LOCAL INTERTWINING RELATIONS AND CO-TEMPERED A-PACKETS OF CLASSICAL GROUPS

HIRAKU ATOBE, WEE TECK GAN, ATSUSHI ICHINO,
TASHO KALETHA, ALBERTO MÍNGUEZ AND SUG WOO SHIN

Abstract. The local intertwining relation is an identity that gives precise information about the action of normalized intertwining operators on parabolically induced representations. We prove several instances of the local intertwining relation for quasi-split classical groups and the twisted general linear group, as