N}_{l/k}(x_1) + \cdots + \lambda_n \mathrm{N}_{l/k}(x_n)$. The quantity

$$\delta = \delta(h) = \lambda_1 \dots \lambda_n \mod \mathrm{N}_{l/k}(l^\times) \in k^\times/\mathrm{N}_{l/k}(l^\times)$$

does not depend on this choice. It is called the discriminant of h. Fix an $\alpha \in k^\times$ such that l is isomorphic to the k-algebra $k[X]/(X^2 - \alpha)$. The image of α in $k^\times/k^{\times 2}$ will still be denoted α.

Up to equivalence, h is uniquely determined by its trace form t, that is, the quadratic form of dimension $2n$ over k obtained by seeing l^n as a k-vector space ([**52**] Theorem 10.1.1).

If l is split, that is, if $l \simeq k \times k$, then $\mathrm{N}_{l/k}(l^\times) = k^\times$ and we may choose $\alpha = 1$. There is, up to equivalence, a unique l/k-Hermitian form of dimension n. Its discriminant is trivial, and the unitary group associated with it is (non-canonically) isomorphic to $\mathrm{GL}_n(k)$. More precisely, if one fixes an isomorphism $l \simeq k \times k$ of k-algebras, h identifies with a non-degenerate bilinear form on $k^n \times k^n$, the group $\mathrm{GL}_n(l)$ identifies with $\mathrm{GL}_n(k) \times \mathrm{GL}_n(k)$ and there is an isomorphism

$$\begin{array}{rcl} \mathrm{GL}_n(k) & \simeq & \mathrm{U}(h) \\ g & \mapsto & (g, g^*) \end{array} \tag{2.3}$$

where g^* is the contragredient of $g \in \mathrm{GL}_n(k)$ with respect to h. (Note that changing the isomorphism $l \simeq k \times k$ has the effect of exchanging g and g^* in (2.3).) Also, the trace form t of h is maximally isotropic, that is, it is the sum of n hyperbolic planes.

If l is a quadratic extension of k, a quadratic form of dimension $2n$ over k is the trace form of an l/k-Hermitian form if and only if $q \otimes_k l$ is maximally isotropic ([**52**] Theorem 10.1.2).

If l/k is a quadratic extension of p-adic fields, there are two equivalence classes of l/k-Hermitian forms of dimension n, in bijection with $k^\times/\mathrm{N}_{l/k}(l^\times)$ through the discriminant.

• If n is odd, the unitary groups associated with these Hermitian forms are isomorphic. More precisely, if $\alpha \neq 1$ and h is a Hermitian form of odd dimension over k, then δh is unequivalent to h for any $\delta \in k^\times$ such that $\delta \notin \mathrm{N}_{l/k}(l^\times)$, and the group $\mathrm{U}(\delta h) = \mathrm{U}(h)$ is quasi-split.

• If n is even, the unitary group corresponding to the discriminant $(-1)^{n/2}$ is quasi-split, and the other one is non-quasi-split. The trace form t of h has discriminant $(-\alpha)^n$ and Hasse invariant

$$\varepsilon(t) = (\alpha, \delta) \cdot (-\alpha, -1)^{n(n-1)/2}. \tag{2.4}$$

If l/k is isomorphic to $\mathbb{C}/\mathbb{R}$, the Hermitian form h is uniquely determined, up to equivalence, by its signature (a, b) with $a + b = n$. Its discriminant is $(-1)^b$. Its trace form t has discriminant 1 and signature $(2a, 2b)$. The unitary group $\mathrm{U}(h)$ is compact if and only if $b \in \{0, n\}$.

If l is a totally imaginary quadratic extension of a totally real number field k (thus $\alpha_v < 0$ for all real places v of k), then h is uniquely determined, up to equivalence, by any one of the following data:

(1) the equivalence class of its trace form t,
(2) the Hasse invariants $\varepsilon(t_v)$ for all finite v and the integers $b(t_v)$ for all real v,
(3) the equivalence classes of its localizations $h_v = h \otimes_k k_v$ for all v,
(4) the discriminants $\delta(h_v)$ for all finite v and the integers $b(h_v)$ for all real v.

We have just seen that (3) and (4) are equivalent, and we have seen that (1) and (2) are equivalent in Paragraph 2.1. Now the fact that (2) and (3) are equivalent follows form the formulas

$$\varepsilon(t_v) = (\alpha_v, \delta(h_v))_v \cdot (-\alpha_v, -1)_v^{n(n-1)/2} \text{ for finite } v, \quad b(t_v) = 2b(h_v) \text{ for real } v,$$

the first one including the case where $l_v = l \otimes_k k_v$ splits over k_v (as $\alpha_v = 1$ in this case) and the fact that, when $\alpha_v \neq 1$, the map $x \mapsto (\alpha_v, x)_v$ is a bijection from $k_v^\times/\mathrm{N}_{l_v/k_v}(l_v^\times)$ to $\{-1, 1\}$. Conversely, a family

$$((\delta_v)_{v \text{ finite}}, (b_v)_{v \text{ real}}), \quad \delta_v \in k_v^\times/\mathrm{N}_{l_v/k_v}(l_v^\times), \quad b_v \in \{0, \dots, n\}, \tag{2.5}$$

corresponds to an l/k-Hermitian form of dimension n if and only if there exists a $\delta \in k^\times/\mathrm{N}_{l/k}(l^\times)$ such that $\delta \equiv \delta_v \bmod \mathrm{N}_{l_v/k_v}(l_v^\times)$ for all v (where we have put $\delta_v = (-1)^{b_v}$ at all real places v), and when it is the case such a Hermitian form is unique. Indeed, this is certainly a necessary condition and, when it is satisfied, the family

$$((\varepsilon_v)_{v \text{ finite}}, (2b_v)_{v \text{ real}}), \quad \varepsilon_v = (\alpha_v, \delta_v)_v \cdot (-\alpha_v, -1)_v^{n(n-1)/2}, \tag{2.6}$$

satisfies $\varepsilon_v = 1$ for almost all finite places v together with

$$\begin{aligned} \prod_{v \text{ finite}} \varepsilon_v \times \prod_{v \text{ real}} (-1)^{b_v} &= \prod_{v \text{ real}} (\alpha_v, \delta_v)_v \times \prod_{v \text{ real}} (-\alpha_v, -1)_v^{n(n-1)/2} \times \prod_{v \text{ real}} (-1)^{b_v} \\ &= \prod_{v \text{ real}} (-1, -1)_v^{b_v} \times \prod_{v \text{ real}} (-1)^{b_v} \end{aligned}$$

which is equal to 1 (thanks to the fact that $\alpha_v < 0$ and $\delta_v = (-1)^{b_v}$ for all real v). Thus there is a unique quadratic form of dimension $2n$ over k and discriminant $(-\alpha)^n$ with local invariants (2.6). One can verify that it is maximally isotropic over l (as it is maximally isotropic over l_v for all v). It is thus the trace form of an l/k-Hermitian form of dimension n, as expected.

Proposition 2.10. — *Let k be a totally real number field of degree r and l be a totally imaginary quadratic extension of k.*

(1) *There is a Hermitian form h of dimension n such that $\mathrm{U}(h)$ is compact at all real places.*

(2) *There is a Hermitian form h of dimension n such that $\mathrm{U}(h)$ is compact at all real places and quasi-split at all finite places if and only if either n is odd, or n and $rn/2$ are both even.*

(3) *Assume that n is odd. For any finite place w and any $\varepsilon \in k_w^\times/\mathrm{N}_{l_w/k_w}(l_w^\times)$, there is a Hermitian form h as in (2) satisfying the extra condition $\delta(h \otimes k_w) = \varepsilon$.*

Proof. — Assertion (1) is verified by any Hermitian form h of dimension n over k such that we have $b(h_v) \in \{0, n\}$ at all real places v.

Assume now that $n = 2m$ for some $m \geqslant 1$. A Hermitian form h of dimension n and discriminant δ satisfies (2) if and only if $b(h_v) \in \{0, n\}$ and $\delta_v > 0$ for all real v, and $\delta_v = (-1)^m$ for all finite v. Such a $\delta \in k^\times/\mathrm{N}_{l/k}(l^\times)$ exists if and only if $\delta_v \in \mathrm{N}_{l_v/k_v}(l_v^\times)$ for almost all finite v, and

$$\prod_v (\alpha_v, \delta_v)_v = 1.$$

The first condition is satisfied since l_v is either split over k_v or an unramified extension of k_v for almost all finite v. The second condition follows from

$$\prod_v (\alpha_v, \delta_v)_v = \prod_{v \text{ finite}} (\alpha_v, -1)_v^m = \prod_{v \text{ real}} (\alpha_v, -1)_v^m = (-1)^{rm}$$

thanks to the Hilbert reciprocity law and the fact that $\alpha_v < 0$ for all real v.

Assume now that n is odd. A Hermitian form h of dimension n and discriminant δ satisfies (2) if and only if $b(h_v) \in \{0, n\}$ at all real places v, and satisfies (3) if and only if $b(h_v) \in \{0, n\}$ at all real places v and $\delta\varepsilon^{-1} \in \mathrm{N}_{l_w/k_w}(l_w^\times)$. Fix a finite place $y \neq w$ and a $\kappa \in l_y^\times$. We claim that such an h exists, with the extra conditions

- $b(h_v) = 0$ for all real places v,
- $\delta \in \mathrm{N}_{l_v/k_v}(l_v^\times)$ for all places $v \notin \{w, y\}$ and $\delta\kappa^{-1} \in \mathrm{N}_{l_y/k_y}(l_y^\times)$.

Arguing as in the case when n is even, it suffices to choose a $\kappa \notin \mathrm{N}_{l_y/k_y}(l_y^\times)$, which is possible as soon as y has been chosen such that $\alpha_y \neq 1$. □

2.7. Proof of Theorem 2.1 in the unitary case

We prove Theorem 2.1 in the case where G is unitary, that is, there are a quadratic extension E of F and an E/F-Hermitian form H over F such that G is isomorphic to $\mathrm{U}(H)$. We will prove the following more precise theorem.

Theorem 2.11. — *Let H be an E/F-Hermitian form such that $\mathrm{U}(H)$ is quasi-split. There exist a totally real number field k, a totally imaginary quadratic extension l of k and an l/k-Hermitian form h such that:*

(1) *there is a finite place w of k above p such that*
   (a) *one has $k_w = F$ and $l_w = E$,*
   (b) *the Hermitian forms $h \otimes F$ and H are equivalent,*

(2) *the group $\mathrm{U}(h \otimes k_v)$ is compact for all real v, and quasi-split for all finite v.*

Proof. — By Lemma 2.5 and Remark 2.6, there are a totally real number field k of even degree and a totally imaginary quadratic extension l of k such that $k_w = F$ and $l_w = E$ for some finite place w of k dividing p.

By Proposition 2.10, there exists an l/k-Hermitian form h satisfying (2). Moreover, the Hermitian forms $h \otimes F$ and H define quasi-split unitary groups.

If H has even dimension, they are thus equivalent. If H has odd dimension, one uses Proposition 2.10(3) with $\varepsilon = \delta(H)$ to ensure that $h \otimes F$ and H have the same discriminant: they are thus equivalent. □

Remark 2.12. — In addition to Theorem 2.11, there is always a finite place $u \neq w$ of k such that $\mathrm{U}(h \otimes k_u)$ is split: it suffices to choose any finite place $u \neq w$ such that $l_u \simeq k_u \times k_u$.

2.8. The symplectic case

We now consider the case where G is a symplectic group, that is, there exists a non-degenerate symplectic form A over F such that $G = \mathrm{Sp}(A)$. By Lemma 2.5, there exists a totally real number field k of even degree such that $k_w = F$ for some finite place w of k dividing p. (Moreover, when p is odd, we may further assume that there is a finite place u of k such that $k_u \simeq \mathbb{Q}_2$.) In this case, Theorem 2.1 is given by [**60**] 2.1.1. See also [**61**] Proposition 3.1.2, where the inner form $\mathbf{G}$ is realized as a rigid inner form of Sp_{2n} over k.

3. Congruences of automorphic forms of definite groups

In this section, we fix a prime number ℓ. Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of the field of ℓ-adic integers, $\overline{\mathbb{Z}}_\ell$ be its ring of integers and $\overline{\mathbb{F}}_\ell$ be its residue field. We fix a field isomorphism

$$\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell \tag{3.1}$$

and a number field k. We denote by $\mathbb{A} = \mathbb{A}_f \times \mathbb{A}_\infty$ the ring of adèles of k.

Given a locally compact, totally disconnected group G, a compact open subgroup K of G, a commutative ring R and a smooth R-representation ρ of K, we denote by

$$\mathcal{H}_R(G, \rho)$$

the endomorphism R-algebra of the compact induction of ρ to G, called the Hecke R-algebra of G relative to ρ. When ρ is the trivial R-character of K, it naturally identifies with the convolution R-algebra made of K-bi-invariant, compactly supported R-valued functions on G, and we denote it by $\mathcal{H}_R(G, K)$.

Let F be a p-adic field for some $p \neq \ell$, G be the group of rational points of a reductive group defined over F and π be an irreducible (smooth) representation of G on a $\overline{\mathbb{Q}}_\ell$-vector space V. It is said to be *integral* if V carries a G-stable $\overline{\mathbb{Z}}_\ell$-lattice. Given such a lattice L, the representation of G on the $\overline{\mathbb{F}}_\ell$-vector space $L \otimes \overline{\mathbb{F}}_\ell$ (where $\overline{\mathbb{F}}_\ell$ is the residue field of $\overline{\mathbb{Z}}_\ell$) is smooth and has finite length, and its semi-simplification does not depend on the choice of L ([**66**] Theorem 1). This semi-simplification is denoted $\mathbf{r}_\ell(\pi)$, and called the *reduction mod* ℓ of π. One defines similarly the reduction mod ℓ of an irreducible $\overline{\mathbb{Q}}_\ell$-representation of a compact, open subgroup of G.

3.1.

Let $\mathbf{G}$ be a connected reductive group defined over k. We assume that $\mathbf{G}$ is definite, that is, the group $\mathbf{G}(\mathbb{A}_\infty)$ is compact. We embed diagonally $\mathbf{G}(k)$ in $\mathbf{G}(\mathbb{A}_f)$ and set

$$\mathbf{Y} = \mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_f).$$

The quotient $\mathbf{Y}$ is compact ([**49**] §5) and hence $\mathbf{Y}/\mathrm{K}$ is finite for every open compact subgroup K of $\mathbf{G}(\mathbb{A}_f)$.

We denote by $\mathscr{A}(\mathbf{G})$ the space of functions $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$ which are square integrable with respect to a Haar measure on $\mathbf{G}(\mathbb{A})$. It is endowed with the natural unitary $\mathbb{C}$-representation of $\mathbf{G}(\mathbb{A})$.

3.2.

Let Ω be the set of open compact subgroups of $\mathbf{G}(\mathbb{A}_f)$. For $\mathrm{K} \in \Omega$, let $\mathrm{Alg}(\mathbf{G},\mathrm{K})$ denote the free $\mathbb{Z}$-module of finite rank made of all functions $\mathbf{Y} \to \mathbb{Z}$ which are invariant under right translations by K ([**64**] 3.3). We consider the $\mathbb{Z}$-module

$$\mathrm{Alg}(\mathbf{G}) = \mathcal{C}^\infty(\mathbf{Y},\mathbb{Z}) = \bigcup_{\mathrm{K}\in\Omega} \mathrm{Alg}(\mathbf{G},\mathrm{K})$$

of locally constant functions $\mathbf{Y} \to \mathbb{Z}$, called *trivial-at-infinity algebraic automorphic forms* for the group $\mathbf{G}$ (see Paragraph 3.3 below). It is endowed with the natural $\mathbb{Z}$-representation of $\mathbf{G}(\mathbb{A}_f)$.

Given any commutative ring R, we write

$$\mathrm{Alg}_\mathrm{R}(\mathbf{G}) = \mathrm{Alg}(\mathbf{G}) \otimes_\mathbb{Z} \mathrm{R}, \quad \mathrm{Alg}_\mathrm{R}(\mathbf{G},\mathrm{K}) = \mathrm{Alg}(\mathbf{G},\mathrm{K}) \otimes_\mathbb{Z} \mathrm{R}.$$

The natural R-representation of $\mathbf{G}(\mathbb{A}_f)$ on $\mathrm{Alg}_\mathrm{R}(\mathbf{G})$ is admissible ([**64**] 3.3.2).

If R is the field $\overline{\mathbb{Q}}_\ell$, the representation of $\mathbf{G}(\mathbb{A}_f)$ on $\mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})$ is semi-simple and any of its irreducible components has an $\mathcal{O}_\mathrm{E}$-structure for some finite extension E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ ([**64**] 3.3.2).

3.3.

Let $\mathrm{K} \in \Omega$, and R be a commutative ring. The Hecke R-algebra of $\mathbf{G}(\mathbb{A}_f)$ relative to K is the convolution R-algebra

$$\mathcal{H}_\mathrm{R}(\mathbf{G},\mathrm{K}) = \mathcal{H}_\mathrm{R}(\mathbf{G}(\mathbb{A}_f),\mathrm{K})$$

made of K-bi-invariant, compactly supported functions $\mathbf{G}(\mathbb{A}_f) \to \mathrm{R}$. It naturally acts on the R-module $\mathrm{Alg}_\mathrm{R}(\mathbf{G},\mathrm{K})$.

As $\mathbf{G}$ is definite, there is, by [**20**] Proposition 8.5, an explicit isomorphism

$$\mathrm{Alg}_\mathbb{C}(\mathbf{G},\mathrm{K}) \simeq \mathscr{A}(\mathbf{G})^{\mathrm{K}\times\mathbf{G}(\mathbb{A}_\infty)}$$

of $\mathcal{H}_\mathbb{C}(\mathbf{G},\mathrm{K})$-modules (see [**20**] (8.4) and Proposition 8.3). In particular, there is a bijection

$$\Theta \leftrightarrow \Pi \tag{3.2}$$

between

– the irreducible subrepresentations Θ of $\mathrm{Alg}_\mathbb{C}(\mathbf{G})$ such that the space Θ^K of K-fixed vectors in Θ is non-zero,

– the irreducible automorphic representations $\Pi = \Pi_f \otimes \Pi_\infty$ of $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{A}_f) \times \mathbf{G}(\mathbb{A}_\infty)$, that is, the irreducible subrepresentations of $\mathscr{A}(\mathbf{G})$ such that Π_∞ is trivial and Π_f^{K} is non-zero.

3.4.

Let us fix an irreducible automorphic representation Π of $\mathbf{G}(\mathbb{A})$ which is trivial on $\mathbf{G}(\mathbb{A}_\infty)$. By (3.2), we can see Π as an irreducible subrepresentation of $\mathrm{Alg}_{\mathbb{C}}(\mathbf{G})$, thus as an irreducible factor of $\mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})$ *via* the isomorphism ι fixed in (3.1).

We fix two finite places w and u of k not dividing ℓ and a finite set S of finite places of k such that:

(1) the set S contains w, u and all finite places above ℓ,

(2) for any finite place $v \notin \mathrm{S}$, the group $\mathbf{G}$ is unramified over k_v, and the local component Π_v is unramified with respect to some hyperspecial maximal compact subgroup K_v of $\mathbf{G}(k_v)$.

Any finite place $v \notin \mathrm{S}$ thus defines a character

$$\chi_v : \mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v) \to \overline{\mathbb{Q}}_\ell \tag{3.3}$$

which we call the *Satake parameter* of Π_v.

Recall that Π is admissible and has an $\mathcal{O}_{\mathrm{E}}$-structure for some finite extension E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Let us write $\Pi = \Pi_{(\ell)} \otimes \Pi^{(\ell)}$, where $\Pi_{(\ell)}$ is the tensor product of all Π_v such that v divides ℓ and $\Pi^{(\ell)}$ is the restricted tensor product of all Π_v such that v is finite and does not divides ℓ. By [**64**] A.3, both $\Pi_{(\ell)}$ and $\Pi^{(\ell)}$ have an $\mathcal{O}_{\mathrm{E}}$-structure. By applying [**64**] A.4 to $\Pi^{(\ell)}$, we get that each Π_v, for $v \notin \mathrm{S}$ finite, has an $\mathcal{O}_{\mathrm{E}}$-structure. Fixing such an $\mathcal{O}_{\mathrm{E}}$-structure, the $\mathcal{O}_{\mathrm{E}}$-algebra $\mathcal{H}_{\mathcal{O}_{\mathrm{E}}}(\mathbf{G}(k_v), \mathrm{K}_v)$ acts on it through the character χ_v, which thus has values in $\mathcal{O}_{\mathrm{E}}$. It follows that the restriction of χ_v to $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v)$ has values in $\overline{\mathbb{Z}}_\ell$.

3.5.

We now make the following assumptions on the representation Π of Paragraph 3.4:

– the local component Π_w is cuspidal, and is compactly induced from an irreducible representation of some compact open subgroup K_w of $\mathbf{G}(k_w)$,

– the local component Π_u is cuspidal, and is compactly induced from an irreducible representation η of some compact open subgroup K_u of $\mathbf{G}(k_u)$.

For any finite $v \in \mathrm{S}$ such that $v \notin \{u, w\}$, we fix a compact open subgroup K_v of $\mathbf{G}(k_v)$ such that Π_v has a non-zero K_v-invariant vector. Recall that, for any finite place $v \notin \mathrm{S}$, we have fixed a hyperspecial maximal compact open subgroup K_v of $\mathbf{G}(k_v)$ in Paragraph 3.4. We define

$$\mathrm{K} = \prod_{v \text{ finite}} \mathrm{K}_v.$$

This is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$.

Given an irreducible representation κ of K_w, we define an irreducible representation $\Lambda = \Lambda(\kappa)$ of K by

$$\Lambda = \bigotimes_{v \text{ finite}} \Lambda_v$$

with $\Lambda_w = \kappa$, $\Lambda_u = \eta$ and Λ_v is the trivial character of K_v for $v \notin \{w, u\}$.

We denote by $\mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G},\Lambda)$ the subrepresentation of $\mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})$ generated by its Λ-isotypic component, that is, the subrepresentation formed by the irreducible factors Θ such that

– the local component Θ_w contains κ,
– the local component Θ_u contains η,
– the local component Θ_v has a non-zero K_v-invariant vector for all finite $v \notin \{w,u\}$.

This amounts to considering the space

$$\mathbf{V} = \mathbf{V}(\kappa) = \mathrm{Hom}_{\mathrm{K}}\left(\Lambda, \mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})\right)$$

seen as a module over the endomorphism $\overline{\mathbb{Q}}_\ell$-algebra $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G},\Lambda) = \mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(\mathbb{A}_f),\Lambda)$ of the compact induction of Λ from K to $\mathbf{G}(\mathbb{A}_f)$. We have an $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G},\Lambda)$-module decomposition

$$\mathbf{V} = \bigoplus_{\Theta} \mathrm{Hom}_{\mathrm{K}}(\Lambda,\Theta) \tag{3.4}$$

where Θ ranges over the irreducible factors of $\mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G},\Lambda)$, and each $\mathrm{Hom}_{\mathrm{K}}(\Lambda,\Theta)$ is of finite dimension as Θ is admissible. By admissibility again, the number of Θ contributing to the direct sum of (3.4) is finite.

Denote by X the set of finite places of k. For any non-empty subset T of X and any irreducible factor Θ contributing to the right hand side of (3.4), we write

$$\mathrm{K}_{\mathrm{T}} = \prod_{v\in\mathrm{T}} \mathrm{K}_v, \quad \Lambda_{\mathrm{T}} = \bigotimes_{v\in\mathrm{T}} \Lambda_v, \quad \Theta_{\mathrm{T}} = \bigotimes_{v\in\mathrm{T}} \Theta_v.$$

We thus have $\mathrm{K} = \mathrm{K}_{\mathrm{S}} \times \mathrm{K}_{\mathrm{X}\backslash\mathrm{S}}$, $\Lambda = \Lambda_{\mathrm{S}} \otimes \Lambda_{\mathrm{X}\backslash\mathrm{S}}$ and Θ is isomorphic to $\Theta_{\mathrm{S}} \otimes \Theta_{\mathrm{X}\backslash\mathrm{S}}$. Accordingly, we have an isomorphism of $\overline{\mathbb{Q}}_\ell$-algebras

$$\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G},\Lambda) \simeq \mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{S}}),\mathrm{K}_{\mathrm{S}}) \otimes \mathcal{H}_{\overline{\mathbb{Q}}_\ell}\left(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}),\Lambda_{\mathrm{X}\backslash\mathrm{S}}\right) \tag{3.5}$$

where $\mathbb{A}_{\mathrm{S}}$ and $\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}$ have their obvious meaning, and an isomorphism of $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\Lambda)$-modules

$$\mathrm{Hom}_{\mathrm{K}}(\Lambda,\Theta) \simeq \mathrm{Hom}_{\mathrm{K}_{\mathrm{S}}}(\Lambda_{\mathrm{S}},\Theta_{\mathrm{S}}) \otimes (\Theta_{\mathrm{X}\backslash\mathrm{S}})^{\mathrm{K}_{\mathrm{X}\backslash\mathrm{S}}}$$

via (3.5). The factor $(\Theta_{\mathrm{X}\backslash\mathrm{S}})^{\mathrm{K}_{\mathrm{X}\backslash\mathrm{S}}}$ has dimension 1 over $\overline{\mathbb{Q}}_\ell$ and $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}),\mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$ acts on this line *via* a character denoted $\chi_{\mathrm{S}}(\Theta)$. Let $d_{\mathrm{S}}(\Theta)$ be the dimension of $\mathrm{Hom}_{\mathrm{K}_{\mathrm{S}}}(\Lambda_{\mathrm{S}},\Theta_{\mathrm{S}})$. Denoting by $\mathbf{V}_{\mathrm{S}}$ the restriction of $\mathbf{V}$ to $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}),\mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$, we therefore have an isomorphism

$$\mathbf{V}_{\mathrm{S}} \simeq \bigoplus_{\Theta} d_{\mathrm{S}}(\Theta)\cdot\chi_{\mathrm{S}}(\Theta) \tag{3.6}$$

of $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}),\mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$-modules.

3.6.

Assume now that κ and η are integral. Fix a K_w-stable $\overline{\mathbb{Z}}_\ell$-lattice L_κ of κ and a K_u-stable $\overline{\mathbb{Z}}_\ell$-lattice L_η of η, both with semi-simple reduction (by [**15**] Lemma 6.11). We deduce a K-stable $\overline{\mathbb{Z}}_\ell$-lattice of Λ, denoted by L_Λ. We define a $\overline{\mathbb{Z}}_\ell$-module

$$\mathbf{V}^\circ = \mathrm{Hom}_{\mathrm{K}}\left(\mathrm{L}_\Lambda, \mathrm{Alg}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})\right).$$

It is a module over $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(\mathbb{A}_f), \mathrm{L}_\Lambda)$. Set $\overline{\Lambda} = \mathrm{L}_\Lambda \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell$. By [**64**] Lemma 3.7.3, the $\overline{\mathbb{Z}}_\ell$-module $\mathbf{V}^\circ$ is a $\overline{\mathbb{Z}}_\ell$-lattice of $\mathbf{V}$ and we have

$$\mathbf{V}^\circ \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \simeq \mathrm{Hom}_{\mathrm{K}}\left(\overline{\Lambda}, \mathrm{Alg}_{\overline{\mathbb{F}}_\ell}(\mathbf{G})\right).$$

We denote the left hand side by $\overline{\mathbf{V}}$, and we continue to see it as a module over $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(\mathbb{A}_f), \mathrm{L}_\Lambda)$. Note that $\overline{\mathbf{V}}$ is semi-simple and depends only on $\mathbf{r}_\ell(\kappa)$.

3.7.

We now assume that $\mathbf{G}(k_w)$ is isomorphic to a special orthogonal, unitary or symplectic group, not necessarily quasi-split. In the rest of this section, we will prove the following theorem.

Theorem 3.1. — *Assume that $\mathbf{G}(k_w)$ is isomorphic to a special orthogonal, unitary or symplectic group, and w does not divide 2. Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A})$ such that*

- *Π_∞ is trivial,*
- *Π_w is cuspidal and integral,*
- *Π_u is compactly induced from a compact mod centre, open subgroup of $\mathbf{G}(k_u)$.*

Let π' be an integral irreducible cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathbf{G}(k_w)$ such that

$$\mathbf{r}_\ell(\Pi_w) \leqslant \mathbf{r}_\ell(\pi'). \tag{3.7}$$

There is an irreducible automorphic representation Π' of $\mathbf{G}(\mathbb{A})$ such that

(1) *the Archimedean component Π'_∞ is trivial,*
(2) *the local component Π'_w is isomorphic to π',*
(3) *the local components Π'_u and Π_u are isomorphic,*
(4) *for any finite place $v \notin \mathrm{S}$, the local component Π'_v is K_v-unramified, with Satake parameter $\chi'_v : \mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v) \to \overline{\mathbb{Z}}_\ell$, and χ_v, χ'_v are congruent mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$.*

Proof. — We follow the argument of Khare [**31**] and Vignéras [**64**]. We start with following lemma, which we will prove in Paragraph 3.8.

Lemma 3.2. — *Let p be a prime number different from 2, let F be a p-adic field and G be a special orthogonal, unitary or symplectic group over F. Let π and π' be integral cuspidal $\overline{\mathbb{Q}}_\ell$-representations of G such that*

$$\mathbf{r}_\ell(\pi) \leqslant \mathbf{r}_\ell(\pi').$$

There are a compact open subgroup J of G and irreducible $\overline{\mathbb{Q}}_\ell$-representations τ and τ' of J such that π is isomorphic to the compact induction of τ to G and π' is isomorphic to the compact induction of τ to G'.

Remark 3.3. — It is known ([**58**]) that any cuspidal $\overline{\mathbb{Q}}_\ell$-representation of G is isomorphic to the compact induction of an (irreducible) representation of some compact open subgroup of G. The point here is that one can choose the same compact open subgroup for π and π'.

As $\mathbf{G}(k_w)$ is isomorphic to a special orthogonal, unitary or symplectic group and w does not divide 2, it follows from Lemma 3.2 that there are a compact open subgroup K_w of $\mathbf{G}(k_w)$ and irreducible representations τ and τ' of K_w such that Π_w is isomorphic to the compact induction of τ to $\mathbf{G}(k_w)$ and π' is isomorphic to the compact induction of τ' to $\mathbf{G}(k_w)$. In particular, Π satisfies the conditions of Paragraph 3.5.

Let η be an irreducible representation of some compact open subgroup K_u of $\mathbf{G}(k_u)$ such that the compact induction of η to $\mathbf{G}(k_u)$ is isomorphic to Π_u. By [**64**] A.3, A.4, the representation Π_u is integral. Thus η is integral. Similarly, since Π_w and Π'_w are integral, τ and τ' are integral as well.

As in Paragraph 3.5, we define $\Lambda = \Lambda(\tau)$ and $\mathbf{V} = \mathbf{V}(\tau)$. Associated with a choice of K_w-stable $\overline{\mathbb{Z}}_\ell$-lattice of κ with semi-simple reduction, there are $\mathbf{V}^\circ$ and $\overline{\mathbf{V}}$. Similarly, replacing τ by τ', we define Λ', $\mathbf{V}'$, $\mathbf{V}'^\circ$ and $\overline{\mathbf{V}}'$. Recall that $\overline{\mathbf{V}}$ and $\overline{\mathbf{V}}'$ are semi-simple. The key point is that $\overline{\mathbf{V}}$ is non-zero and contained in $\overline{\mathbf{V}}'$ thanks to (3.7).

The character $\chi_{\mathrm{S}}(\Pi)$ of $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}), \mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$ defined by Π occurs in $\mathbf{V}^\circ_{\mathrm{S}}$. By reduction, we get a character $\overline{\chi}_{\mathrm{S}}$ of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}), \mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$ occurring in $\overline{\mathbf{V}}_{\mathrm{S}}$, and therefore in $\overline{\mathbf{V}}'_{\mathrm{S}}$.

By Deligne-Serre's lemma ([**18**] Lemma 6.11), there is a character χ'_{S} of $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}), \mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$ occurring in $\mathbf{V}'^\circ_{\mathrm{S}}$ such that its reduction equals $\overline{\chi}_{\mathrm{S}}$.

Therefore, there is an irreducible factor Π' of $\mathrm{Alg}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})$ contributing to $\mathbf{V}'_{\mathrm{S}}$ such that $\chi_{\mathrm{S}}(\Pi')$ and χ'_{S} coincide on $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}(\mathbb{A}_{\mathrm{X}\backslash\mathrm{S}}), \mathrm{K}_{\mathrm{X}\backslash\mathrm{S}})$. Such a Π' satisfies the conditions of the theorem. □

3.8.

In remains to prove Lemma 3.2.

Proof. — According to [**58**] Theorem 7.14 (and [**41**] Appendix A), there are a compact open subgroup J of G and an irreducible $\overline{\mathbb{Q}}_\ell$-representation τ of J such that π is isomorphic to the compact induction of τ to G. More precisely, the pair (J, τ) can be chosen among *cuspidal types* of G in the sense of [**41**] Appendix A. It then has the following properties:

– There is a normal pro-p-subgroup J^1 of J such that J/J^1 is isomorphic to the group of rational points of a reductive group $\mathscr{G}$ defined over the residue field of F.

– The representation τ decomposes as $\kappa\otimes\xi$ where κ is a representation of J whose restriction to J^1 is irreducible and ξ is an irreducible representation of J whose restriction to J^1 is trivial.

– The representation ξ is the inflation of a representation of J/J^1 whose restriction to the rational points of the neutral component of $\mathscr{G}$ is cuspidal.

– The representation κ is a standard beta-extension ([**58**] §4.2) of a skew semisimple character θ ([**58**] §3.1) defined with respect to a skew semisimple stratum $[\Lambda, \beta]$ ([**58**] §2.1).

– The centre of the centralizer G_E of $E = F[\beta]$ in G is compact, and the parahoric subgroup $P^\circ(\Lambda_E)$ of G_E associated with $[\Lambda, \beta]$ (see [**58**] §2.1) is a maximal parahoric subgroup of G_E.

Lemma 3.4. — *The character θ occurs in π'.*

Proof. — By definition, θ is a character of an open pro-p-subgroup $H^1 = H^1(\Lambda, \beta)$ of G. Since θ occurs in the restriction of π to H^1, its reduction mod ℓ occurs in $\mathbf{r}_\ell(\pi')|_{H^1}$. Let V be an irreducible summand of $\pi'|_{H^1}$ such that $\mathbf{r}_\ell(V)$ contains $\mathbf{r}_\ell(\theta)$. Since H^1 is a pro-p-group, V is isomorphic to θ. □

Now let $\mathcal{C}$ be the set of skew semisimple characters $\theta' \in \mathcal{C}(\Lambda', \beta)$ occurring in π' such that

$$P^\circ(\Lambda'_E) \subseteq P^\circ(\Lambda_E)$$

and $\mathcal{C}_{\min}$ be the subset of $\mathcal{C}$ made of all $\theta' \in \mathcal{C}$ such that $P^\circ(\Lambda'_E)$ is minimal among all parahoric subgroups of G_E occurring this way.

Let us prove that $\mathcal{C}_{\min} = \mathcal{C}$. Let $\theta' \in \mathcal{C}_{\min}$. Then [**58**] §7.2 (in particular Lemma 7.4) and [**41**] Appendix A imply that π' contains a cuspidal type $(J', \kappa' \otimes \xi')$ where $J' = J(\Lambda', \beta)$ for some skew semisimple stratum $[\Lambda', \beta]$, κ' is any standard beta-extension of θ' and $P^\circ(\Lambda'_E)$ is a maximal parahoric subgroup of G_E. This implies that $P^\circ(\Lambda'_E)$ is equal to $P^\circ(\Lambda_E)$.

It follows that $\theta \in \mathcal{C}_{\min}$. We thus may choose $\theta' = \theta$ (and $\kappa' = \kappa$). Thus π' contains a cuspidal type $(J, \kappa \otimes \xi')$. It follows from [**58**] Corollary 6.19 that the compact induction of the representation $\tau' = \kappa \otimes \xi'$ from J to G is isomorphic to π'. □

Remark 3.5. — Applying the functor $\mathrm{Hom}_{J^1}(\kappa, -)$ from representations of G to representations of J which are trivial on J^1, which is compatible to reduction mod ℓ if we assume moreover that κ is integral, we deduce from [**34**] Corollary 8.5 that $\mathbf{r}_\ell(\xi) \leqslant \mathbf{r}_\ell(\xi')$.

4. Globalizing representations

In this section, we fix a p-adic field F and a quasi-split special orthogonal, unitary or symplectic group G over F. Let k, w and $\mathbf{G}$ be as in Theorem 2.1, and $\jmath : \mathbf{G}(F) \simeq G$ be an isomorphism of locally compact groups which we use to identify $\mathbf{G}(F)$ with G.

Let ℓ denote a prime number different from p, and fix a field isomorphism ι as in (3.1). Let u be a finite place of k different from w, not dividing ℓ.

In Paragraph 4.2 only, the prime number p will be assumed to be odd.

4.1.

The next proposition is the first step towards Theorem 4.4. (See also Paragraph 1.3.)

Proposition 4.1. — *Let π be a unitary cuspidal irreducible complex representation of G, and let ρ be a unitary cuspidal irreducible complex representation of $\mathbf{G}(k_u)$. There is an irreducible automorphic representation Π of $\mathbf{G}(\mathbb{A})$ such that*

(1) *the local component Π_u is isomorphic to ρ,*
(2) *the local component Π_w is isomorphic to π,*
(3) *the local component Π_v is the trivial character of $\mathbf{G}(k_v)$ for any real place v of k.*

Remark 4.2. — When the centre of G is compact, any cuspidal irreducible representation π of G is unitarizable. The only case where a quasi-split classical group G has a non-compact centre is when G is isomorphic to the split special orthogonal group $\mathrm{SO}_2(F) \simeq F^\times$ (see [**41**] 4.2).

Proof. — Let $\mathbf{Z}$ be the centre of $\mathbf{G}$. We start the proof by the following lemma.

Lemma 4.3. — *There is a unitary automorphic character $\Omega : \mathbf{Z}(\mathbb{A})/\mathbf{Z}(k) \to \mathbb{C}^\times$ such that*

(1) *the local component Ω_u is equal to the central character ω_ρ of ρ,*
(2) *the local component Ω_w is equal to the central character ω_π of π,*
(3) *the local component Ω_v is the trivial character of $\mathbf{Z}(k_v)$ for any real place v of k.*

Proof. — Let U denote the subgroup $\mathbf{Z}(k_u) \times \mathbf{Z}(k_w) \times \mathbf{Z}(\mathbb{A}_\infty)$ of $\mathbf{Z}(\mathbb{A})$. The intersection $\mathrm{U} \cap \mathbf{Z}(k)$ is trivial, thus U identifies with a locally compact subgroup of $\mathbf{Z}(\mathbb{A})/\mathbf{Z}(k)$. By Pontryagin duality, any unitary character of U extends to $\mathbf{Z}(\mathbb{A})/\mathbf{Z}(k)$. (Note that ω_ρ and ω_π are unitary.) □

We now follow the proof of [**23**]. Let $\Omega : \mathbf{Z}(\mathbb{A})/\mathbf{Z}(k) \to \mathbb{C}^\times$ be a unitary automorphic character as in Lemma 4.3. Let y be a finite place different from u and w.

Let us choose coefficients f_u and f_w of ρ and π, respectively, which are non-zero at 1.

For all real places v of k, let f_v be the constant function equal to 1 on $\mathbf{G}(k_v)$. As this group is compact, f_v is smooth and compactly supported.

For all finite places $v \neq y$ such that $\mathbf{G}$ is unramified over k_v and Ω_v is unramified, let f_v be the complex function on $\mathbf{G}(k_v)$ supported on $\mathbf{Z}(k_v)\mathrm{K}_v$ such that $f_v(zg) = \Omega_v(z)$ for all $z \in \mathbf{Z}(k_v)$ and all g in a fixed hyperspecial maximal compact subgroup K_v of $\mathbf{G}(k_v)$.

For any other place x, we choose a smooth complex function f_x on $\mathbf{G}(k_x)$, non-zero at 1, compactly supported modulo $\mathbf{Z}(k_x)$ with restriction to this later group equal to Ω_x.

We let f be the product of all these f_v. It is smooth and compactly supported on $\mathbf{G}(\mathbb{A})$. We may and will assume that

– the support of f_y is small enough so that

$$f(g^{-1})f(\gamma g) = 0 \quad \text{for all } g \in \mathbf{G}(\mathbb{A}),\ \gamma \in \mathbf{G}(k) \text{ such that } \gamma \notin \mathbf{Z}(k),$$

– and $f_v(g) = \overline{f_v(g^{-1})}$ for all places v of k and all $g \in \mathbf{G}(k_v)$.

We construct as in [**23**] the Poincaré series

$$Pf(g) = \sum_{\gamma \in \mathbf{Z}(k)\backslash \mathbf{G}(k)} f(\gamma g), \qquad \text{for } g \in \mathbf{G}(\mathbb{A}).$$

We are in a particular case of the proof of [**23**] Appendice 1, so in particular this is well defined, non-zero, square-integrable and even cuspidal. There is thus an irreducible automorphic representation Π of $\mathbf{G}(\mathbb{A})$ such that f_v acts non-trivially on Π_v for each place v of k. In particular, the local components Π_u and Π_w are isomorphic to ρ and π, respectively. At any real place v, Π_v contains a vector which is $\mathbf{G}(k_v)$-invariant, so Π_v is trivial. □

4.2.

We now assume that $p \neq 2$.

Theorem 4.4. — *Let π_1, π_2 be integral cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representations of G such that*

$$\mathbf{r}_\ell(\pi_1) \leqslant \mathbf{r}_\ell(\pi_2).$$

Let ρ be a unitary cuspidal irreducible complex representation of $\mathbf{G}(k_u)$ which is compactly induced from some compact mod centre, open subgroup of $\mathbf{G}(k_u)$. Assume that G is not the split special orthogonal group $\mathrm{SO}_2(F) \simeq F^{\times}$. There are irreducible automorphic representations Π_1 and Π_2 of $\mathbf{G}(\mathbb{A})$ such that

(1) *$\Pi_{1,u}$ and $\Pi_{2,u}$ are both isomorphic to ρ,*
(2) *$\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_1 and $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_2,*
(3) *$\Pi_{1,v}$ and $\Pi_{2,v}$ are trivial for any real place v,*
(4) *there is a finite set S of places of k, containing all real places, such that for all $v \notin \mathrm{S}$:*

(a) *the local components $\Pi_{1,v}$ and $\Pi_{2,v}$ are unramified with respect to some hyperspecial maximal compact subgroup K_v of $\mathbf{G}(k_v)$,*

(b) *the restrictions of the Satake parameters of $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ to the Hecke $\overline{\mathbb{Z}}_\ell$-algebra $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v)$ are congruent mod the maximal ideal $\mathfrak{m}$ of $\overline{\mathbb{Z}}_\ell$.*

Remark 4.5. — The assumption on G implies that the centre of G is compact, thus any cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representation of G is integral.

Proof. — First, let us apply Proposition 4.1 with $\pi = \pi_1 \otimes_{\overline{\mathbb{Q}}_\ell} \mathbb{C}$. (Since the centre of G is compact, the central character of π has finite order, thus π is unitarizable.) We obtain an irreducible automorphic representation Π_1 of $\mathbf{G}(\mathbb{A})$ such that

(1) the local component $\Pi_{1,u}$ is isomorphic to ρ,
(2) the local component $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_1,
(3) the local component $\Pi_{1,v}$ is the trivial character of $\mathbf{G}(k_v)$ for any real place v.

We then choose for S a set of finite places of k as in Paragraph 3.4, that is, S contains u, w and all places dividing ℓ, and, for any finite place $v \notin \mathrm{S}$, the local component $\Pi_{1,v}$ is unramified with respect to some hyperspecial maximal compact subgroup K_v of $\mathbf{G}(k_v)$. For such v, this defines a $\overline{\mathbb{Z}}_\ell$-character $\chi_{1,v}$ of $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v)$.

We now apply Theorem 3.1 with $\pi' = \pi_2$. The conditions of Paragraph 3.5 are automatically satisfied for $\Pi_{1,w}$ thanks to [**58**]. We get an irreducible automorphic representation Π_2 of the group $\mathbf{G}(\mathbb{A})$, trivial at infinity, such that

(1) the local component $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_2,
(2) the local component $\Pi_{2,u}$ is isomorphic to ρ,
(3) for all finite places $v \notin \mathrm{S}$, the local component $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is K_v-unramified with associated $\overline{\mathbb{Z}}_\ell$-character $\chi_{2,v} : \mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v) \to \overline{\mathbb{Z}}_\ell$, and $\chi_{1,v}$ and $\chi_{2,v}$ are congruent mod $\mathfrak{m}$.

This proves Theorem 4.4. □

5. Global transfer

5.1. Quasi-split classical groups

Let k be either a p-adic field for some prime number p, or a real Archimedean local field, or a totally real number field. We will consider the following families of quasi-split reductive groups over k:

(1) For $n \geqslant 1$, the (split) symplectic group Sp_{2n} defined as $\mathrm{Sp}(f)$, where f is the alternating form on $k^{2n} \times k^{2n}$ defined by

$$f(x_1, \dots, x_{2n}, y_1, \dots, y_{2n}) = x_1 y_{2n} - x_{2n} y_1 + \dots + x_n y_{n+1} - x_{n+1} y_n. \tag{5.1}$$

(2) For $n \geqslant 1$ and $\alpha \in k^\times$, the (split) special orthogonal group SO_{2n+1} defined as $\mathrm{SO}(q)$, where q is the quadratic form on k^{2n+1} of discriminant $(-1)^n \alpha$ defined by

$$q(x_1, \dots, x_{2n+1}) = x_1 x_2 + \dots + x_{2n-1} x_{2n} + \alpha x_{2n+1}^2. \tag{5.2}$$

(3) For $n \geqslant 1$ and $\alpha \in k^\times$, the special orthogonal group SO_{2n}^α defined as $\mathrm{SO}(q)$, where q is the quadratic form on k^{2n} of discriminant $(-1)^n \alpha$ defined by

$$q(x_1, \dots, x_{2n}) = x_1 x_2 + \dots + x_{2n-3} x_{2n-2} + x_{2n-1}^2 - \alpha x_{2n}^2. \tag{5.3}$$

(4) For $n \geqslant 1$ and $\alpha \in k^\times$, the unitary group U_n^α defined as $\mathrm{U}(h)$, where h is the l/k-Hermitian form on l^n of discriminant $(-1)^{n(n-1)/2}$ defined by

$$h(x_1, \dots, x_n) = x_1^c x_n - x_2^c x_{n-1} + \dots + (-1)^{n-1} x_n^c x_1 \tag{5.4}$$

where l is the k-algebra $k[\mathrm{X}]/(\mathrm{X}^2 - \alpha)$ and c is the non-trivial automorphism of l/k. If $\alpha \in k^{\times 2}$, the k-group U_n^α is thus isomorphic to GL_n.

In the even orthogonal and unitary cases, the image of α in $k^\times / k^{\times 2}$ will still be denoted α.

5.2. The dual group

In this paragraph, k is either a p-adic field or a totally real number field and $\mathbf{G}^*$ is one of the quasi-split special orthogonal, unitary or symplectic k-groups of 5.1. We define its dual group

$$\widehat{\mathbf{G}} = \begin{cases} \mathrm{SO}_{2n+1}(\mathbb{C}) & \text{if } \mathbf{G}^* = \mathrm{Sp}_{2n}, \\ \mathrm{Sp}_{2n}(\mathbb{C}) & \text{if } \mathbf{G}^* = \mathrm{SO}_{2n+1}, \\ \mathrm{SO}_{2n}(\mathbb{C}) & \text{if } \mathbf{G}^* = \mathrm{SO}_{2n}^\alpha, \\ \mathrm{GL}_n(\mathbb{C}) & \text{if } \mathbf{G}^* = \mathrm{U}_n^\alpha. \end{cases}$$

In the even orthogonal case, the groups $\mathrm{SO}_{2n}(\mathbb{C}) \subseteq \mathrm{O}_{2n}(\mathbb{C})$ are defined with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{2n}$ given by

$$\langle \mathsf{e}_i, \mathsf{e}_j \rangle = \begin{cases} 0 & \text{if } i + j \neq 2n+1, \\ 1 & \text{otherwise,} \end{cases}$$

where $(\mathsf{e}_1, \dots, \mathsf{e}_{2n})$ is the canonical basis of $\mathbb{C}^{2n}$.

5.3. The local Langlands correspondence

In this paragraph, k is a p-adic field and $\mathbf{G}^*$ is either the general linear k-group GL_n for some $n \geqslant 1$ (whose dual group is $\mathrm{GL}_n(\mathbb{C})$) or one of the quasi-split classical k-groups of 5.1. We denote by W_k the Weil group of $\overline{\mathbb{Q}}_p$ over k, and define the semi-direct product ${}^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes \mathrm{W}_k$, where

- the action of W_k on $\widehat{\mathbf{G}}$ is trivial when $\mathbf{G}^*$ is split (that is, when $\mathbf{G}^*$ is general linear, symplectic, odd orthogonal, even orthogonal with $\alpha = 1$ or unitary with $\alpha = 1$),

• when $\mathbf{G}^*$ is even orthogonal and $\alpha \neq 1$, and if l denotes the quadratic extension of k in $\overline{\mathbb{Q}}_p$ generated by a square root of α, the action of W_k on $\widehat{\mathbf{G}}$ factors through $\mathrm{Gal}(l/k)$, the generator c of which acts by conjugacy by the element $w \in \mathrm{O}_{2n}(\mathbb{C})$ fixing $\mathsf{e}_1, \dots, \mathsf{e}_{n-1}, \mathsf{e}_{n+2}, \dots, \mathsf{e}_{2n}$ and exchanging e_n and e_{n+1} (thus $\widehat{\mathbf{G}} \rtimes \mathrm{Gal}(l/k)$ identifies with $\mathrm{O}_{2n}(\mathbb{C})$),

• when $\mathbf{G}^*$ is unitary and $\alpha \neq 1$, and if l denotes the quadratic extension of k in $\overline{\mathbb{Q}}_p$ generated by a square root of α, the action of W_k on $\widehat{\mathbf{G}}$ factors through the group $\mathrm{Gal}(l/k)$ whose generator c acts by

$$g \mapsto g^* = \mathrm{J} \cdot {}^{\mathrm{t}}g^{-1} \cdot \mathrm{J}^{-1}$$

where ${}^{\mathrm{t}}g$ denotes the transpose of $g \in \mathrm{GL}_n(\mathbb{C})$ and J is the antidiagonal matrix in $\mathrm{GL}_n(\mathbb{C})$ defined by $\mathrm{J}_{i,j} = 0$ if $i + j \neq n + 1$ and $\mathrm{J}_{i,n+1-i} = (-1)^{i-1}$.

Let $\mathrm{WD}_k = \mathrm{W}_k \times \mathrm{SL}_2(\mathbb{C})$ denote the Weil-Deligne group of k. A *(local) Langlands parameter* for $\mathbf{G}(k)$ is a group homomorphism

$$\varphi : \mathrm{WD}_k \to \widehat{\mathbf{G}} \rtimes \mathrm{W}_k$$

such that

– its restriction to W_k is smooth,
– its restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic,
– the projection of $\varphi(\mathrm{W}_k)$ onto $\widehat{\mathbf{G}}$ is made of semi-simple elements, and
– the projection of $\varphi(w, x)$ onto W_k is equal to w for all $(w, x) \in \mathrm{WD}_k$.

When $\mathbf{G}^*$ is split, this is the same as a morphism $\mathrm{WD}_k \to \widehat{\mathbf{G}}$ satisfying the first three points. In the even orthogonal case with $\alpha \neq 1$, this is the same as a morphism $\mathrm{WD}_k \to \mathrm{O}_{2n}(\mathbb{C})$ satisfying the first three points and whose determinant is the quadratic character

$$x \mapsto (\alpha, x)$$

of $k^\times$, which can be seen as a character of W_k *via* the Artin reciprocity map of local class field theory. We say a local Langlands parameter φ is *bounded* if $\varphi(\mathrm{W}_k)$ is relatively compact in $\widehat{\mathbf{G}}$.

Let

– $\Phi(\mathbf{G}^*, k)$ be the set of $\widehat{\mathbf{G}}$-conjugacy classes of local Langlands parameters for $\mathbf{G}^*$ over k,
– $\Pi(\mathbf{G}^*(k))$ be the set of isomorphism classes of irreducible representations of $\mathbf{G}^*(k)$.

When $\mathbf{G}^*$ is the general linear group GL_n, the local Langlands correspondence ([**22, 24**]) is a bijection from $\Pi(\mathrm{GL}_n(k))$ to $\Phi(\mathrm{GL}_n, k)$.

When $\mathbf{G}^*$ is classical, the local Langlands correspondence ([**2**] Theorem 2.2.1, [**46**] Theorems 2.5.1, 3.2.1, see also [**4**] Theorems 3.2, 3.6 and Remarks 3.3, 3.7) defines

(1) (symplectic, odd orthogonal and unitary cases) a partition

$$\Pi(\mathbf{G}^*(k)) = \coprod_{\varphi \in \Phi(\mathbf{G}^*, k)} \Pi_\varphi(\mathbf{G}^*(k)) \tag{5.5}$$

into non-empty finite sets $\Pi_\varphi(\mathbf{G}^*(k))$ if $\mathbf{G}^*$ is symplectic, odd special orthogonal or unitary,

(2) (even orthogonal case) a partition

$$\Pi(\mathrm{SO}^\alpha_{2n}(k)) = \coprod_{\varphi \in \Phi(\mathrm{SO}^\alpha_{2n}, k)/\mathrm{O}_{2n}(\mathbb{C})} \Pi_\varphi(\mathrm{SO}^\alpha_{2n}(k)) \tag{5.6}$$

where each $\Pi_\varphi(\mathrm{SO}^\alpha_{2n}(k))$ is non-empty, finite and stable under $\mathrm{O}^\alpha_{2n}(k)$-conjugacy.

In each case, we have the following properties:

• $\Pi_\varphi(\mathbf{G}^*(k))$ contains a tempered representation if and only if φ is bounded. When this is the case, all representations in $\Pi_\varphi(\mathbf{G}^*(k))$ are tempered.

• $\Pi_\varphi(\mathbf{G}^*(k))$ contains a discrete series representation if and only if φ is bounded and the quotient of the centralizer of the image of φ in $\widehat{\mathbf{G}}$ by $\mathbf{Z}(\widehat{\mathbf{G}})^{\mathrm{W}_k}$ is finite. When this is the case, all representations in $\Pi_\varphi(\mathbf{G}^*(k))$ are discrete series representations. (See for instance [**68**] Theorem 2.2 for symplectic and special orthogonal groups, and [**30**] Theorem 1.6.1 for unitary groups.)

5.4. The local transfer

In this paragraph, k is a p-adic field and $\mathbf{G}^*$ is one of the quasi-split classical k-groups of 5.1. If $\mathbf{G}^*$ is symplectic or special orthogonal, there is a morphism $\mathrm{Std}:\widehat{\mathbf{G}}\to\mathrm{GL}_N(\mathbb{C})$ with

$$N = N(\mathbf{G}^*) = \begin{cases} 2n & \text{if } \mathbf{G}^* = \mathrm{SO}_{2n+1} \text{ or } \mathbf{G}^* = \mathrm{SO}^\alpha_{2n}, \\ 2n+1 & \text{if } \mathbf{G}^* = \mathrm{Sp}_{2n}, \end{cases} \tag{5.7}$$

given by the natural inclusion. We extend it to a morphism $\mathrm{Std}:\widehat{\mathbf{G}}\rtimes\mathrm{W}_k\to\mathrm{GL}_N(\mathbb{C})$ as follows:

• Std is trivial on W_k when $\mathbf{G}^*$ is split,

• when $\mathbf{G}^*$ is even orthogonal and $\alpha\neq 1$, Std is trivial on W_l and $\mathrm{Std}(c) = w \in \mathrm{O}_{2n}(\mathbb{C})$, thus Std factors through

$$\mathrm{SO}_{2n}(\mathbb{C})\rtimes\mathrm{W}_k \twoheadrightarrow \mathrm{SO}_{2n}(\mathbb{C})\rtimes\mathrm{Gal}(l/k)\simeq\mathrm{O}_{2n}(\mathbb{C})\subseteq\mathrm{GL}_{2n}(\mathbb{C})$$

(see also [**4**] 3.2).

In the unitary case ($\mathbf{G}^* = \mathrm{U}^\alpha_n$), we need to introduce the k-group GL^α_n, the restriction of GL_n with respect to l/k. Its dual group is $\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})$, and we define the semi-direct product

$${}^L\mathrm{GL}^\alpha_n = (\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C}))\rtimes\mathrm{W}_k$$

where

• the action of W_k is trivial when l/k is split,

• otherwise, the action of W_k factors through $\mathrm{Gal}(l/k)$ and c acts by $(g,h)\mapsto(h,g)$.

It will be convenient to set

$$N = N(\mathrm{U}^\alpha_n) = n. \tag{5.8}$$

Let Std be the morphism $\widehat{\mathbf{G}}\rtimes\mathrm{W}_k\to(\mathrm{GL}_N(\mathbb{C})\times\mathrm{GL}_N(\mathbb{C}))\rtimes\mathrm{W}_k$ defined by $g\rtimes w\mapsto(g,g^*)\rtimes w$.

Given an irreducible representation $\pi\in\Pi(\mathbf{G}^*(k))$, let $\varphi\in\Phi(\mathbf{G}^*,k)$ be a Langlands parameter such that $\pi\in\Pi_\varphi(\mathbf{G}^*(k))$. (In the even orthogonal case, φ is determined up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy only.)

If $\mathbf{G}^*$ is symplectic or special orthogonal, then, composing with Std, we get a local Langlands parameter $\phi = \mathrm{Std}\circ\varphi\in\Phi(\mathrm{GL}_N,k)$, uniquely determined up to $\mathrm{GL}_N(\mathbb{C})$-conjugacy.

If $\mathbf{G}^*$ is unitary, then, composing with Std, we obtain a Langlands parameter

$$\mathrm{Std}\circ\varphi:\mathrm{WD}_k\to(\mathrm{GL}_N(\mathbb{C})\times\mathrm{GL}_N(\mathbb{C}))\rtimes\mathrm{W}_k.$$

• If l is non-split, its restriction to WD_l has the form $(w,x) \mapsto (\phi(w,x), \phi(w,x)^*) \rtimes w$ for a local Langlands parameter $\phi \in \Phi(\mathrm{GL}_N, l)$, uniquely determined up to $\mathrm{GL}_N(\mathbb{C})$-conjugacy.

• If l is split, it is of the form $(w,x) \mapsto (\phi(w,x), \phi(w,x)^*) \rtimes w$ for a local Langlands parameter $\phi \in \Phi(\mathrm{GL}_N, k)$, uniquely determined up to $\mathrm{GL}_N(\mathbb{C})$-conjugacy.

Definition 5.1. — The *local transfer* of π, denoted $\mathbf{t}(\pi)$, is the isomorphism class of irreducible representations associated with ϕ through the local Langlands correspondence. It is

(1) a class of representations of $\mathrm{GL}_N(k)$ if $\mathbf{G}^*$ is symplectic or special orthogonal,

(2) a class of representations of $\mathrm{GL}_N(l)$ if $\mathbf{G}^*$ is unitary,

which is uniquely determined by the isomorphism class of π.

Remark 5.2. — If $\mathbf{G}^*$ is unitary and l is split over k, and if we fix an isomorphism of k-algebras $l \simeq k \times k$, which we use to identify $\mathrm{U}_n^\alpha(k)$ with $\mathrm{GL}_n(k)$ and $\mathrm{GL}_N(l)$ with $\mathrm{GL}_N(k) \times \mathrm{GL}_N(k)$, then

$$\mathbf{t}(\pi) = \pi \otimes \pi^\vee \tag{5.9}$$

(where $\pi^\vee$ is the contragredient of π). This does not depend on the choice of $l \simeq k \times k$. Indeed, making the other choice twists the isomorphism $\mathrm{U}_n^\alpha(k) \simeq \mathrm{GL}_n(k)$ by $g \mapsto g^*$ (see (2.3) and the explanation thereafter) and the isomorphism $\mathrm{GL}_N(l) \simeq \mathrm{GL}_N(k) \times \mathrm{GL}_N(k)$ by $(g,h) \mapsto (h,g)$, which gives (5.9) again since $g \mapsto \pi(g^*)$ is isomorphic to $\pi^\vee$.

In Section 6, we will describe explicitly the local transfer for unramified representations when $\mathbf{G}^*$ is unramified over k, and will describe its congruence properties.

5.5. Arthur parameters in the symplectic and orthogonal cases

In this paragraph, k is a totally real number field and $\mathbf{G}^*$ is symplectic or quasi-split special orthogonal. We write $\mathbb{A}$ for the ring of adèles of k and $N = N(\mathbf{G}^*)$ (see (5.7)).

Definition 5.3. — A *discrete global Arthur parameter* (for $\mathbf{G}^*$) is a formal sum

$$\psi = \Pi_1[d_1] \oplus \cdots \oplus \Pi_r[d_r] \tag{5.10}$$

for some integer $r \geqslant 1$, where, for each $i \in \{1, \dots, r\}$, d_i is a positive integer and Π_i is a self-dual cuspidal automorphic irreducible representation of $\mathrm{GL}_{N_i}(\mathbb{A})$ for some $N_i \geqslant 1$, such that

(1) one has $N_1 d_1 + \cdots + N_r d_r = N$,

(2) if $r \geqslant 2$ and $\Pi_i \simeq \Pi_j$ for some $i \neq j$ in $\{1, \dots, r\}$, then $d_i \neq d_j$,

(3) the self-dual representation Π_i has the same parity as $\widehat{\mathbf{G}}$ if d_i is odd, and has the opposite parity if d_i is even, where the parity of Π_i is defined to be orthogonal if $L(s, \Pi_i, \mathrm{Sym}^2)$ has a pole at $s = 1$, and symplectic if $L(s, \Pi_i, \wedge^2)$ has a pole at $s = 1$,

(4) the character $\omega_{\Pi_1}^{d_1} \dots \omega_{\Pi_r}^{d_r}$ is trivial if $\mathbf{G}^* = \mathrm{Sp}_{2n}$ or $\mathbf{G}^* = \mathrm{SO}_{2n+1}$, and is equal to the quadratic character

$$\chi_\alpha : x \mapsto \prod_v (\alpha_v, x_v)_v \in \{-1, 1\}$$

of $\mathbb{A}^\times / k^\times$ if $\mathbf{G}^* = \mathrm{SO}_{2n}^\alpha$, where ω_{Π_i} is the central character of Π_i.

A discrete global Arthur parameter $\Sigma_1[e_1]\oplus\cdots\oplus\Sigma_s[e_s]$ is said to be *equivalent to* (5.10) if we have $s = r$ and there is a permutation $\varepsilon \in \mathfrak{S}_r$ such that $e_i = d_{\varepsilon(i)}$ and $\Sigma_i \simeq \Pi_{\varepsilon(i)}$ for each i. Let

$$\widetilde{\Psi}_2(\mathbf{G}^*)$$

be the set of equivalence classes of discrete global Arthur parameters for $\mathbf{G}^*$.

Associated with a discrete global Arthur parameter $\psi \in \Psi_2(\mathbf{G}^*)$ given by (5.10), there are a local Arthur parameter ψ_v and a local Arthur packet $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$ for each finite place v of k: see (5.11) and (5.12) below.

Let v be a finite place of k, and consider the local component $\Pi_{i,v}$ for some i. It is a unitarisable irreducible representation of $\mathrm{GL}_{N_i}(k_v)$. Associated with it through the local Langlands correspondence for $\mathrm{GL}_{N_i}(k_v)$, there is a local Langlands parameter

$$\phi_{i,v} : \mathrm{WD}_{k_v} \to \mathrm{GL}_{N_i}(\mathbb{C}),$$

uniquely determined up to $\mathrm{GL}_{N_i}(\mathbb{C})$-conjugacy. Since one does not know whether $\Pi_{i,v}$ is tempered, the parameter $\phi_{i,v}$ might not be bounded.

We define a morphism

$$\psi_v = (\phi_{1,v} \boxtimes \mathrm{S}_{d_1}) \oplus \cdots \oplus (\phi_{r,v} \boxtimes \mathrm{S}_{d_r}) : \mathrm{WD}_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C}) \tag{5.11}$$

where $\mathrm{S}_d = \mathrm{Sym}^{d-1}$ denotes the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension $d \geqslant 1$. Recall that we have defined a morphism Std in 5.3. By [**2**] Theorem 1.4.2, there is a local Arthur parameter $\xi : \mathrm{WD}_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}} \rtimes \mathrm{W}_{k_v}$ such that ψ_v is $\mathrm{GL}_N(\mathbb{C})$-conjugate to $\mathrm{Std} \circ \xi$. The parameter ξ is uniquely determined up to $\widehat{\mathbf{G}}$-conjugacy, except if $\mathbf{G}^* = \mathrm{SO}_{2n}^{\alpha}$ and all $N_1d_1, \dots, N_rd_r$ are even, in which case there are two $\widehat{\mathbf{G}}$-conjugacy classes of such ξ.

Associated with ψ_v, there is a multiset $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$ of irreducible smooth representations of $\mathbf{G}^*(k_v)$, that is, a map

$$\Pi(\mathbf{G}^*(k_v)) \to \mathbb{Z}_{\geqslant 0} \tag{5.12}$$

with finite support, where $\Pi(\mathbf{G}^*(k_v))$ is the set of isomorphism classes of irreducible smooth representations of $\mathbf{G}^*(k_v)$. If $\psi_v(\mathrm{W}_{k_v})$ is relatively compact in $\mathrm{GL}_N(\mathbb{C})$, this comes from [**2**] Theorems 1.5.1, 2.2.1, 2.2.4 and (5.12) is supported in the subset $\Pi_{\mathrm{unit}}(\mathbf{G}^*(k_v))$ of unitarisable representations. Thanks to Moeglin ([**43**], see also [**67**] Theorem 8.12), it does not take any value > 1, that is, $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$ can be regarded as a finite subset of $\Pi_{\mathrm{unit}}(\mathbf{G}^*(k_v))$.

When $\psi_v(\mathrm{W}_{k_v})$ is not relatively compact, $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$ is obtained from the relatively compact case by a parabolic induction process: see [**2**] 1.5 in the symplectic and orthogonal cases and [**4**] 6.5 in the even orthogonal case. For our purpose, it will be enough to make the following remark.

Remark 5.4. — Let v be a finite place of k, and assume that $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$ contains a cuspidal representation. Then $\psi_v(\mathrm{W}_{k_v})$ is relatively compact in $\mathrm{GL}_N(\mathbb{C})$.

When ψ_v is trivial on the $\mathrm{SL}_2(\mathbb{C})$-factor, that is, ψ_v is a local Langlands parameter for $\mathbf{G}^*(k_v)$, the local Arthur packet $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$ coincides with the L-packet associated with ψ_v by the local Langlands correspondence in (5.5) and (5.6). (See [**4**] top of Paragraph 6.3.)

5.6. Transfer

In this paragraph, k is a totally real number field, $\mathbf{G}^*$ is one of the quasi-split special orthogonal, unitary or symplectic k-groups of 5.1 and $\mathbf{G}$ is an inner form of $\mathbf{G}^*$ over k such that $\mathbf{G}(k_v)$ is compact for any real place v and quasi-split for any finite place v.

In order to state the following theorem, we need more than the group $\mathbf{G}$. Following [**61**] and [**30**], we realize $\mathbf{G}$ as

- a rigid inner twist of $\mathbf{G}^*$ in the symplectic case (see Paragraph 2.8),
- a pure inner twist of $\mathbf{G}^*$ in the special orthogonal and unitary cases, that is, we fix a quadratic form q such that $\mathbf{G} = \mathrm{SO}(q)$ or a Hermitian form h such that $\mathbf{G} = \mathrm{U}(h)$. (See for instance [**32**] Sections 29.D, 29.E.)

If $\mathbf{G}^*$ is special orthogonal, let q^* be the quadratic form (5.2) or (5.3) such that $\mathbf{G}^* = \mathrm{SO}(q^*)$, and let $\alpha = (-1)^{n(n-1)/2}\delta(q^*)$ be its normalized discriminant. Let v be a finite place of k:

- if $q\otimes k_v$ is equivalent to $q^*\otimes k_v$, any choice of k_v-isomorphism f such that $q = q^*\circ f$ defines a group isomorphism $\jmath : \mathbf{G}(k_v) \simeq \mathbf{G}^*(k_v)$, and changing f changes $\jmath$ by an inner automorphism, which does not affect isomorphism classes of representations of these groups;
- if $q\otimes k_v$ is not equivalent to $q^*\otimes k_v$, which can only happen when $\mathbf{G}^* = \mathrm{SO}_{2n}^{\alpha}$ with $\alpha \neq 1$, there is a $\lambda \in k_v^{\times}$ such that $q\otimes k_v$ is equivalent to $\lambda\cdot(q^*\otimes k_v)$. We thus have (canonically upto an inner automorphism) $\mathbf{G}(k_v) \simeq \mathrm{SO}(\lambda\cdot(q^*\otimes k_v)) = \mathbf{G}^*(k_v)$.

If $\mathbf{G}^*$ is unitary, let h^* be the l/k-Hermitian form (5.4) such that $\mathbf{G}^* = \mathrm{U}(h^*)$. Let v be a finite place of k:

- if $h\otimes k_v$ is equivalent to $h^*\otimes k_v$, any choice of isomorphism f such that $h = h^*\circ f$ defines a group isomorphism $\jmath : \mathbf{G}(k_v) \simeq \mathbf{G}^*(k_v)$, and changing f changes $\jmath$ by an inner automorphism, which does not affect isomorphism classes of representations of these groups;
- if $h\otimes k_v$ is not equivalent to $h^*\otimes k_v$, which can only happen when $\mathbf{G}^* = \mathrm{U}_{2n+1}^{\alpha}$ with $\alpha \neq 1$, there is a $\delta \in k_v^{\times}$ such that $h\otimes k_v$ is equivalent to $\delta\cdot(h^*\otimes k_v)$. We thus have (canonically up to an inner automorphism) $\mathbf{G}(k_v) \simeq \mathrm{U}(\delta\cdot(h^*\otimes k_v)) = \mathbf{G}^*(k_v)$.

If $\mathbf{G}^*$ is the symplectic group Sp_{2n}, then $\mathbf{G}(k)$ is the group made of all $g \in \mathbf{M}_n(D)$ such that $g^*g = 1$, where D is a quaternion k-algebra which is split at each finite place and definite at each real place, and g^* in the matrix whose (i,j)-entry is the conjugate of g_{ji}. (See 2.8 and [**60**] 2.1.1.) Let v be a finite place of k, and fix an isomorphism of k_v-algebras $u : D\otimes_k k_v \simeq \mathbf{M}_2(k_v)$. Through u, the group $\mathbf{G}(k_v)$ identifies with $\mathrm{Sp}(f_v)$ for some alternating form f_v on $k_v^{2n}\times k_v^{2n}$. Changing u changes this identification by an inner automorphism. We thus have (canonically up to an inner automorphism) a group isomorphism $\mathbf{G}(k_v) \simeq \mathbf{G}^*(k_v)$.

In all cases, we have explained how to canonically identify representations of $\mathbf{G}(k_v)$ with those of $\mathbf{G}^*(k_v)$. This thus defines a local transfer for irreducible representations of $\mathbf{G}(k_v)$.

Theorem 5.5. — *Assume that the group $\mathbf{G}^*$ is symplectic or special orthogonal. Let π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A})$ and suppose that there is a finite place u of k such that both the local component π_u and its local transfer to $\mathrm{GL}_N(k_u)$ are cuspidal. There is a unique self-dual cuspidal automorphic representation Π of $\mathrm{GL}_N(\mathbb{A})$ such that*

(1) *for all finite places v of k, the local transfer of π_v to $\mathrm{GL}_N(k_v)$ is Π_v,*

(2) *for all real places v of k, the infinitesimal character of Π_v is algebraic regular.*

Proof. — First note that, associated with any discrete global Arthur parameter $\psi \in \Psi_2(\mathbf{G}^*)$ and any finite place v of k, there is a local Arthur packet $\Pi_{\psi_v}(\mathbf{G}^*(k_v))$. We explained how to canonically identify representations of $\mathbf{G}(k_v)$ with those of $\mathbf{G}^*(k_v)$. This thus defines a local Arthur packet $\Pi_{\psi_v}(\mathbf{G}(k_v))$.

Now, as $\mathbf{G}$ is compact at all real places and quasi-split at all finite places, [**61**] Theorem 4.0.1 and Remark 4.0.2 apply. We thus get a global Arthur parameter ψ for $\mathbf{G}^*$ such that

(1) $\pi_v \in \Pi_{\psi_v}(\mathbf{G}(k_v))$ for all finite places v of k,

(2) the infinitesimal character of ψ_v is algebraic regular for all real places v of k.

In the remainder of the proof, we follow an argument which has been suggested to us by A. Moussaoui, whom we thank for this. First, at $v = u$, we have

$$\pi_u \in \Pi_{\psi_u}(\mathbf{G}(k_u))$$

and it follows from Remark 5.4 that $\psi_u(\mathrm{W}_{k_u})$ is relatively compact in $\mathrm{GL}_N(\mathbb{C})$. Associated with ψ_u in [**42**] 4.1, there is its *extended cuspidal support* (or *infinitesimal character*), denoted λ_u. It is the N-dimensional representation of W_{k_u} defined by

$$\lambda_u(w) = \psi_u(w, \mathsf{d}_w, \mathsf{d}_w), \quad \mathsf{d}_w = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), \quad w \in \mathrm{W}_{k_u},$$

where $w \mapsto |w|$ is the character $\mathrm{W}_{k_u} \to \mathbb{R}_+^\times$ defined by $|w| = q^{-v(w)}$, where q is the cardinality of the residue field of k_u and $v(w) \in \mathbb{Z}$ is the valuation of w, normalized so that any geometric Frobenius element has valuation 1. If we write explicitly

$$\psi_u = \bigoplus_{i=1}^{m} \sigma_i \boxtimes \mathrm{S}_{a_i} \boxtimes \mathrm{S}_{b_i}$$

for some $m \geqslant 1$, with $a_i, b_i \geqslant 1$ and where σ_i is an irreducible representation of W_{k_u}, then

$$\lambda_u = \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{b_i-1} \bigoplus_{k=0}^{a_i-1} \sigma_i |\cdot|^{(b_i-1)/2+(a_i-1)/2-j-k}. \tag{5.13}$$

On the other hand, by [**42**] 4.1 again, the *extended cuspidal support* (or *infinitesimal character*) of π_u is the representation λ of W_{k_u} defined by $\lambda(w) = \phi(w, \mathsf{d}_w)$ for all $w \in \mathrm{W}_{k_u}$, where $\phi = \mathrm{Std} \circ \varphi$ and φ is the Langlands parameter associated with π_u (up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy in the even orthogonal case). Given the assumption that we made on π_u, the extended cuspidal support λ is irreducible. By [**42**] Proposition 4.1, the extended cuspidal supports of ψ_u and π_u coincide. It follows that (5.13) is irreducible, which implies that $m = 1$ and $a_1 = b_1 = 1$. Thus ψ satisfies $r = 1$ and $d_1 = 1$.

We thus have $\psi = \Pi[1]$ for a uniquely determined self-dual cuspidal automorphic irreducible representation Π of $\mathrm{GL}_N(\mathbb{A})$. Given a finite place v of k, the local component π_v is in the Arthur packet $\Pi_{\psi_v}(\mathbf{G}(k_v))$. Since ψ_v is a Langlands parameter (as $d_1 = 1$), this Arthur packet is an L-packet, thus Π_v is the local transfer of π_v to $\mathrm{GL}_N(k_v)$. □

We now consider the case of unitary groups.

Theorem 5.6. — *Assume that the group $\mathbf{G}^*$ is unitary. Let π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A})$, and suppose that there is a finite place u of k such that $\mathbf{G}(k_u)$ is split and π_u is cuspidal. There exists a unique conjugate-self-dual cuspidal automorphic representation Π of $\mathrm{GL}_N(\mathbb{A}_l)$ such that*

(1) *for all finite places v of k, the local transfer of π_v to $\mathrm{GL}_N(l_v)$ is Π_v,*
(2) *for all real places v of k, the infinitesimal character of Π_v is algebraic regular.*

Proof. — Since $\mathbf{G}$ is compact at all real places, the assumptions of [**35**] Corollaire 5.3 are satisfied (see the paragraph following [**35**] Remarque 5.2 regarding Property $(*)$). By [**35**] Corollaire 5.3, there is an integer $r \geqslant 1$ and, for each $i \in \{1, \dots, r\}$, there is a conjugate-self-dual discrete automorphic representation Π_i of $\mathrm{GL}_{N_i}(\mathbb{A}_l)$ for some $N_i \geqslant 1$, such that

– one has $N_1 + \cdots + N_r = N$,
– if Π is the irreducible automorphic representation of $\mathrm{GL}_N(\mathbb{A}_l)$ obtained by parabolic induction from $\Pi_1 \otimes \cdots \otimes \Pi_r$, then Π_v is the local transfer of π_v for all finite places v which are either unramified or split. (The local base change of [**35**] is the same as the local transfer of Paragraph 5.4: see [**35**] 4.10.)

In particular, for $v = u$, the group $\mathbf{G}(k_u)$ is split, thus Π_u is isomorphic to $\pi_u \otimes \pi_u^\vee$ *via* the choice of a k_u-algebra isomorphism $l_u \simeq k_u \times k_u$ (see Remark 5.2). Since π_u is cuspidal, Π_u is cuspidal as well. It follows that $r = 1$ and ψ is cuspidal. By [**35**] Théorème 5.9, we get that

- Π_v is the base change of the trivial character of $\mathbf{G}(k_v)$, thus its infinitesimal character is algebraic regular, for all real places v of k,
- and the local transfer of π_v to $\mathrm{GL}_N(l_v)$ is Π_v for all finite places v of k.

This finishes the proof of Theorem 5.6. □

6. Unramified local transfer

In this section, we examine the congruence properties of the local transfer (as defined in Paragraph 5.4) for unramified representations of unramified classical groups.

6.1.

Let F be a non-Archimedean locally compact field of residue characteristic p, and G be the group of rational points of an unramified reductive group $\mathbf{G}$ defined over F. Let $\mathbf{S}$ be a maximal F-split torus in $\mathbf{G}$, $\mathbf{T}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$ and K be a hyperspecial maximal compact subgroup of G corresponding to a hyperspecial point in the apartment associated with $\mathbf{S}$ in the reduced Bruhat-Tits building of $(\mathbf{G}, F)$. Let W be the Weyl group associated with $T = \mathbf{T}(F)$ and Λ be the $\mathbb{Z}$-lattice $T/(T \cap K)$. We have the Satake isomorphism ([**51**]) of $\mathbb{C}$-algebras

$$\begin{array}{rcl} \mathbb{C}[K\backslash G/K] & \to & \mathbb{C}[\Lambda]^W \\ f & \mapsto & \left(t \mapsto \delta^{1/2}(t) \displaystyle\int_U f(tu)\ \mathrm{d}u \right) \end{array}$$

where U is the group of rational points of the unipotent radical of a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$, du is the Haar measure on U giving measure 1 to $U \cap K$ and $\delta^{1/2}$ is the square root of the modulus character δ of $B = \mathbf{B}(F)$ defined with respect to the positive square root $\sqrt{q} \in \mathbb{R}_{>0}$ of q, the cardinality of the residue field of F.

The same formula applies when one replaces $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$. We then get a Satake isomorphism of $\overline{\mathbb{Q}}_\ell$-algebras $\overline{\mathbb{Q}}_\ell[K\backslash G/K] \to \overline{\mathbb{Q}}_\ell[\Lambda]^W$ depending on the choice of a square root $q^{1/2}$ of q in $\overline{\mathbb{Q}}_\ell$. By [**27**] §7.10–15, as this square root and its inverse are contained in $\overline{\mathbb{Z}}_\ell$, this isomorphism induces by restriction an isomorphism

$$\overline{\mathbb{Z}}_\ell[K\backslash G/K] \to \overline{\mathbb{Z}}_\ell[\Lambda]^W \tag{6.1}$$

of $\overline{\mathbb{Z}}_\ell$-algebras.

6.2.

Let π be a K-unramified irreducible $\overline{\mathbb{Q}}_\ell$-representation of G, that is, π has a non-zero K-fixed vector. Recall that its *Satake parameter* is the character χ of $\overline{\mathbb{Q}}_\ell[K\backslash G/K]$ through which this algebra acts on the 1-dimensional space π^K of K-invariant vectors of π. Through the Satake isomorphism, it defines a character of $\overline{\mathbb{Q}}_\ell[\Lambda]^W$. Such a character is of the form

$$f \mapsto \int_T f(t)\omega(t)\ \mathrm{d}t \tag{6.2}$$

for some unramified $\overline{\mathbb{Q}}_\ell$-character ω of T – which we may consider as a character of Λ – uniquely determined up to W-conjugacy. (Here dt is the Haar measure giving measure 1 to $T \cap K$.) By [**51**], the W-conjugacy class of ω is the cuspidal support of π, that is, π occurs as an irreducible component of the representation obtained by parabolically inducing ω to G along B, *where parabolic induction is normalized by the same square root of the* $\overline{\mathbb{Q}}_\ell$*-modulus* δ *as the one used to define the Satake* $\overline{\mathbb{Q}}_\ell$*-isomorphism.*

Now assume that the restriction of χ to $\overline{\mathbb{Z}}_\ell[K\backslash G/K]$ has values in $\overline{\mathbb{Z}}_\ell$. Thanks to (6.1), it defines a $\overline{\mathbb{Z}}_\ell$-character of $\overline{\mathbb{Z}}_\ell[\Lambda]^W$, still denoted χ. Let us prove that ω has values in $\overline{\mathbb{Z}}_\ell^\times$. For this, let μ be the $\overline{\mathbb{Q}}_\ell$-character of $\overline{\mathbb{Q}}_\ell[\Lambda]$ defined by (6.2). Its restriction to $\overline{\mathbb{Z}}_\ell[\Lambda]^W$ is equal to χ. According to [**8**] Chapter 5, §1, n°9, Proposition 22, the ring $\overline{\mathbb{Z}}_\ell[\Lambda]$ is integral over $\overline{\mathbb{Z}}_\ell[\Lambda]^W$. As χ takes values in $\overline{\mathbb{Z}}_\ell$ on $\overline{\mathbb{Z}}_\ell[\Lambda]^W$, and as $\overline{\mathbb{Z}}_\ell$ is integrally closed, it follows that μ takes values in $\overline{\mathbb{Z}}_\ell$ on $\overline{\mathbb{Z}}_\ell[\Lambda]$. By evaluating μ at the characteristic function of any $\lambda \in \Lambda$, we get $\omega(\lambda) \in \overline{\mathbb{Z}}_\ell$. So far, we proved the following result.

Proposition 6.1. — *Let G be the group of rational points of an unramified group defined over F, let K be a hyperspecial maximal compact subgroup of G and π be a K-unramified $\overline{\mathbb{Q}}_\ell$-representation of G with Satake parameter χ. Then π is integral if and only if χ is integral (that is, it takes integral values on $\overline{\mathbb{Z}}_\ell[K\backslash G/K]$).*

Proof. — Indeed, using the notation above, the cuspidal support of π is the W-conjugacy class of the unramified character ω of T, and π is integral if and only if ω is. □

Finally, assume that χ_1 and χ_2 are congruent $\overline{\mathbb{Z}}_\ell$-characters of $\overline{\mathbb{Z}}_\ell[K\backslash G/K]$. One can see them *via* (6.1) as congruent $\overline{\mathbb{Z}}_\ell$-characters of $\overline{\mathbb{Z}}_\ell[\Lambda]^W$, still denoted χ_1 and χ_2. For $i = 1, 2$, let μ_i be a character of $\overline{\mathbb{Z}}_\ell[\Lambda]$ extending χ_i. It takes the form (6.2) for a uniquely determined unramified character ω_i of T, which is integral thanks to the previous paragraph. Reducing mod the maximal ideal of $\overline{\mathbb{Z}}_\ell$, the characters μ_1 and μ_2 define $\overline{\mathbb{F}}_\ell$-characters $\overline{\mu}_1$ and $\overline{\mu}_2$ of $\overline{\mathbb{F}}_\ell[\Lambda]$ which, by assumption, coincide on $\overline{\mathbb{F}}_\ell[\Lambda]^W$. Applying the corollary of [**8**] Chapter 5, §2, n°2, Theorem 2, it follows that the characters $\mathbf{r}_\ell(\omega_1)$ and $\mathbf{r}_\ell(\omega_2)$ are W-conjugate. We thus proved:

Proposition 6.2. — *Let G be the group of rational points of an unramified group defined over F, let K be a hyperspecial maximal compact subgroup of G, let π_1 and π_2 be K-unramified irreducible $\overline{\mathbb{Q}}_\ell$-representations of G whose Satake parameters χ_1 and χ_2 define congruent $\overline{\mathbb{Z}}_\ell$-characters of $\overline{\mathbb{Z}}_\ell[K\backslash G/K]$ and let ω_1 and ω_2 be unramified $\overline{\mathbb{Q}}_\ell$-characters of T such that π_i occurs in the parabolic induction of ω_i to G along B, for $i = 1, 2$. Then $\mathbf{r}_\ell(\omega_1)$ and $\mathbf{r}_\ell(\omega_2)$ are W-conjugate.*

6.3.

From now on and until the end of this section, we assume that $\mathbf{G}$ is an unramified special orthogonal, unitary or symplectic group among the groups of Paragraph 5.1. The associated dual group $\widehat{\mathbf{G}}$ has been defined in Paragraph 5.2. Recall that $G = \mathbf{G}(F)$.

Let π be an integral K-unramified $\overline{\mathbb{Q}}_\ell$-representation of G. Its cuspidal support is the W-orbit of an unramified $\overline{\mathbb{Z}}_\ell$-character ω of T. Its Satake parameter is a character $\chi : \overline{\mathbb{Z}}_\ell[K\backslash G/K] \to \overline{\mathbb{Z}}_\ell$. They are related through the Satake isomorphism by the formula (6.2).

Restriction from T to $S = \mathbf{S}(F)$ induces an isomorphism $\Lambda \simeq S/(S \cap K)$, thus between unramified characters of T and unramified characters of S. The later is the dual group $\widehat{\mathbf{S}}(\overline{\mathbb{Q}}_\ell)$.

Let Φ be a Frobenius element in the Weil group W_F. By [**6**] 6.4, 6.5, the surjection of $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell)$ onto $\widehat{\mathbf{S}}(\overline{\mathbb{Q}}_\ell)$ induces a bijection between

– N-conjugacy classes in $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell) \rtimes \Phi$, and
– W-conjugacy classes in $\widehat{\mathbf{S}}(\overline{\mathbb{Q}}_\ell)$,

where N is the inverse image of W in the normalizer of $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell)$ in $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$, and the embedding of of $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell)$ in $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$ induces a bijection between

– N-conjugacy classes in $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell) \rtimes \Phi$, and
– $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$-conjugacy classes of semi-simple elements in $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell) \rtimes \Phi$.

The W-orbit of ω thus determines the W-conjugacy class of a point $s \in \widehat{\mathbf{S}}(\overline{\mathbb{Z}}_\ell)$, then the $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$-conjugacy class of a semi-simple element $t \rtimes \Phi \in \widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell) \rtimes \Phi$. We are going to prove that $t \rtimes \Phi$ may be chosen in $\widehat{\mathbf{T}}(\overline{\mathbb{Z}}_\ell) \rtimes \Phi \subseteq \widehat{\mathbf{G}}(\overline{\mathbb{Z}}_\ell) \rtimes \Phi$. Let us fix a uniformizer ϖ of F.

When $\mathbf{G}$ is split, we have $\mathbf{T} = \mathbf{S}$, thus $t = s$ is in $\widehat{\mathbf{T}}(\overline{\mathbb{Z}}_\ell) \subseteq \widehat{\mathbf{G}}(\overline{\mathbb{Z}}_\ell)$. Explicitly, if we identify T with $(F^\times)^m$ for some integer $m \geqslant 1$, then ω identifies with the tensor product of m unramified characters $\omega_1, \dots, \omega_m$ of $F^\times$ and $t \rtimes \Phi$ is $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$-conjugate to

- $\mathrm{diag}(\omega_1(\varpi), \dots, \omega_m(\varpi), 1, \omega_m(\varpi)^{-1}, \dots, \omega_1(\varpi)^{-1}) \in \mathrm{GL}_{2n+1}(\overline{\mathbb{Z}}_\ell)$ if $\mathbf{G} = \mathrm{Sp}_{2n}$,
- $\mathrm{diag}(\omega_1(\varpi), \dots, \omega_m(\varpi), \omega_m(\varpi)^{-1}, \dots, \omega_1(\varpi)^{-1}) \in \mathrm{GL}_{2n}(\overline{\mathbb{Z}}_\ell)$ if $\mathbf{G} = \mathrm{SO}_{2n+1}$ or $\mathbf{G} = \mathrm{SO}_{2n}^1$,

with $m = n$ in all cases.

Now assume that $\mathbf{G}$ is non-split, thus either $\mathbf{G} = \mathrm{SO}_{2n}^\alpha$ or $\mathbf{G} = \mathrm{U}_n^\alpha$, with $\alpha \neq 1$.

• In the even orthogonal case, we have $S \simeq (F^\times)^m$ and $T \simeq S \times \mathrm{SO}_2^\alpha(F)$ with $m = n-1$ (see [**7**] §23.4), thus $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell^{m+1}$ surjects onto $\widehat{\mathbf{S}}(\overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell^m$ through

$$(t_1, t_2, \ldots, t_{m+1}) \mapsto (t_1, t_2, \ldots, t_m)$$

and $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{W}_F$ embeds into $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{W}_F$ through

$$(t_1, t_2, \ldots, t_n) \rtimes w \mapsto \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \rtimes w$$

thus the image of $t \rtimes \Phi$ in $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell) \rtimes \Phi$ is $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$-conjugate to

$$\mathrm{diag}(\omega_1(\varpi), \ldots, \omega_m(\varpi), 1, 1, \omega_m(\varpi)^{-1}, \ldots, \omega_1(\varpi)^{-1}) \rtimes \Phi \in \mathrm{GL}_{2n}(\overline{\mathbb{Z}}_\ell) \rtimes \Phi.$$

• In the unitary case, we have $S \simeq (F^\times)^m$ and $T \simeq (E^\times)^m$ where E is the quadratic extension of F generated by a square root of α (note that it is unramified since $\mathbf{G}$ is assumed to be unramified), and $m = \lfloor n/2 \rfloor$ is the Witt index of $\mathbf{G}$ (see [**7**] §23.9), thus $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell^{2m}$ surjects onto $\widehat{\mathbf{S}}(\overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell^m$ through

$$(t_1, t_2, \ldots, t_{2m}) \mapsto (t_1 t_{2m}, t_2 t_{2m-1}, \ldots, t_m t_{m+1})$$

and $\widehat{\mathbf{T}}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{W}_F$ embeds into $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell) \rtimes \mathrm{W}_F$ through

$$(t_1, t_2, \ldots, t_{2m}) \rtimes w \mapsto \begin{cases} \mathrm{diag}(t_1, \ldots, t_m, t_{m+1}^{-1}, \ldots, t_{2m}^{-1}) \rtimes w & \text{if } n = 2m \text{ is even,} \\ \mathrm{diag}(t_1, \ldots, t_m, 1, t_{m+1}^{-1}, \ldots, t_{2m}^{-1}) \rtimes w & \text{if } n = 2m+1 \text{ is odd,} \end{cases}$$

thus the image of $t \rtimes \Phi$ in $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell) \rtimes \Phi$ is $\widehat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$-conjugate to

$$\begin{cases} \mathrm{diag}(\omega_1(\varpi)^{1/2}, \ldots, \omega_m(\varpi)^{1/2}, \omega_m(\varpi)^{-1/2}, \ldots, \omega_1(\varpi)^{-1/2}) \rtimes \Phi & \text{if } n = 2m \text{ is even,} \\ \mathrm{diag}(\omega_1(\varpi)^{1/2}, \ldots, \omega_m(\varpi)^{1/2}, 1, \omega_m(\varpi)^{-1/2}, \ldots, \omega_1(\varpi)^{-1/2}) \rtimes \Phi & \text{if } n = 2m+1 \text{ is odd,} \end{cases}$$

which is in $\mathrm{GL}_n(\overline{\mathbb{Z}}_\ell) \rtimes \Phi$ in both cases.

We now define an unramified local Langlands parameter $\varphi : \mathrm{WD}_F \to \widehat{\mathbf{G}}(\overline{\mathbb{Z}}_\ell) \rtimes \mathrm{W}_F$ by

– $\varphi(\Phi) = t \rtimes \Phi$, and
– φ is trivial on the inertia subgroup I_F of W_F and on $\mathrm{SL}_2(\mathbb{C})$.

It is uniquely determined by the K-unramified representation π, or equivalently by its Satake parameter χ. Composing with Std (or just restricting to WD_E in the unitary case), we get an unramified Langlands parameter $\phi \in \Phi(\mathrm{GL}_N, E)$, where $E = F$ in the symplectic and orthogonal cases and E is the quadratic extension of F generated by a square root of α in the unitary case. This ϕ uniquely determines an unramified $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_N(E)$, denoted $\mathbf{t}_\ell(\pi)$.

In the symplectic and special orthogonal cases, $\mathbf{t}_\ell(\pi)$ is the unique unramified irreducible component of

$$\begin{cases} \omega_1 \times \cdots \times \omega_n \times \omega_n^{-1} \times \cdots \times \omega_1^{-1} & \text{if } \mathbf{G} = \mathrm{SO}_{2n+1} \text{ or } \mathbf{G} = \mathrm{SO}_{2n}^1, \\ \omega_1 \times \cdots \times \omega_n \times 1 \times \omega_n^{-1} \times \cdots \times \omega_1^{-1} & \text{if } \mathbf{G} = \mathrm{Sp}_{2n}, \\ \omega_1 \times \cdots \times \omega_{n-1} \times 1 \times 1 \times \omega_{n-1}^{-1} \times \cdots \times \omega_1^{-1} & \text{if } \mathbf{G} = \mathrm{SO}_{2n}^\alpha \text{ with } \alpha \neq 1, \end{cases}$$

where $\times$ denotes the parabolic induction to $\mathrm{GL}_N(F)$ normalized with respect to $q^{1/2}$.

In the unitary case, the Weil group W_E is generated by Φ^2 and I_F (since E/F is unramified), thus ϕ is uniquely determined by $\phi(\Phi^2) = tt^* \rtimes \Phi^2$, with

$$tt^* = \begin{cases} \mathrm{diag}(\omega_1(\varpi), \dots, \omega_m(\varpi), \omega_m(\varpi)^{-1}, \dots, \omega_1(\varpi)^{-1}) & \text{if } n = 2m \text{ is even,} \\ \mathrm{diag}(\omega_1(\varpi), \dots, \omega_m(\varpi), 1, \omega_m(\varpi)^{-1}, \dots, \omega_1(\varpi)^{-1}) & \text{if } n = 2m+1 \text{ is odd,} \end{cases}$$

which gives $\mathbf{t}_\ell(\pi)$ explicitly. Namely, $\mathbf{t}_\ell(\pi)$ is the unique unramified irreducible component of

$$\begin{cases} \omega_1 \times \dots \times \omega_m \times \omega_m^{-1} \times \dots \times \omega_1^{-1} & \text{if } \mathbf{G} = \mathrm{U}^\alpha_{2m}, \\ \omega_1 \times \dots \times \omega_m \times 1 \times \omega_m^{-1} \times \dots \times \omega_1^{-1} & \text{if } \mathbf{G} = \mathrm{U}^\alpha_{2m+1}, \end{cases}$$

where $\times$ denotes the parabolic induction to $\mathrm{GL}_n(E)$ normalized with respect to $(q^{1/2})^2 = q$ (as E is quadratic and unramified over F). (See also [**37**].) We have:

Proposition 6.3. — *Let G be the group of rational points of an unramified special orthogonal, unitary or symplectic F-group among the groups of Paragraph 5.1. Let K be a hyperspecial maximal compact subgroup of G and let π_1, π_2 be K-unramified irreducible $\overline{\mathbb{Q}}_\ell$-representations of G whose Satake parameters χ_1, χ_2 define congruent $\overline{\mathbb{Z}}_\ell$-characters of $\overline{\mathbb{Z}}_\ell[K\backslash G/K]$. Then*

(1) *the representations $\mathbf{t}_\ell(\pi_1)$ and $\mathbf{t}_\ell(\pi_2)$ of $\mathrm{GL}_N(E)$ are integral,*
(2) *their Langlands parameters are integral and congruent.*

Remark 6.4. — Note that the reductions mod ℓ of $\mathbf{t}_\ell(\pi_1)$ and $\mathbf{t}_\ell(\pi_2)$ may not have any irreducible component in common. However, if τ_i denotes the unique unramified irreducible component of $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_i))$ for $i = 1, 2$, then τ_1 and τ_2 have the same cuspidal support.

For instance, assume that $\mathbf{G} = \mathrm{SO}_5$ and ℓ divides $q^2 - 1$. Let π_1 (resp. π_2) be the unramified representation of $G = \mathrm{SO}_5(F)$ with respect to some hyperspecial maximal compact group, with cuspidal support the W-conjugacy class of $|\cdot|^{-1/2} \otimes |\cdot|^{1/2}$ (resp. $|\cdot|^{3/2} \otimes |\cdot|^{1/2}$), where $|\cdot|$ is the absolute value of $F^\times$. By assumption, these cuspidal supports are congruent (for $|\cdot|^{3/2}$ and $|\cdot|^{-1/2}$ have the same reduction mod ℓ). Then $\mathbf{t}_\ell(\pi_1)$ is the unique unramified irreducible component of

$$|\cdot|^{-1/2} \times |\cdot|^{1/2} \times |\cdot|^{-1/2} \times |\cdot|^{1/2}$$

that is, $\mathbf{t}_\ell(\pi_1) = 1_2 \times 1_2$ (where 1_2 is the trivial character of $\mathrm{GL}_2(F)$). Similarly, $\mathbf{t}_\ell(\pi_2)$ is equal to $|\det| \times |\det|$ (where det is the determinant of $\mathrm{GL}_2(F)$). Now assume further that $\ell \neq 2$ and ℓ divides $q + 1$, thus ℓ does not divide $q - 1$. Then $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_1))$ and $\mathbf{r}_\ell(\mathbf{t}_\ell(\pi_2))$ are both irreducible and twists of each other by the non-trivial character $|\cdot|$, thus non-isomorphic.

Remark 6.5. — Let ι be an isomorphism of fields $\mathbb{C} \to \overline{\mathbb{Q}}_\ell$ taking the positive square root of q in $\mathbb{R}$ to the square root $q^{1/2} \in \overline{\mathbb{Q}}_\ell$ of Paragraph 6.1. According to Arthur [**2**] 6.1 (see p. 304) and Mok [**46**] 7.1 (see also Labesse [**35**] p. 38-39), which describe the local transfer map $\mathbf{t}$ of Definition 5.1 for unramified representations of unramified groups, we have

$$\mathbf{t}(\pi) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell = \mathbf{t}_\ell(\pi \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell)$$

for any K-unramified complex representation π of G, where tensor products are taken with respect to ι. Proposition 6.7 and Remark 6.8 will describe the dependency of $\mathbf{t}_\ell$ in $q^{1/2}$.

6.4.

In this paragraph, G is an unramified classical group and K is a hyperspecial maximal compact subgroup of G as in Paragraph 6.3.

Unlike Paragraph 6.3 however, we will consider *complex* representations rather than $\overline{\mathbb{Q}}_\ell$-representations. We examine the dependency of the unramified transfer from G to $\mathrm{GL}_N(E)$ with respect to the choice of a square root of q, and to the action of $\mathrm{Aut}(\mathbb{C})$. This will be useful in Paragraph 9.2.

Let π be a K-unramified irreducible complex representation of G. Associated with π, there is the W-conjugacy class of an unramified character ω of T, such that π is the unique K-unramified irreducible component of the normalized parabolic induction of ω to G along B. Writing Ind_B^G for unnormalized parabolic induction from T to G along B and $\boldsymbol{i}_B^G$ for normalized parabolic induction, we have

$$\boldsymbol{i}_B^G(\omega) = \mathrm{Ind}_B^G(\delta^{1/2}\omega).$$

Let us write $T \simeq E^{\times m} \times T_0$, where

- T_0 is trivial if G is split or $G \simeq \mathrm{U}_{2m}^{\alpha}(F)$ with $\alpha \neq 1$,
- $T_0 = \mathrm{SO}_2^{\alpha}(F)$ if $G \simeq \mathrm{SO}_{2n}^{\alpha}(F)$ with $\alpha \neq 1$,
- $T_0 = \mathrm{U}_1^{\alpha}(F)$ if $G \simeq \mathrm{U}_{2m+1}^{\alpha}(F)$ with $\alpha \neq 1$.

The character ω can thus be written $\omega_1 \otimes \cdots \otimes \omega_m \otimes 1$, where each ω_i is an unramified character of $E^\times$ and 1 is the trivial character of T_0. By [**45**] IV.4 p. 69, the modulus character of the parabolic subgroup $P \supseteq B$ with Levi component $\mathrm{GL}_m(E) \times T_0$ is equal to

$$\nu_m^{d-m-e} \otimes 1$$

where ν_m is the unramified character "absolute value of the determinant" of $\mathrm{GL}_m(E)$ and

- $d = 2n$ and $e = -1$ if $G = \mathrm{Sp}_{2n}(F)$,
- $d = 2n+1$ and $e = 1$ if $G = \mathrm{SO}_{2n+1}(F)$,
- $d = 2n$ and $e = 1$ if $G = \mathrm{SO}_{2n}^{\alpha}(F)$,
- $d = n$ and $e = 0$ if $G = \mathrm{U}_n^{\alpha}(F)$ with $\alpha \neq 1$.

By using the transivity property of parabolic induction, we deduce that

$$\begin{aligned} \delta^{1/2} &= |\cdot|_E^{(d-m-e)/2+(m-1)/2} \otimes \cdots \otimes |\cdot|_E^{(d-m-e)/2-(m-1)/2} \otimes 1 \\ &= |\cdot|_E^{(d-e-1)/2} \otimes \cdots \otimes |\cdot|_E^{(d-e-1)/2-m+1} \otimes 1 \end{aligned}$$

(where $|\cdot|_E$ is the absolute value of E). Replacing $\sqrt{q}$ by the opposite square root changes $|\cdot|_E$ to $\eta|\cdot|_E$, where η is the unramified character of $E^\times$ of order 2. It thus changes ω_i to $\omega_i\eta^{(d-e-1)f}$, where f is the residual degree of E over F (which is 1 or 2 depending whether $E = F$ or not).

Similarly, replacing $\sqrt{q}$ by its opposite square root has the effect of twisting normalized parabolic induction from $E^{\times N}$ to $\mathrm{GL}_N(E)$ (along the Borel subgroup made of upper triangular matrices) by $\eta^{(1-N)f}$.

Consequently, considering the explicit formulas of Paragraph 6.3, replacing $\sqrt{q}$ by the opposite square root has the effect of twisting $\mathbf{t}(\pi)$ by the character $\eta^{(d-e-N)f}$. We have

- $d - e - N = 2n + 1 - (2n+1) = 0$ if $G = \mathrm{Sp}_{2n}(F)$,

- $d - e - N = (2n+1) - 1 - 2n = 0$ if $G = \mathrm{SO}_{2n+1}(F)$,
- $d - e - N = 2n - 1 - 2n = -1$ if $G = \mathrm{SO}_{2n}^{\alpha}(F)$,

and $f = 2$ if G is unitary. The integer $(d-e-N)f$ is thus even, except if G is even orthogonal.

Example 6.6. — If $G = \mathrm{SO}_2^1(F) \simeq F^\times$, the transfer of any unramified character ω of $F^\times$ is the unique unramified irreducible component of $\omega \times \omega^{-1}$. If G is the compact group $\mathrm{SO}_2^{\alpha}(F)$ with $\alpha \neq 1$, the transfer of the trivial character of G is 1×1. In both cases, the transfer depends on the choice of a square root of q.

Now consider an automorphism $\gamma \in \mathrm{Aut}(\mathbb{C})$. Given a representation π of a group H on a complex vector space V, we write π^γ for the representation of H on $V \otimes_{\mathbb{C}} \mathbb{C}_\gamma$, where $\mathbb{C}_\gamma$ is the field $\mathbb{C}$ considered as a $\mathbb{C}$-algebra *via* γ. Consider the map $\pi \mapsto \mathbf{t}(\pi^{\gamma^{-1}})^\gamma$ from K-unramified irreducible representations of G to irreducible representations of $\mathrm{GL}_N(E)$. It is the unramified local transfer map from G to $\mathrm{GL}_N(E)$ with respect to the square root $\gamma(\sqrt{q})$. We thus have:

Proposition 6.7. — *Let G and K be as above, and let π be a K-unramified irreducible representation of G. Let $\gamma \in \mathrm{Aut}(\mathbb{C})$.*

(1) *If G is not even orthogonal, then $\mathbf{t}(\pi^\gamma) = \mathbf{t}(\pi)^\gamma$.*
(2) *If G is even orthogonal, then $\mathbf{t}(\pi^\gamma) = \mathbf{t}(\pi)^\gamma \cdot \varepsilon_\gamma$, where ε_γ is the unramified character*

$$x \mapsto \left(\frac{\gamma(\sqrt{q})}{\sqrt{q}}\right)^{\mathrm{val}_F(x)}$$

of $F^\times$.

Remark 6.8. — We now go back to $\overline{\mathbb{Q}}_\ell$-representations. We deduce that the map

$$\pi \mapsto \mathbf{t}_\ell(\pi) \tag{6.3}$$

from (isomorphism classes of) unramified $\overline{\mathbb{Q}}_\ell$-representations of G to those of $\mathrm{GL}_N(E)$ is insensitive to the choice of a square root of q in $\overline{\mathbb{Q}}_\ell$, except when G is even orthogonal, in which case changing this square root to its opposite has the effect of twisting (6.3) by η.

7. Representations of local Galois and Weil groups

In this section, F is a p-adic field. We write Γ for the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and W for the associated Weil group, considered as a subgroup of Γ. It is endowed with a smooth character $w \mapsto |w|$ with kernel I, the inertia subgroup of W, taking any geometric Frobenius element to q^{-1}, where q is the cardinality of the residue field of F.

All representations of Γ and W considered in this section will be finite-dimensional.

Let ℓ be a prime number different from p.

7.1.

For this paragraph, the reader may refer to [**10**] Chapter 7 and [**59**] 4.2.

If σ is a smooth representation of Γ, then its restriction $\sigma|_{\mathrm{W}}$ to W is smooth.

Restriction from Γ to W induces an injection from isomorphism classes of irreducible smooth representations of Γ to isomorphism classes of irreducible smooth representations of W. The image is made of those representations of W whose determinant has finite order (see [**10**] 28.6 Proposition).

If ρ is a smooth ℓ-adic representation (that is, $\overline{\mathbb{Q}}_\ell$-representation) of W on a vector space V, and if $\Phi \in \mathrm{W}$ is a Frobenius element, the following assertions are equivalent:

(1) ρ is semi-simple,

(2) $\rho(\Phi)$ is a semi-simple element in GL(V),

(3) $\rho(w)$ is a semi-simple element in GL(V) for any $w \in \mathrm{W}$

(see [**10**] 28.7 Proposition).

If σ is a continuous ℓ-adic representation of Γ, its restriction to W is a continuous ℓ-adic representation of W, which is irreducible if and only if σ is irreducible.

Fix a continuous surjective group homomorphism

$$t : I \to \mathbb{Z}_\ell. \tag{7.1}$$

For the following proposition, see [**10**] 32.5 Theorem and [**54**] Appendix.

Proposition 7.1. — *Let σ be a continuous ℓ-adic representation of* W *on a $\overline{\mathbb{Q}}_\ell$-vector space* V.

(1) *There is a unique nilpotent endomorphism $N \in \mathrm{End(V)}$ such that there is an open subgroup U of the inertia subgroup I such that*

$$\sigma(x) = e^{t(x)N}, \quad x \in U.$$

(2) *We have $\sigma(w)N\sigma(w)^{-1} = |w| \cdot N$ for all $w \in \mathrm{W}$.*

Note that $N = 0$ if and only if σ is smooth.

The subspaces $\mathrm{Ker}(N^i)$, $i \geqslant 0$ of V are stable by σ. Thus, if σ is irreducible, then $N = 0$ and σ is smooth. More generally, a semi-simple representation of Γ is smooth, and its restriction to W is smooth semi-simple (see also [**59**] 4.2.3).

Fix a Frobenius element $\Phi \in \mathrm{W}$. Associated with σ, there is a smooth ℓ-adic representation ρ of W defined by

$$\rho(\Phi^a x) = \sigma(\Phi^a x)e^{-t(x)N}, \quad a \in \mathbb{Z}, \quad x \in I.$$

The pair (ρ, N) is called the *Deligne representation* of W associated with σ. Up to isomorphism, it does not depend on the choices of t and Φ (see [**10**] 32.6 Theorem).

The element $\rho(\Phi) = \sigma(\Phi)$ decomposes uniquely in GL(V) as $su = us$, with s semi-simple and u unipotent. Define a smooth ℓ-adic representation ρ^* of W by

$$\rho^*(\Phi^a x) = s^a \rho(x), \quad a \in \mathbb{Z}, \quad x \in I.$$

This defines a Deligne representation (ρ^*, N), called the *Frobenius-semi-simplification* of (ρ, N). By Paragraph 7.1, the representation ρ^* is a semi-simple smooth representation of W.

7.2.

In this paragraph, if κ is a continuous ℓ-adic representation of W or Γ, we will write κ^{ss} for its semisimplification.

Lemma 7.2. — *If ρ is a smooth ℓ-adic representation of* W *such that ρ^{ss} is integral, then ρ is integral.*

Proof. — We prove it by induction on the dimension n of ρ. If ρ is irreducible, there is nothing to prove. Otherwise, let τ be an irreducible subrepresentation of ρ, of dimension $k \geqslant 1$, and let λ be the quotient of ρ by τ, which is of dimension $l = n - k$. Since $\rho^{\mathrm{ss}} = \tau \oplus \lambda^{\mathrm{ss}}$, we may apply the inductive hypothesis to λ, from which we deduce that λ is integral. We therefore fix a basis of the vector space of ρ such that

$$\rho(w) = \begin{pmatrix} \tau(w) & \alpha(w) \\ 0 & \lambda(w) \end{pmatrix} \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell), \quad \tau(w) \in \mathrm{GL}_k(\overline{\mathbb{Z}}_\ell), \quad \lambda(w) \in \mathrm{GL}_l(\overline{\mathbb{Z}}_\ell), \quad w \in \mathrm{W},$$

and $\alpha : \mathrm{W} \to \mathbf{M}_{k,l}(\overline{\mathbb{Q}}_\ell)$ satisfies $\alpha(xy) = \tau(x)\alpha(y) + \alpha(x)\lambda(y)$ for all $x, y \in \mathrm{W}$.

Since ρ is smooth, we may consider it as a representation of the discrete group W/U for some open subgroup U of W. Since this quotient is a finitely generated group, we may consider ρ as a representation of the free group F with r generators $\mathsf{f}_1, \dots, \mathsf{f}_r$ for some $r \geqslant 1$. Assume α is not identically zero, and let $-v$ denote the minimum of the ℓ-adic valuations of all the entries of all the $\alpha(\mathsf{f}_i)$. Conjugating ρ by $\mathrm{diag}(\ell^v \cdot \mathrm{id}_t, \mathrm{id}_l)$, we may and will assume that $v = 0$.

We are going to prove that α takes values in $\mathbf{M}_{t,l}(\overline{\mathbb{Z}}_\ell)$. We prove it by induction on the length of the words in F. Given $x \in \mathsf{F}$, write it yf with $f = \mathsf{f}_i$ for some $i \in \{1, \dots, r\}$ and the length of y is smaller than that of x. Then

$$\alpha(x) = \tau(y)\alpha(f) + \alpha(y)\lambda(f) \in \mathbf{M}_{k,l}(\overline{\mathbb{Z}}_\ell)$$

thanks to the inductive hypothesis. □

Lemma 7.3. — *Let σ be a continuous ℓ-adic representation of Γ, with associated Deligne representation (ρ, N). The restriction of σ^{ss} to* W *is equal to ρ^{ss}.*

Proof. — Note that semi-simplification and restriction from Γ to W commute, that is

$$\sigma^{\mathrm{ss}}|_{\mathrm{W}} = (\sigma|_{\mathrm{W}})^{\mathrm{ss}}.$$

If N is zero, then σ is smooth and ρ is the restriction of σ to W, thus σ^{ss} is smooth semi-simple, and its restriction to W is smooth semi-simple as well. Otherwise, if $n = \dim(\sigma)$, there is a basis of $\overline{\mathbb{Q}}_\ell^n$ such that

$$\sigma(g) = \begin{pmatrix} \alpha(g) & \gamma(g) \\ 0 & \beta(g) \end{pmatrix} \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell), \quad g \in \Gamma, \quad \text{and} \quad N = \begin{pmatrix} 0 & C \\ 0 & M \end{pmatrix},$$

where

- α is a smooth ℓ-adic representation of Γ of dimension $k = \dim(\mathrm{Ker}\ N)$,
- β is a continuous ℓ-adic representation of Γ of dimension $l = n - k$,
- γ is a continuous map from Γ to $\mathbf{M}_{k,l}(\overline{\mathbb{Q}}_\ell)$,
- M is nilpotent in $\mathbf{M}_l(\overline{\mathbb{Q}}_\ell)$ and C is a matrix in $\mathbf{M}_{k,l}(\overline{\mathbb{Q}}_\ell)$.

We have

$$e^{t(x)N} = \begin{pmatrix} \mathrm{id} & \varepsilon(x) \\ 0 & e^{t(x)M} \end{pmatrix} \quad \text{with} \quad \varepsilon(x) = \sum_{i\geqslant 1} \frac{t(x)^i}{i!} CM^{i-1}, \quad x \in I.$$

Writing (ρ_1, N_1) for the Deligne representation associated with β, we get $N_1 = M$ and

$$\rho(w) = \begin{pmatrix} \alpha(w) & \delta(w) \\ 0 & \rho_1(w) \end{pmatrix} \in \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell), \quad w \in \mathrm{W},$$

for some smooth map δ from W to $\mathbf{M}_{k,l}(\overline{\mathbb{Q}}_\ell)$ which can be explicitly described by

$$\delta(w) = (\gamma(w) - \alpha(w)\varepsilon(w))e^{-t(x)M}, \quad w = \Phi^a x, \quad a \in \mathbb{Z}, \quad x \in I.$$

By the inductive hypothesis, we get

$$\sigma^{\mathrm{ss}}|_{\mathrm{W}} = (\alpha|_{\mathrm{W}}) \oplus (\beta^{\mathrm{ss}}|_{\mathrm{W}}) = (\alpha|_{\mathrm{W}}) \oplus \rho_1^{\mathrm{ss}} = \rho^{\mathrm{ss}}.$$

This proves the lemma. □

Corollary 7.4. — *Let σ be a continuous ℓ-adic representation of Γ, with associated Deligne representation (ρ, N). Then ρ is (smooth) integral and $\mathbf{r}_\ell(\rho) = \mathbf{r}_\ell(\sigma)|_{\mathrm{W}}$.*

Proof. — Since σ is integral (for Γ is compact), $\sigma^{\mathrm{ss}}|_{\mathrm{W}}$ is integral. We deduce from Lemma 7.3 that ρ^{ss} is integral, then from Lemma 7.2 that ρ is integral. Now write

$$\mathbf{r}_\ell(\rho) = \mathbf{r}_\ell(\rho^{\mathrm{ss}}) = \mathbf{r}_\ell(\sigma^{\mathrm{ss}}|_{\mathrm{W}}) = \mathbf{r}_\ell(\sigma^{\mathrm{ss}})|_{\mathrm{W}} = \mathbf{r}_\ell(\sigma)|_{\mathrm{W}}.$$

This proves the corollary. □

8. Galois representations associated with automorphic representations

Recall that we have fixed an isomorphism of fields $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. Fix a positive integer N.

Let k be a totally real number field, and l be either k or a quadratic totally imaginary extension of k in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For any place v of l, let l_v denote the completion of l at v.

For any finite place v, fix a decomposition subgroup Γ_v of $\mathrm{Gal}(\overline{\mathbb{Q}}/l)$ at v and write W_v for the associated Weil group. For any finite place v not dividing ℓ, write

– $\mathrm{WD}(\sigma)$ for the Deligne representation of W_v associated with a continuous ℓ-adic representation σ of Γ_v and $\mathrm{WD}^*(\sigma)$ for its Frobenius-semi-simplification,

– rec_v for the local Langlands correspondence ([**22**] Theorem A) between irreducible smooth complex representations of $\mathrm{GL}_N(l_v)$ and N-dimensional Frobenius-semi-simple complex Deligne representations of W_v.

8.1.

A cuspidal irreducible automorphic representation Π of $\mathrm{GL}_N(\mathbb{A}_l)$ is said to be

– *polarized* if its contragredient $\Pi^\vee$ is isomorphic to Π^c, where c is the generator of $\mathrm{Gal}(l/k)$ (thus $\Pi^c = \Pi$ when $l = k$),

– *algebraic regular* if the Harish-Chandra module Π_∞ associated with Π has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from l to $\mathbb{Q}$ of GL_N.

Recall the following result of Barnet-Lamb–Geraghty–Harris–Taylor ([**5**] Theorems 1.1, 1.2).

Theorem 8.1. — *Let Π be an algebraic regular, polarized, cuspidal irreducible automorphic representation of $\mathrm{GL}_N(\mathbb{A}_l)$. There is a continuous semi-simple ℓ-adic representation*

$$\Sigma : \mathrm{Gal}(\overline{\mathbb{Q}}/l) \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$

such that, for any finite place v of l not dividing ℓ, we have

$$\mathrm{WD}^*(\Sigma|_{\Gamma_v}) \simeq \mathrm{rec}_v(\Pi_v \otimes |\det|_v^{(1-N)/2}) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell.$$

Note that the representation Σ depends on the choice of ι.

8.2.

The main result of this section is the following. Let $\mathfrak{m}$ denote the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

Theorem 8.2. — *Let Π_1 and Π_2 be algebraic regular, polarized, cuspidal irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_l)$. Suppose that there is a finite set S of places of l, containing all infinite places, such that for all $v \notin \mathrm{S}$:*

(1) *the local components $\Pi_{1,v}$ and $\Pi_{2,v}$ are unramified,*

(2) *the characteristic polynomials of the conjugacy classes of semisimple elements in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ associated with $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ have coefficients in $\overline{\mathbb{Z}}_\ell$ and are congruent mod $\mathfrak{m}$.*

Then, for any finite place v of l not dividing ℓ, the representations $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral, their reductions mod $\mathfrak{m}$ share a common generic irreducible component, and such a generic component is unique.

Proof. — Applying Theorem 8.1 to Π_1 and Π_2, we get continuous ℓ-adic representations

$$\Sigma_i : \mathrm{Gal}(\overline{\mathbb{Q}}/l) \to \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell), \quad i = 1, 2,$$

such that, for any finite place v of l not dividing ℓ, we have

$$\mathrm{WD}^*(\Sigma_{i,v}) \simeq \mathrm{rec}_v(\Pi_{i,v} \otimes |\det|_v^{(1-N)/2}) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$$

where $\Sigma_{i,v}$ denotes the restriction of Σ_i to Γ_v and the tensor product over $\mathbb{C}$ is taken with respect to ι. For all $v \notin \mathrm{S}$, the ℓ-adic representation $\Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is unramified, generic and integral, thus

$$\mathrm{rec}_v(\Pi_{i,v} \otimes |\det|_v^{(1-N)/2}) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell \simeq (\phi_{i,v}, 0)$$

where $\phi_{i,v}$ is an integral semi-simple ℓ-adic representation of W_v trivial on I_v. It is thus entirely determined by the semi-simple matrix

$$\phi_{i,v}(\Phi_v) \in \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell)$$

where Φ_v is a Frobenius element in W_v. By assumption, its characteristic polynomial has coefficients in $\overline{\mathbb{Z}}_\ell$, that is, its eigenvalues are in $\overline{\mathbb{Z}}_\ell^\times$. That the nilpotent operator is 0 implies that $\Sigma_{i,v}$ is smooth, thus

$$\phi_{i,v} = (\Sigma_{i,v}|_{\mathrm{W}_v})^{\mathrm{ss}}.$$

Thus $\Sigma_{i,v}$ is trivial on the inertia subgroup I_v, that is, $\Sigma_{i,v}$ is unramified.

Given $v \notin \mathrm{S}$ and $i \in \{1,2\}$, let $P_{i,v}(\mathrm{T})$ be the characteristic polynomial of $\Sigma_{i,v}(\Phi_v) \otimes \overline{\mathbb{F}}_\ell$, that is, the characteristic polynomial of $\phi_{i,v}(\Phi_v) \otimes \overline{\mathbb{F}}_\ell$. By assumption, $P_{1,v}(\mathrm{T}) = P_{2,v}(\mathrm{T})$ at all $v \notin \mathrm{S}$. Applying Deligne-Serre [**18**] Lemma 3.2 to the semi-simple $\overline{\mathbb{F}}_\ell$-representations $(\Sigma_1 \otimes \overline{\mathbb{F}}_\ell)^{\mathrm{ss}}$ and $(\Sigma_2 \otimes \overline{\mathbb{F}}_\ell)^{\mathrm{ss}}$, which at $v \notin \mathrm{S}$ give $\phi_{1,v} \otimes \overline{\mathbb{F}}_\ell$ and $\phi_{2,v} \otimes \overline{\mathbb{F}}_\ell$ respectively, we deduce that $(\Sigma_1 \otimes \overline{\mathbb{F}}_\ell)^{\mathrm{ss}}$ and $(\Sigma_2 \otimes \overline{\mathbb{F}}_\ell)^{\mathrm{ss}}$ are isomorphic. In particular, we deduce that

$$(\Sigma_1 \otimes \overline{\mathbb{F}}_\ell)^{\mathrm{ss}}|_{\Gamma_w} \simeq (\Sigma_2 \otimes \overline{\mathbb{F}}_\ell)^{\mathrm{ss}}|_{\Gamma_w}$$

thus the continuous ℓ-adic representations $\Sigma_{1,w}$ and $\Sigma_{2,w}$ of Γ_w are congruent mod ℓ.

Now write $\mathrm{WD}^*(\Sigma_{i,w}) = (\rho_i, N_i)$ for $i = 1,2$. Thanks to Corollary 7.4, we know that ρ_1 and ρ_2 are integral and have same reduction mod ℓ. By [**64**] Theorem 1.6, we deduce that

$$\mu_1 = (\Pi_{1,w} \otimes |\det|_w^{(1-N)/2}) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad \mu_2 = (\Pi_{2,w} \otimes |\det|_w^{(1-N)/2}) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell,$$

are integral and *have the same mod ℓ supercuspidal support*, that is, the supercuspidal support of any irreducible component v of $\mathbf{r}_\ell(\mu_i)$ is independent of i (and of the choice of v).

Since μ_i is generic (as $\Pi_{i,w}$ is a local component of a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_l)$), the $\overline{\mathbb{F}}_\ell$-representation $\mathbf{r}_\ell(\mu_i)$ contains a generic irreducible component δ_i. It occurs in $\mathbf{r}_\ell(\mu_i)$ with multiplicity 1, and any generic irreducible representation occurring in $\mathbf{r}_\ell(\mu_i)$ is isomorphic to δ_i. Since δ_i only depends on the mod ℓ supercuspidal support of μ_i ([**64**] III.5.10), we deduce that δ_1 and δ_2 are isomorphic. ☐

Remark 8.3. — We expect Theorem 8.2 to hold without assuming that Π_1, Π_2 are polarized.

9. Proof of the main theorem

9.1.

We prove our main theorem 1.1.

Let p be a prime number different from 2, let F be a p-adic field and G be a quasi-split special orthogonal, unitary or symplectic group over F. We thus have

- either $G = \mathrm{SO}(Q)$ for some non-degenerate quadratic form Q over F,
- or $G = \mathrm{U}(H)$ for some non-degenerate E/F-Hermitian form H,
- or $G = \mathrm{Sp}(A)$ for some non-degenerate symplectic form A over F.

As usual, we write $E = F$ in the symplectic and orthogonal cases.

In this paragraph and the next one, we assume that the group G is not the split special orthogonal group $\mathrm{SO}_2(F) \simeq F^\times$. The case of split $\mathrm{SO}_2(F)$ will be treated in Paragraph 9.3.

Let π_1, π_2 be integral cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representations of G such that

$$\mathbf{r}_\ell(\pi_1) \leqslant \mathbf{r}_\ell(\pi_2).$$

First, let k, w, $\mathbf{G}$ be as in Theorem 2.1. More precisely, we have

- either $\mathbf{G} = \mathrm{SO}(q)$ for a quadratic form q as in Theorem 2.8 if G is special orthogonal,
- or $\mathbf{G} = \mathrm{U}(h)$ for an l/k-Hermitian form h as in Theorem 2.11 if G is unitary,
- or $\mathbf{G}$ is as in Paragraph 2.8 if G is symplectic (see also Paragraph 5.6).

In particular, we have $k_w = F$ and $l_w = E$, and the group $\mathbf{G}(F)$ naturally identifies with G. As usual, we write $l = k$ in the symplectic and orthogonal cases.

Let $\mathbf{G}^*$ be the quasi-split inner form of $\mathbf{G}$ over k, and write $N = N(\mathbf{G}^*)$. We thus have:

- either $\mathbf{G}^* = \mathrm{SO}(q^*)$ where q^* is a quadratic form over k as in (5.2) or (5.3),
- or $\mathbf{G}^* = \mathrm{U}(h^*)$ where h^* is an l/k-Hermitian form as in (5.4),
- or $\mathbf{G}^* = \mathrm{Sp}(f^*)$ where f^* is a symplectic form over k as in (5.1).

Let $\mathbf{t}$ be the local transfer from $\mathbf{G}^*(F)$ to $\mathrm{GL}_N(E)$ given by Definition 5.1. We explained how to canonically identify representations of $\mathbf{G}(F)$ with those of $\mathbf{G}^*(F)$ in Paragraph 5.6. (In the symplectic case, we identified $\mathbf{G}(F)$ with $\mathrm{Sp}(f_w)$ for some symplectic form f_w over $k_w = F$.) This gives us a local transfer from $\mathbf{G}(F) = G$ to $\mathrm{GL}_N(E)$, still denoted $\mathbf{t}$.

Lemma 9.1. — *There is a finite place u of k different from w, not dividing ℓ, such that there is a unitary cuspidal irreducible complex representation ρ of $\mathbf{G}(k_u)$ with the following properties:*

(1) *ρ is compactly induced from some compact mod centre, open subgroup of $\mathbf{G}(k_u)$,*
(2) *the local transfer of ρ to $\mathrm{GL}_N(l_u)$ is cuspidal.*

Proof. — If G is special orthogonal, it suffices to choose $u \neq w$ such that $\mathbf{G}(k_u)$ is split (Remark 2.9), and then apply Proposition B.1.

If G is unitary, it suffices to choose $u \neq w$ such that $\mathbf{G}(k_u)$ is split (Remark 2.12).

If G is symplectic, it suffices to choose a place u such that k_u is isomorphic to $\mathbb{Q}_2$ (see Lemma 2.5) and then apply Theorem C.1. □

Now fix an isomorphism of fields $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. As in Paragraph 1.2, let $\mathbf{t}_\ell$ denote the ℓ-adic local transfer from G to $\mathrm{GL}_N(E)$ obtained from $\mathbf{t}$ thanks to ι, that is, $\mathbf{t}_\ell(V \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell) = \mathbf{t}(V) \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ for any complex representation V of G, where tensor products are taken with respect to ι. Let u and ρ be as in Lemma 9.1. By Theorem 4.4, there are irreducible automorphic representations Π_1 and Π_2 of $\mathbf{G}(\mathbb{A})$ such that

(1) $\Pi_{1,u}$ and $\Pi_{2,u}$ are both isomorphic to ρ,
(2) $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_1 and $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ is isomorphic to π_2,
(3) $\Pi_{1,v}$ and $\Pi_{2,v}$ are trivial for any real place v,
(4) there is a finite set S of places of k, containing all real places, such that for all $v \notin \mathrm{S}$:

(a) the group $\mathbf{G}$ is unramified over k_v, and the local components $\Pi_{1,v}$ and $\Pi_{2,v}$ are unramified with respect to some hyperspecial maximal compact subgroup K_v of $\mathbf{G}(k_v)$,

(b) the restrictions of the Satake parameters of $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ to the Hecke $\overline{\mathbb{Z}}_\ell$-algebra $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}(k_v), \mathrm{K}_v)$ are congruent mod the maximal ideal $\mathfrak{m}$ of $\overline{\mathbb{Z}}_\ell$.

Applying Theorems 5.5 and 5.6 to Π_1, Π_2, we get algebraic regular, polarized, cuspidal irreducible automorphic representations $\widetilde{\Pi}_1, \widetilde{\Pi}_2$ of $\mathrm{GL}_N(l)$ such that, for $i = 1, 2$ and all finite places v of k, the local transfer of $\Pi_{i,v}$ to $\mathrm{GL}_N(l_v)$ is $\widetilde{\Pi}_{i,v}$. Writing $\mathbf{t}_v$ for the local transfer over k_v, we thus have $\widetilde{\Pi}_{i,v} = \mathbf{t}_v(\Pi_{i,v})$, or equivalently $\widetilde{\Pi}_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell = \mathbf{t}_{v,\ell}(\Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell)$ where $\mathbf{t}_{v,\ell}$ is obtained from $\mathbf{t}_v$ thanks to ι.

In particular, for all $v \notin \mathrm{S}$, it follows from Proposition 6.3 that $\widetilde{\Pi}_{1,v}$ and $\widetilde{\Pi}_{2,v}$ are unramified and that the characteristic polynomials of the conjugacy classes of semisimple elements in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ associated with $\widetilde{\Pi}_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\widetilde{\Pi}_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ have coefficients in $\overline{\mathbb{Z}}_\ell$ and are congruent mod $\mathfrak{m}$.

Now apply Theorem 8.2 at w: the representations $\widetilde{\Pi}_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\widetilde{\Pi}_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral, their reductions mod ℓ share a common generic irreducible component, and such a generic component is unique. The result now follows from the fact that $\Pi_{i,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell \simeq \pi_i$ for $i = 1, 2$.

9.2.

We now describe how the map $\mathbf{t}_\ell$ depends on the choice of ι. Equivalently, since any two isomorphisms ι, ι' between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ give rise to a field automorphism $\iota^{-1} \circ \iota'$ of $\mathbb{C}$, we will describe the behavior of $\mathbf{t}$ under the action of $\mathrm{Aut}(\mathbb{C})$. More precisely, we prove the following result.

Proposition 9.2. — *Let π be a cuspidal complex representation of G. Let $\gamma \in \mathrm{Aut}(\mathbb{C})$.*

(1) *If G is not even orthogonal, then $\mathbf{t}(\pi^\gamma) = \mathbf{t}(\pi)^\gamma$.*

(2) *If G is even orthogonal, then $\mathbf{t}(\pi^\gamma) = \mathbf{t}(\pi)^\gamma \cdot \varepsilon_\gamma$, where ε_γ is the unramified character*

$$x \mapsto \left(\frac{\gamma(\sqrt{q})}{\sqrt{q}}\right)^{\mathrm{val}_F(x)} \tag{9.1}$$

of $F^\times$, where q is the cardinality of the residue field of F.

Let π be a cuspidal complex representation of G. As in Lemma 9.1, let u be a finite place of k different from w, not dividing ℓ, and ρ be a unitary cuspidal irreducible complex representation of $\mathbf{G}(k_u)$ with cuspidal transfer. By Proposition 4.1, we have an irreducible automorphic representation Π of $\mathbf{G}(\mathbb{A})$ such that

(1) the local component Π_u is isomorphic to ρ,

(2) the local component Π_w is isomorphic to π,

(3) the local component Π_v is the trivial character of $\mathbf{G}(k_v)$ for any real place v of k.

Associated with Π by Theorems 5.5 and 5.6, there is an algebraic regular, polarized, cuspidal irreducible automorphic representation $\widetilde{\Pi}$ of $\mathrm{GL}_N(\mathbb{A}_l)$ such that $\widetilde{\Pi}_v = \mathbf{t}_v(\Pi_v)$ for all finite places v of k, where $\mathbf{t}_v$ is as in Paragraph 9.1. Now let $\gamma \in \mathrm{Aut}(\mathbb{C})$. Then Π^γ satisfies

(1) the local component Π_u^γ is isomorphic to ρ^γ,

(2) the local component Π_w^γ is isomorphic to π^γ,

(3) the local component Π_v^γ is the trivial character of $\mathbf{G}(k_v)$ for any real place v of k.

Associated with it by Theorems 5.5 and 5.6, there is an algebraic regular, polarized, cuspidal irreducible automorphic representation $\widetilde{\Pi}'$ of $\mathrm{GL}_N(\mathbb{A}_l)$ such that $\widetilde{\Pi}'_v = \mathbf{t}_v(\Pi_v^\gamma)$ for all finite v.

Let S be a finite set of places of k, containing all real places, such that for all $v \notin \mathrm{S}$ the group $\mathbf{G}$ is unramified over k_v and the local component Π_v is unramified with respect to some hyperspecial maximal compact subgroup of $\mathbf{G}(k_v)$.

Assume first that G is not an even special orthogonal group. For $v \notin \mathrm{S}$, Proposition 6.7 gives us $\mathbf{t}_v(\Pi_v^\gamma) = \mathbf{t}_v(\Pi_v)^\gamma$, thus $\widetilde{\Pi}'$ and $\widetilde{\Pi}^\gamma$ coincide at almost all finite places. By strong multiplicity 1, we deduce that $\widetilde{\Pi}' = \widetilde{\Pi}^\gamma$. It follows that $\mathbf{t}_w(\Pi_w^\gamma) = \mathbf{t}_w(\Pi_w)^\gamma$, that is, $\mathbf{t}(\pi^\gamma) = \mathbf{t}(\pi)^\gamma$.

Assume now that G is even special orthogonal (thus $l = k$). For all finite places v of k, let $\mathbf{t}_v^*$ be the map $\pi \mapsto \mathbf{t}_v(\pi)|\det|_v^{1/2}$, where $|\cdot|_v$ is the absolute value of $k_v^\times$ and $|\cdot|_v^{1/2}$ is its square root with respect to $q_v^{1/2}$, where q_v is the cardinality of the residue field of k_v. An argument similar to that of the non even orthogonal case gives us $\widetilde{\Pi}'|\det|^{1/2} = (\widetilde{\Pi}|\det|^{1/2})^\gamma$ where $|\cdot|$ is the absolute value of $\mathbb{A}^\times$. Looking at the local component at w, we deduce that $\mathbf{t}(\pi^\gamma) = \mathbf{t}(\pi)^\gamma \cdot \varepsilon_\gamma$, where ε_γ is defined as in (9.1). We have proved Proposition 9.2.

Remark 9.3. — The same argument shows that Proposition 9.2 holds for all discrete series representations π of G (it suffices to replace Proposition 4.1 by [**56**] Theorem 5.13). Let us explain how this implies that the set of isomorphism classes of discrete series representations of G is stable under $\mathrm{Aut}(\mathbb{C})$. Let φ be the local Langlands parameter of a discrete series representation π (up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy in the even orthogonal case) and let $\phi = \mathrm{Std} \circ \varphi$ be the Langlands parameter of $\mathbf{t}(\pi)$. On the one hand, the fact that π is a discrete series representation implies that the quotient of the centralizer of the image of ϕ in $\widehat{\mathbf{G}}$ by $\mathbf{Z}(\widehat{\mathbf{G}})^{\mathrm{W}_k}$ is finite (see the end of Paragraph 5.3). On the other hand, the Langlands parameter of $\mathbf{t}(\pi^\gamma)$ is $\phi' = \phi^\gamma \cdot \eta\chi$ (where η is the unramified character of $F^\times$ of order 2 and χ is either the character ε_γ defined by (9.1) if G is even orthogonal, or the trivial character otherwise), which has the same finiteness property. Thus the L-packet of π^γ is discrete. Thus π^γ is a discrete series representation.

Remark 9.4. — Let us examine how the local transfer map behaves under automorphisms of the base field F, for discrete series representations. Let π be a discrete series representation of G, and let ϕ be the Langlands parameter of its transfer $\mathbf{t}(\pi)$. By Moeglin [**44**], an irreducible Langlands parameter $\sigma \boxtimes \mathrm{S}_a$, where σ is an irreducible representation of dimension $k \geqslant 1$ of W_F and a is a positive integer, occurs in ϕ if and only if:

(1) the cuspidal representation ρ of $\mathrm{GL}_k(E)$ associated with σ by the Langlands correspondence is c-selfdual,

(2) if s is the unique non-negative real number such that the normalized parabolically induced representation $\rho\nu^s \rtimes \pi$ is reducible, then $2s-1$ is a positive integer and $2s-1-a$ is a non-negative even integer.

Now let $\varkappa \in \mathrm{Aut}(F)$, which extends to an automorphism of E still denoted $\varkappa$. Then

– the cuspidal representation $\rho^\varkappa$ is $\varkappa^{-1}c\varkappa$-selfdual,

– the irreducible representation of W_E associated with it by the Langlands correspondence is $\sigma^\varkappa$ (see [**25**] Propriété 1),

– the normalized parabolically induced representation $\rho^\varkappa\nu^s \rtimes \pi^\varkappa$ is reducible,

– the representation $\sigma \boxtimes \mathrm{S}_a$ occurs in ϕ if and only if $\sigma^{\varkappa} \boxtimes \mathrm{S}_a$ occurs in $\phi^{\varkappa}$.

It follows that the Langlands parameter of $\mathbf{t}(\pi^{\varkappa})$ is $\phi^{\varkappa}$. Applying [**25**] Propriété 1 again, $\phi^{\varkappa}$ is the Langlands parameter of $\mathbf{t}(\pi)^{\varkappa}$. Thus $\mathbf{t}(\pi^{\varkappa})$ is equal to $\mathbf{t}(\pi)^{\varkappa}$.

9.3.

In this paragraph, we discuss the case of the split special orthogonal group $\mathrm{SO}_2(F) \simeq F^{\times}$. Let χ be a $\overline{\mathbb{Q}}_\ell$-character of this group. Its transfer to $\mathrm{GL}_2(F)$ is

– either the normalized parabolically induced representation $\chi \times \chi^{-1}$ when the character χ^2 is different from the absolute value $|\cdot|$ and its inverse $|\cdot|^{-1}$,
– or the unique character occurring as a component of $\chi \times \chi^{-1}$ when $\chi^2 \in \{|\cdot|, |\cdot|^{-1}\}$.

Properties (1) and (2) of Theorem 1.1 thus hold, since

– an irreducible $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_2(F)$ is integral if and only if its cuspidal support is integral (see [**63**] II.4.14 and [**15**] Proposition 6.7),
– if χ is integral, the supercuspidal support of any irreducible component of $\mathbf{r}_\ell(\chi \times \chi^{-1})$ is the $\mathrm{GL}_2(F)$-conjugacy class of the cuspidal pair $(F^{\times} \times F^{\times}, \chi \otimes \chi^{-1})$.

However, if ξ is any non-trivial character of $F^{\times}$ with values in $1+\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal of $\overline{\mathbb{Z}}_\ell$) such that $\xi^2 \notin \{1, |\cdot|^{-2}\}$, the characters $|\cdot|^{1/2}$ and $\xi|\cdot|^{1/2}$ are congruent, but the transfer of the first one is the trivial character of $\mathrm{GL}_2(F)$, which is not generic. Property (3) thus does not hold. Also, the transfer of the second one is $\xi|\cdot|^{1/2} \times \xi^{-1}|\cdot|^{-1/2}$, whose reduction mod ℓ contains the trivial character with multiplicity 1 (if $\ell \neq 2$) or 2 (if $\ell = 2$) by [**62**] Théorème 3.

Assume further that q has order 2 mod ℓ, that is, ℓ divides q^2-1 but not $q-1$, and let η be the unique unramified $\overline{\mathbb{Q}}_\ell$-character of order 2 of $F^{\times}$. Then the transfer of $\eta|\cdot|^{-1/2}$ (which is congruent to $|\cdot|^{1/2}$) is $\eta \circ \det$, whose reduction mod ℓ is a character of order 2. We thus have two congruent characters of $F^{\times}$ whose transfers to $\mathrm{GL}_2(F)$ have reductions mod ℓ with no component in common.

A
Cyclic base change

Let F be a p-adic field, and let K be a cyclic finite extension of F of degree d. Fix an integer $n \geqslant 1$ and write $G = \mathrm{GL}_n(F)$ and $H = \mathrm{GL}_n(K)$. By [**3**], there exists a map from isomorphism classes of irreducible (smooth) complex representations of G to those of H called the local *base change*, denoted $\mathbf{b} = \mathbf{b}_{K/F}$.

Now let us fix a prime number ℓ different from p and an isomorphism of fields ι between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$. Replacing $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$ thanks to ι, one obtains a local base change $\mathbf{b}_{K/F,\ell}$ for irreducible smooth $\overline{\mathbb{Q}}_\ell$-representations.

In this appendix, we investigate the dependency of $\mathbf{b}_{K/F,\ell}$ in the choice of ι, or equivalently the behavior of $\mathbf{b}_{K/F}$ with respect to automorphisms of $\mathbb{C}$.

A.1.

Let $\mathbf{a}_F$ denote the local Langlands correspondence from the set of isomorphism classes of irreducible complex representations of G to the set $\Phi(G)$ of $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes of local Langlands parameters for G ([**22, 24**]).

Replacing $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$ thanks to ι, one obtains a local Langlands correspondence $\mathbf{a}_{F,\ell}$ for irreducible $\overline{\mathbb{Q}}_\ell$-representations. The dependency of $\mathbf{a}_{F,\ell}$ in ι, or equivalently the behavior of $\mathbf{a}_F$ with respect to automorphisms of $\mathbb{C}$, has been studied in [**25, 14**]: the map $\pi \mapsto \mathbf{a}_F(\pi|\det|^{(1-n)/2})$ is insensitive to automorphisms of $\mathbb{C}$. It follows that

$$\mathbf{a}_F(\pi^\gamma) = \mathbf{a}_F(\pi)^\gamma \cdot \eta_{F,\gamma}^{1-n} \tag{A.1}$$

for all $\gamma \in \mathrm{Aut}(\mathbb{C})$ and all irreducible complex representations π of G, where

$$\eta_{F,\gamma}(w) = \left(\frac{\gamma(\sqrt{q})}{\sqrt{q}}\right)^{\upsilon_F(w)} \tag{A.2}$$

for all $w \in \mathrm{W}_F$, where υ_F is the valuation map taking any Frobenius element to 1.

A.2.

Let $\mathbf{res}_{K/F}$ be the map from $\Phi(G)$ to $\Phi(H)$ defined by restricting local Langlands parameters from WD_F to WD_K. The local base change $\mathbf{b}_{K/F}$ is characterized by the identity

$$\mathbf{a}_K \circ \mathbf{b}_{K/F} = \mathbf{res}_{K/F} \circ \mathbf{a}_F.$$

Now let us prove that $\mathbf{b} = \mathbf{b}_{K/F}$ is insensitive to the action of $\mathrm{Aut}(\mathbb{C})$.

Proposition A.1. — *For all $\gamma \in \mathrm{Aut}(\mathbb{C})$ and all irreducible complex representations π of G, we have $\mathbf{b}_{K/F}(\pi^\gamma) = \mathbf{b}_{K/F}(\pi)^\gamma$.*

Proof. — Let π be an irreducible complex representation of G. We have

$$\begin{aligned}
\mathbf{a}_K(\mathbf{b}_{K/F}(\pi^\gamma)) &= \mathbf{res}_{K/F}(\mathbf{a}_F(\pi^\gamma)) \\
&= \mathbf{res}_{K/F}(\mathbf{a}_F(\pi)^\gamma \cdot \eta_{F,\gamma}^{1-n}) \\
&= \mathbf{a}_K(\mathbf{b}_{K/F}(\pi))^\gamma \cdot (\eta_{F,\gamma}|_{\mathrm{W}_K})^{1-n} \\
&= \mathbf{a}_K(\mathbf{b}_{K/F}(\pi)^\gamma) \cdot (\eta_{K,\gamma} \cdot \eta_{F,\gamma}|_{\mathrm{W}_K})^{1-n}.
\end{aligned}$$

We are thus reduced to compare $\eta_{F,\gamma}|_{\mathrm{W}_K}$ with $\eta_{K,\gamma}$. Using the explicit formula (A.2), we get

$$\eta_{F,\gamma}|_{\mathrm{W}_K} = \left(\frac{\gamma(\sqrt{q})}{\sqrt{q}}\right)^{\upsilon_F|_{\mathrm{W}_K}}, \quad \eta_{K,\gamma} = \left(\frac{\gamma(\sqrt{q'})}{\sqrt{q'}}\right)^{\upsilon_K},$$

where q' is the cardinality of the residue field of K. Since $q' = q^{f_{K/F}}$ and $\upsilon_F|_{\mathrm{W}_K} = f_{K/F}\upsilon_K$, we deduce that $\eta_{F,\gamma}|_{\mathrm{W}_K} = \eta_{K,\gamma}$, thus $\mathbf{b}_{K/F}(\pi^\gamma) = \mathbf{b}_{K/F}(\pi)^\gamma$. □

A.3.

Let us prove that the map $\mathbf{b}_{K/F,\ell}$ preserves the fact of being integral.

Let π be an integral irreducible $\overline{\mathbb{Q}}_\ell$-representation of G. By [**15**] Proposition 6.7, its cuspidal support is integral, or equivalently, the central character of its cuspidal support is integral. The semisimple representation of W_F corresponding to this cuspidal support is thus integral, because the determinant of each of its irreducible components is integral, and the same holds for its restriction to W_K. Since this restriction corresponds to the cuspidal support of $\mathbf{b}_{K/F,\ell}(\pi)$, it follows from [**63**] II.4.14 that the base change $\mathbf{b}_{K/F,\ell}(\pi)$ is integral.

A.4.

We now review the congruence properties of $\mathbf{b}_{K/F,\ell}$, after J. Zou's PhD thesis [**69**] 1.10.

Associated with an irreducible representation τ of $\mathrm{GL}_n(K)$, with coefficients in $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$, there is a partition

$$\lambda(\tau) = (k_1 \geqslant k_2 \geqslant \dots)$$

of n defined inductively as follows. Let k_1 denote the largest integer $k \in \{1, \dots, n\}$ such that the kth derivative $\tau^{(k)}$ is non-zero. If $k_1 = n$, then $\lambda(\tau) = (n)$. Otherwise, $(k_2 \geqslant \dots)$ is the partition of $n - k_1$ associated with the representation $\tau^{(k_1)}$ of $\mathrm{GL}_{n-k_1}(K)$.

By [**64**] V.9.2, if τ is an integral irreducible $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(K)$, its reduction mod ℓ has a unique irreducible component π such that $\lambda(\pi) = \lambda(\tau)$. This component is denoted $\mathbf{j}_\ell(\tau)$.

Theorem A.2 **([69] Theorem 1.10.17)**. — *Let π_1 and π_2 be integral irreducible $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(F)$. If $\mathbf{j}_\ell(\pi_1) = \mathbf{j}_\ell(\pi_2)$, then $\mathbf{j}_\ell(\mathbf{b}_{K/F,\ell}(\pi_1)) = \mathbf{j}_\ell(\mathbf{b}_{K/F,\ell}(\pi_2))$.*

In particular, if π_1, π_2 are cuspidal, which implies that $\lambda(\pi_1) = \lambda(\pi_2) = (n)$, their base changes $\mathbf{b}_{K/F,\ell}(\pi_1)$ and $\mathbf{b}_{K/F,\ell}(\pi_2)$ are generic. This theorem thus says that, if $\mathbf{r}_\ell(\pi_1) = \mathbf{r}_\ell(\pi_2)$, then $\mathbf{r}_\ell(\mathbf{b}_{K/F,\ell}(\pi_1))$ and $\mathbf{r}_\ell(\mathbf{b}_{K/F,\ell}(\pi_2))$ have a unique generic irreducible component in common. This can be seen as an analogue of Theorem 1.1 for the cyclic base change from G to H.

A.5.

In this paragraph, we give an example of congruent integral cuspidal $\overline{\mathbb{Q}}_\ell$-representations π_1, π_2 of G such that $\mathbf{b}_{K/F,\ell}(\pi_1)$ and $\mathbf{b}_{K/F,\ell}(\pi_2)$ are not congruent.

First, assume that π is an integral cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representation of G. Let m denote the cardinality of the set of isomorphism classes of $\pi\chi$, where χ runs over the characters of $F^\times$ trivial on $\mathrm{N}_{K/F}(K^\times)$, and set $e = d/m$. Then there exists a cuspidal irreducible representation ρ of $\mathrm{GL}_{n/e}(K)$ such that

$$\mathbf{b}_{K/F}(\pi) = \rho \times \rho^\alpha \times \dots \times \rho^{\alpha^{e-1}}$$

where α is a generator of $\mathrm{Gal}(K/F)$ and $\times$ denotes normalized parabolic induction with respect to a choice of square root of q, the cardinality of the residue field of F (see [**3**] Chapter 1, §6.4).

Now assume that $n = 2$ and that K is a ramified quadratic extension of F, and let $\omega_{K/F}$ be the character of $F^\times$ with kernel $\mathrm{N}_{K/F}(K^\times)$. Let π_1 be an integral cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $G = \mathrm{GL}_2(F)$ of level 0. By [**13**], it is compactly induced from a representation $\boldsymbol{\lambda}_1$ of $F^\times\mathrm{GL}_2(\mathcal{O}_F)$

whose restriction to $\mathrm{GL}_2(\mathcal{O}_F)$ is the inflation of a cuspidal irreducible representation σ_1 of the group $\mathrm{GL}_2(\boldsymbol{k})$, where $\boldsymbol{k}$ is the residue field of F. Associated with σ_1, there is ([**19**]) a character

$$\xi_1 : \boldsymbol{l}^\times \to \overline{\mathbb{Z}}_\ell^\times \tag{A.3}$$

such that $\xi_1^q \neq \xi_1$, where $\boldsymbol{l}$ is a quadratic extension of $\boldsymbol{k}$ and q is the cardinality of $\boldsymbol{k}$.

The representation $\pi_1\omega_{K/F}$ is isomorphic to π_1 if and only if $\boldsymbol{\lambda}_1\omega_{K/F}$ is isomorphic to $\boldsymbol{\lambda}_1$. As these representations all have the same central character, this is equivalent to $\sigma_1\eta \simeq \sigma_1$, where η is the unique character of order 2 of $\boldsymbol{k}^\times$ (note that the restriction of $\omega_{K/F}$ to $\mathcal{O}_F^\times$ is the inflation of η), which is equivalent to $\xi_1(\eta \circ \mathrm{N}_{\boldsymbol{l}/\boldsymbol{k}}) = \xi_1^q$, that is, ξ_1^{q-1} has order 2. Assume that this is the case. Thus $e_1 = 2$ and we may write $\mathbf{b}_{K/F}(\pi_1) = \rho_1 \times \rho_1^\alpha$ for some (tamely ramified, integral) character ρ_1 of $K^\times$.

Assume further that ℓ is a prime divisor of q^2-1 not dividing $q-1$, that is, ℓ is an odd prime divisor of $q+1$. Let μ be a character of $\boldsymbol{l}^\times$ of order ℓ and set $\xi_2 = \xi_1\mu$. Since $\xi_2^q \neq \xi_2$, there is a cuspidal $\overline{\mathbb{Q}}_\ell$-representation σ_2 of $\mathrm{GL}_2(\boldsymbol{k})$ associated with ξ_2. Since ξ_2 and ξ_1 are congruent, σ_2 and σ_1 are congruent (see for instance [**39**] 2.6). Let us inflate and extend σ_2 to a representation $\boldsymbol{\lambda}_2$ of $F^\times\mathrm{GL}_2(\mathcal{O}_F)$ which is congruent to $\boldsymbol{\lambda}_1$, then compactly induce $\boldsymbol{\lambda}_2$ to a representation π_2 of $\mathrm{GL}_2(F)$. This is an integral cuspidal representation of level 0 which is congruent to π_1.

Since $\mu^q \neq \mu$, we have $e_2 = 1$, thus $\mathbf{b}_{K/F}(\pi_1)$ is a cuspidal representation ρ_2 of $\mathrm{GL}_2(K)$. Its reduction mod ℓ is an irreducible cuspidal $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_2(K)$. It is the unique generic component of $\mathbf{r}_\ell(\rho_1 \times \rho_1^\alpha)$.

B

Cuspidal representations of split p-adic orthogonal groups with irreducible Galois parameter

B.1.

Let F be a p-adic field with $p \neq 2$, and let G be a split special orthogonal group over F, that is, $G = \mathrm{SO}(Q)$ where Q is a maximally isotropic quadratic form over F. Let n be the dimension of Q. In this section, we assume that $n \neq 2$. Let $m = \lfloor n/2 \rfloor$ be the Witt index of Q. With the notation of Paragraph 5.1, we have $G = \mathrm{SO}_{2m+1}(F)$ if n is odd, $G = \mathrm{SO}_{2m}^1(F)$ if n is even. We will prove the following result.

Proposition B.1. — *There exists a cuspidal representation of level* 0 *of* G *whose transfer to* $\mathrm{GL}_N(F)$ *is cuspidal.*

B.2.

In this paragraph, we refer to [**36**] §2 (see p. 1090 in particular). Let V be the n-dimensional F-vector space on which Q is defined. Write

$$V = V^{\mathrm{an}} \oplus V^{\mathrm{iso}}$$

where V^{an} is anisotropic (thus $\dim(V^{\mathrm{an}}) \leqslant 1$) and V^{iso} is a sum of m hyperbolic planes.

Let q denote the cardinality of the residue field of F. The anisotropic group $G^{\mathrm{an}} = \mathrm{SO}(V^{\mathrm{an}})$ has a unique (up to conjugacy) maximal parahoric subgroup. Its finite reductive quotient $\mathcal{G}^{\mathrm{an}}$ has neutral component the finite special orthogonal group

$$\mathrm{SO}_a(q)$$

with $a = \dim(V^{\mathrm{an}})$.

For any choice of integers $m_1, m_2 \geqslant 0$ such that $m_1 + m_2 = m$, there is a maximal parahoric subgroup $J = J_{m_1,m_2}$ whose finite reductive quotient $\mathcal{G} = \mathcal{G}_{m_1,m_2}$ has neutral component

$$\mathrm{SO}_{a+m_1,m_1}(q) \times \mathrm{SO}_{m_2,m_2}(q)$$

where $\mathrm{SO}_{u,v}(q)$ is the special orthogonal group over $\mathbb{F}_q$ associated with a quadratic space of dimension $u+v$ and Witt index v. Choose $m_2 = 0$, so that $\mathcal{G}$ has neutral component $\mathrm{SO}_{m,m}(q)$ if $n = 2m$, and $\mathrm{SO}_{m+1,m}(q)$ if $n = 2m+1$. In other words, $\mathcal{G}^\circ$ is split.

B.3.

Let σ be a self-dual cuspidal irreducible representation of $\mathrm{GL}_{2r}(q)$ and $s \in \mathbb{F}_{q^{2r}}^\times$ be a parameter corresponding to σ. In particular, s has degree $2r$ over $\mathbb{F}_q$ and $s^{-1} = s^{q^r}$. Its characteristic polynomial $P(X)$ is thus irreducible, of degree $2r$, and self-dual (that is, reciprocal).

The parameter s can be seen in the dual group $\mathcal{G}^{\circ,*} \subseteq \mathrm{GL}_{2r}(q)$. It then defines a Lusztig series $\mathcal{E}(\mathcal{G}^\circ, s)$.

Lemma B.2. — *The Lusztig series $\mathcal{E}(\mathcal{G}^\circ, s)$ contains a cuspidal representation.*

Proof. — If m is odd, see [**36**] §7.2 (p. 1098). Assume now that m is even. We follow [**36**] §7.3. Consider the group with connected centre $\widetilde{\mathcal{G}} = \mathrm{GSO}_m^\pm$ of which $\mathcal{G}^\circ$ is a subgroup. The scalars 1 and -1 are not eigenvalues of s. The centralizer of s is thus connected and the two Lusztig series associated with s are the same. A cuspidal representation of $\mathcal{G}^\circ$ associated with s is an irreducible component of the restriction to $\mathcal{G}^\circ$ of a cuspidal representation of $\widetilde{\mathcal{G}}$ associated with a semisimple element $\tilde{s} \in \widetilde{\mathcal{G}}^*$ lifting s. To prove the lemma, it thus suffices to prove that the Lusztig series $\mathcal{E}(\widetilde{\mathcal{G}}, \tilde{s})$ contains a cuspidal representation.

The two groups $\widetilde{\mathcal{G}}$ and $\mathcal{G}^\circ$ act naturally on the same space, thus $\tilde{s}$ and s have the same characteristic polynomial $P(X)$. It follows from [**36**] §7.2 (p. 1098) that $\mathcal{E}(\widetilde{\mathcal{G}}, \tilde{s})$ contains a cuspidal representation. □

B.4.

Let τ be a cuspidal representation in the Lusztig series $\mathcal{E}(\mathcal{G}^\circ, s)$. Let λ be an irreducible representation of J whose restriction to J° (the preimage of $\mathcal{G}^\circ$ in J) is a direct sum of conjugates (under J) of the inflation of τ. Let π be the representation obtained by compactly inducing λ to G. It is a cuspidal irreducible representation of level 0 of G.

As G is split, it follows from Moeglin [**44**] that the Langlands parameter φ associated with π is described by the reducibility set $\mathrm{Red}(\pi)$ and the Jordan set $\mathrm{Jord}(\pi)$ (see for instance the introduction of [**36**] for a definition).

In our situation, it follows from [**36**] §8 that the sets $\mathrm{Red}(\pi)$ and $\mathrm{Jord}(\pi)$ are equal and both reduced to a single element $(\rho, 1)$, where ρ is a selfdual cuspidal representation of $\mathrm{GL}_N(F)$ (with $N = n-1$ if n is odd and $N = n$ if n is even), which proves Proposition B.1.

Remark B.3. — More precisely, ρ has level 0, and is obtained by compactly inducing a representation of $F^\times \mathrm{GL}_N(\mathcal{O}_F)$ which is trivial on $1 + \mathbf{M}_N(\mathfrak{p}_F)$ and whose restriction to $\mathrm{GL}_N(\mathcal{O}_F)$ is the inflation of σ.

C

Cuspidal representations of $\mathrm{Sp}_{2n}(\mathbb{Q}_2)$ with irreducible Galois parameter

(by Guy HENNIART at ORSAY)

C.1.

Let p be a prime number and F a finite extension of $\mathbb{Q}_p$. Let $\overline{F}$ be an algebraic closure of F and W_F the Weil group of $\overline{F}/F$. Let n be a positive integer, and π a cuspidal (complex) representation of $\mathrm{Sp}_{2n}(F)$. Let σ be the Galois parameter attached to π by Arthur [**2**], which one sees as an orthogonal representation of $\mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C})$, of dimension $2n+1$. The following result is used in the main text, in Section 9.

Theorem C.1. — *Assume that $F = \mathbb{Q}_2$, and take for π the (unique) simple supercuspidal representation of* $\mathrm{Sp}_{2n}(F)$. *Then σ is an irreducible representation of* W_F.

Here *simple* is in the sense of Gross and Reeder [**21**]. The point of the result is that π is compactly induced from a compact open subgroup of $\mathrm{Sp}_{2n}(F)$, as we describe below. Indeed when $p = 2$ there is at least one irreducible orthogonal representation σ of W_F of dimension $2n+1$ [**11**], only one if $F = \mathbb{Q}_2$, and by [**2**] it is the parameter of a cuspidal representation π of $\mathrm{Sp}_{2n}(F)$, but it is not clear *a priori* that π is compactly induced.

Our method is inspired by work of Oi [**47**]. When p is odd, Oi determines the parameter σ of a simple cuspidal representation π of $\mathrm{Sp}_{2n}(F)$. In his case σ is always reducible, but a number of techniques and results remain valid when $p = 2$, and, with extra information given by Adrian and Kaplan [**1**] when $F = \mathbb{Q}_2$, that is enough for us. It is quite likely that one can describe σ explicitly whenever π is simple cuspidal, not only when p is odd or $F = \mathbb{Q}_2$. Indeed many of our arguments work more generally, and until C.6 we make no special assumption on F, except that in C.3 we start assuming that(1) $p = 2$.

C.2.

We now proceed. We use customary notation, $\mathcal{O}_F$ for the ring of integers of F, $\mathfrak{p}_F$ for the maximal ideal of $\mathcal{O}_F$. We fix a uniformizer ϖ of F, and write k for the residue field $\mathcal{O}_F/\mathfrak{p}_F$ and q for its cardinality. We also fix a non-trivial character ψ of k. If $\mathbf{H}$ is an algebraic group over F, we usually put $H = \mathbf{H}(F)$.

(1) Oi and the author ([**26**]) can now extend Theorem C.1 to any 2-adic field F.

We use the usual explicit model of $\mathbf{G} = \mathrm{Sp}_{2n}$, see [**47**] §2.4, so elements of $G = \mathrm{Sp}_{2n}(F)$ are symplectic $2n \times 2n$ matrices. By cuspidal representation of G we mean an irreducible smooth complex cuspidal representation. We are interested in *simple* cuspidal representations of G, in the sense of Gross and Reeder [**21**]. Let us describe them.

The choice in [**21**] of a root basis and an affine root basis determines an Iwahori subgroup I of G, with its first two congruence subgroups I^+ and I^{++}. The Iwahori subgroup I is the subgroup of $\mathrm{Sp}_{2n}(\mathcal{O}_F)$ made out of the matrices which are upper triangular modulo $\mathfrak{p}_F$, I^+ is made out of the matrices which are further upper unipotent modulo $\mathfrak{p}_F$, and I^{++} is made out of the matrices $(x_{i,j})$ in I^+ with $x_{i,i+1} \in \mathfrak{p}_F$ for $i = 1, \ldots, 2n-1$, and $x_{2n,1} \in \mathfrak{p}_F^2$. The quotient I^+/I^{++} is isomorphic to a product of $n+1$ copies of k, *via* the surjective homomorphism

$$(x_{i,j}) \mapsto (x_{1,2} \bmod \mathfrak{p}_F, \ldots, x_{n,n+1} \bmod \mathfrak{p}_F, x_{2n,1}/\varpi \bmod \mathfrak{p}_F)$$

from I^+ to k^{n+1}.

A character of I^+ is *simple* if it is trivial on I^{++}, and is the inflation of a character of k^{n+1} which is non-trivial on each factor k. The normalizer in G of a simple character θ of I^+ is ZI^+, where Z is the centre of G, and ZI^+ is also the intertwining of θ in G, so that any extension of θ to ZI^+ gives by compact induction to G a cuspidal representation of G: see [**47**] §2.4, Proposition 2.6. Note that when p is 2, the centre Z of G is actually contained in I^{++}. The cuspidal representations of G thus obtained are the *simple cuspidal* representations of [**21**].

The normalizer of I^+ in G is ZI, and I acts on I^+/I^{++} *via* I/I^+; identifying I/I^+ with $k^{\times n}$ *via*

$$(x_{i,j}) \mapsto (x_{1,1} \bmod \mathfrak{p}_F, \ldots, x_{n,n} \bmod \mathfrak{p}_F),$$

the conjugation action of $(\chi_1, \ldots, \chi_n) \in k^{\times n}$ on I^+/I^{++} (identified with k^{n+1}) sends the family $(u_1, \ldots, u_{n+1}) \in k^{n+1}$ to

$$\left(u_1\chi_1\chi_2^{-1}, u_2\chi_2\chi_3^{-1}, \ldots, u_{n-1}\chi_{n-1}\chi_n^{-1}, u_n\chi_n^2, u_{n+1}\chi_1^{-2}\right).$$

In particular, when $p = 2$, a given simple character θ of I^+ can always be conjugated in I to the character

$$\theta(a) : (u_1, \ldots, u_{n+1}) \mapsto \psi(u_1 + \cdots + u_n + au_{n+1})$$

for some a in $k^\times$, uniquely determined by θ. More precisely if θ sends $(u_1, \ldots, u_{n+1})$ to $\psi(a_1u_1 + \cdots + a_nu_n + a_{n+1}u_{n+1})$ for some a_i's in $k^\times$, then $a = (a_1 \cdots a_{n-1})^2 \cdot a_n \cdot a_{n+1}$. Thus when $p = 2$ there are only $q-1$ isomorphism classes of simple cuspidal representations of G, whereas by a similar analysis ([**47**] §2.4), there are $4(q-1)$ such classes when p is odd. Note that when $q = 2$ all that is obvious since k has only one non-trivial character.

C.3.

The group Sp_{2n} is split, and its dual group is $\mathrm{SO}_{2n+1}(\mathbb{C})$. To a cuspidal representation π of G, Arthur attaches the conjugacy class of a discrete parameter, that is (the conjugacy class of) a continuous homomorphism from $\mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C})$ into $\mathrm{SO}_{2n+1}(\mathbb{C})$ which, as a representation of dimension $2n+1$, is a direct sum of inequivalent irreducible orthogonal representations $\sigma_1, \ldots, \sigma_r$ with the product $\det\sigma_1 \cdots \det\sigma_r$ trivial. What our theorem says is that when $F = \mathbb{Q}_2$ and π is simple cuspidal, then $r = 1$ and σ_1 is trivial on $\mathrm{SL}_2(\mathbb{C})$, *i.e.* is in fact a representation of W_F.

Note that [**11**] shows that when p is odd, there is no irreducible orthogonal representation of W_F of odd dimension > 1, contrary to the case $p = 2$, where [**11**] gives a complete classification.

From now on we assume $p = 2$. For a in $k^\times$ let us denote by $\pi(a)$ the isomorphism class of the representation of G compactly induced from the character $\theta(a)$ of I^+. We let $\phi(a)$ be the parameter of $\pi(a)$, $r(a)$ the number of irreducible components of $\phi(a)$, and $\Pi(a)$ the L-packet of $\pi(a)$, that is the set of isomorphism classes of tempered (in fact, discrete series) representations of G with parameter $\phi(a)$; it is known that $\Pi(a)$ has $2^{r(a)-1}$ elements, so one of our goals is to show that $r(a) = 1$. Let $\mathbf{G}_{\mathrm{ad}}$ be the adjoint group of $\mathbf{G}$, and ι the quotient map from $\mathbf{G}$ to $\mathbf{G}_{\mathrm{ad}}$.

Lemma C.2. — *$\pi(a)$ is stable under the action of G_{ad}.*

Proof. — We follow the proof of [**47**] Proposition 5.2. As there, one gets a description of the quotient $G_{\mathrm{ad}}/\iota(G)$. It is isomorphic to $\mathrm{Hom}(F^\times, \mu_2)$, itself isomorphic, by Kummer theory, to $F^\times/F^{\times 2}$. More concretely if T is the diagonal torus of G made out of elements

$$t(b) = (b, b, \dots, b, b^{-1}, \dots, b^{-1})$$

(with n times b and n times b^{-1}), then for any b in $\overline{F}$ with b^2 in $F^\times$ the image of $t(b)$ in $\mathbf{G}_{\mathrm{ad}}(\overline{F})$ is actually in G_{ad}, and the set of such $t(b)$'s covers $G_{\mathrm{ad}}/\iota(G)$.

If b^2 is a unit in F then $t(b)$ actually normalizes I and its congruence subgroups, and sends $\theta(a)$ to the character given by

$$(u_1, \dots, u_{n+1}) \mapsto \psi(u_1 + \dots + u_{n-1} + b^2 u_n + (a/b^2) u_{n+1}),$$

conjugate in I to $\theta(a)$. If b^2 is the uniformizer ϖ, $t(b)$ conjugates I to another Iwahori subgroup, but if s is the matrix in G with four blocks of size n, first line $(0, I_n)$ and second line $(-I_n, 0)$, then $st(b)$ normalizes I and its congruence subgroups, and sends $\theta(a)$ to the character given by

$$(u_1, \dots, u_{n+1}) \mapsto \psi(u_1 + \dots + u_{n-1} + a u_n + u_{n+1})$$

(recall that $p = 2$, so $-1 = 1$ in k), which is conjugate to $\theta(a)$. Since the stabilizer in G_{ad} of $\pi(a)$ is a subgroup containing all of $\iota(G)$, it follows that it is all of G_{ad}. □

An important point is the genericity of simple cuspidal representations. We fix the same Whittaker datum as Oi [**47**] §6.3(2) to define genericity. By [**29**] Proposition 5.1, the G_{ad}-orbit of $\pi(a)$ contains a single generic representation, so by the previous lemma the representation $\pi(a)$ is generic. Reasoning as in [**47**] Corollary 4.9 and Corollary 5.7, we get:

Proposition C.3. — *The parameter $\phi(a)$ is trivial on $\mathrm{SL}_2(\mathbb{C})$, every element of $\Pi(a)$ is cuspidal, and among them only $\pi(a)$ is a simple cuspidal representation.*

It only remains to prove that $r(a) = 1$.

C.4.

Still following [**47**] we prove:

Proposition C.4. — *$\Pi(a)$ does not contain any level 0 cuspidal representation.*

Proof. — By [**55**] Corollary 9.10, all elements of $\Pi(a)$ have the same formal degree. If $\mathrm{d}g$ is a Haar measure on G/Z, then the formal degree of $\pi(a)$ is $\mathrm{d}g/\mathrm{vol}(I^+/Z, \mathrm{d}g)$ (by [**47**] Lemma 5.10), whereas the formal degree of a level 0 cuspidal representation of G is strictly smaller, by the following reasoning inspired by *loc. cit.*, Proposition 5.11. A level 0 cuspidal representation π' of G is compactly induced from an irreducible representation ρ of a maximal parahoric subgroup P of G, trivial on the pro-p radical P^+ of P, and coming via inflation from a cuspidal representation of the finite (connected here) reductive group $\overline{P} = P/P^+$. The formal degree of π' is

$$\frac{\dim(\rho)}{\mathrm{vol}(P/Z, \mathrm{d}g)}\mathrm{d}g.$$

One can assume that P contains I and I^+ contains P^+. Since $p = 2$, the group P^+ contains Z, so what we have to prove is that $\dim(\rho) < \mathrm{card}(P/I^+)$. But I^+/P^+ is the unipotent radical $\overline{U}$ of the Borel subgroup $\overline{B} = I/P^+$ of $\overline{P}$, and obviously $\dim(\rho)^2$ is at most $\mathrm{card}(\overline{P})$, so it is enough to check $\mathrm{card}(\overline{P}) < \mathrm{card}(\overline{P}/\overline{U})^2$ or $\mathrm{card}(\overline{U})^2 < \mathrm{card}(\overline{P})$, which is a consequence of the existence of the big cell $\overline{B}w\overline{U}$ in the Bruhat decomposition for $\overline{P}$. □

Remark C.5. — It is highly plausible that for a cuspidal representation π' of G which is not of level 0 and is not a simple cuspidal either, the formal degree of π' is bigger than the formal degree of $\pi(a)$. But nothing explicit is known about such π'.

C.5.

Now we compute the character $\xi(a)$ of $\pi(a)$ at an affine generic element g of I^+, where g generic means that modulo I^{++}, g gives an $(n+1)$-tuple $(u_1, \ldots, u_{n+1})$ in k^{n+1} with all coordinates non-zero. As in [**47**] Lemma 2.5, we see that an element y conjugating g into I^+ belongs to I, so that by the usual formula for the character of compactly induced representations (see e. g. *loc. cit.* Theorem 3.2), the character $\xi(a)$ of $\pi(a)$ at g is the sum

$$\sum_{(\chi_1,\ldots,\chi_n)\in k^{\times n}} \psi\left(u_1\chi_1\chi_2^{-1} + u_2\chi_2\chi_3^{-1} + \cdots + u_{n-1}\chi_{n-1}\chi_n^{-1} + u_n\chi_n^2 + au_{n+1}\chi_1^{-2}\right),$$

which is a kind of Kloosterman sum, the sum

$$\sum_{(\eta_1,\ldots,\eta_n)\in k^{\times n}} \psi\left(u_1\eta_1 + u_2\eta_2 + \cdots + u_{n-1}\eta_{n-1} + u_n\eta_n^2 + au_n\eta_{n+1}\right)$$

with η_{n+1} given by $(\eta_1 \ldots \eta_{n-1})^2\eta_n\eta_{n+1} = 1$. Noting that ψ takes only the values 1 and -1, we conclude:

Proposition C.6. — *The value of $\xi(a)$ at a generic element g of I^+ is an odd integer depending only on g modulo I^{++}.*

C.6.

Still following [**47**] 5.3, we now show:

Proposition C.7. — *$r(a) = 1$ or 2, and, seen as a representation of W_F of dimension $2n+1$, $\phi(a)$ is either irreducible or the direct sum of a character ω with $\omega^2 = 1$ and an irreducible (orthogonal) representation with determinant ω.*

Proof. — Put $s = 2^{r(a)-1}$ and enumerate the elements of $\pi(a)$ as $\pi_1 = \pi(a), \dots, \pi_s$, and let ξ_i be the character of π_i. Let g be a generic element of I^+. Choose $\varepsilon(i) = 1$ or -1 for $i = 1, \dots, s$. Exactly as in the proof of Claim in *loc. cit.*, we get that $\varepsilon(1)\xi_1 + \cdots + \varepsilon(s)\xi_s$ does not vanish at g. Using that the characteristic polynomial of g is irreducible of degree $2n$ (*loc. cit.*, Lemma 7.5, still valid when $p = 2$), the proofs of Theorem 5.1 and Corollary 5.13 in *loc. cit.* give the result. □

C.7.

To get the remaining assertion that $r(a)$ is in fact 1, we use new information given by Adrian and Kaplan [**1**]. Unfortunately that information is only available presently when $F = \mathbb{Q}_2$, hence the restriction in our main result, but we expect that the computation in *loc. cit.* can be carried over to the general case. When $F = \mathbb{Q}_2$ there is only $a = 1$, so we put $\pi = \pi(1)$. In [**1**] Theorem 3.13, the authors, taking $\varpi = 2$, compute the Rankin-Selberg γ-factor $\gamma(\pi \times \tau, \psi')$ (a rational function in 2^s for a complex parameter s) for any tame character τ of $\mathbb{Q}_2^\times$ with $\tau^2 = 1$ and a character ψ' of $\mathbb{Q}_2$ trivial on $2\mathbb{Z}_2$ but not on $\mathbb{Z}_2$. They find

$$\gamma(\pi \times \tau, \psi') = \tau(2)2^{1/2-s}. \tag{C.1}$$

On the other hand if ϕ is the parameter of π, seen as a representation of W_F of dimension $2n+1$, and λ the character of W_F corresponding to τ *via* class field theory, then

$$\gamma(\pi \times \tau, \psi') = \gamma(\phi \otimes \lambda, \psi') \tag{C.2}$$

where the right-hand side is the Deligne-Langlands factor[(2)].

That gives new information on ϕ which, we recall, is by Proposition C.7 either irreducible or the direct sum of a character ω with $\omega^2 = 1$ and an irreducible representation, say α, with

$$\omega \det \alpha = 1.$$

But the factor $\gamma(\pi \times \tau, \psi')$ has no zero nor pole, so is equal to the factor $\varepsilon(\pi \times \tau, \psi') = \varepsilon(\phi \otimes \lambda, \psi')$ which has the form

$$u \cdot 2^{\mathrm{Art}(\phi\otimes\lambda)-\dim(\phi\otimes\lambda))(1/2-s)}$$

for some non-zero complex number u: the exact value of the exponent comes from the fact that ψ' is trivial on $2\mathbb{Z}_2$ but not on $\mathbb{Z}_2$. This implies that $\mathrm{Art}(\phi \otimes \lambda) = 2n + 2$, and taking λ trivial yields $\mathrm{Art}(\phi) = 2n + 2$.

Assume we are in the case where $\phi = \omega \oplus \alpha$. Taking $\lambda = \omega$ gives a pole to $\gamma(\phi \otimes \omega, \psi)$ which contradicts (C.1) if ω is tame (that is, since $F = \mathbb{Q}_2$, unramified). Thus ω is wildly ramified, so its Artin exponent is at least 2, and the Artin exponent of α is at most $2n$. That implies that α is tamely ramified, and in fact $\mathrm{Art}(\alpha) = 2n$, $\mathrm{Art}(\omega) = 2$. But then $\det \alpha$ is also tamely ramified, which contradicts $\omega \det \alpha = 1$. That contradiction shows that ϕ is irreducible, as desired.

(2) That is to say, Arthur's correspondence is compatible with Rankin-Selberg γ-factors. It can be proved by a local-global argument. Detail will appear in joint work with Oi [**26**].

C.8.

One can describe ϕ explicitly. By the main result of [**11**] an orthogonal irreducible representation of $\mathrm{W}_{\mathbb{Q}_2}$ is induced from an order 2 wildly ramified character β of W_K, where K is a totally ramified extension of $\mathbb{Q}_2$ degree $2n+1$. Such an extension is unique up to isomorphism, generated by a uniformizer z with $z^{2n+1} = 2$. Let $\widetilde{\beta}$ be the character of $K^\times$ corresponding to β *via* class field theory. Since $\mathrm{Art}(\phi) = 2n+2$, we have $\mathrm{Art}(\widetilde{\beta}) = 2$, and moreover $\det(\phi) = 1$ is the restriction of $\widetilde{\beta}$ to $\mathbb{Q}_2^\times$ times the determinant of the representation of $\mathrm{W}_{\mathbb{Q}_2}$ induced from the trivial character of W_K. That determinant is an unramified quadratic character of W_K, computed in [**9**] as the unramified character taking value at Frobenius elements the Jacobi symbol of 2 modulo $2n+1$. That imposes $\widetilde{\beta}(z)$, and with $\mathrm{Art}(\widetilde{\beta}) = 2$ and $\widetilde{\beta}(1+z) = -1$ it determines $\widetilde{\beta}$ hence β.

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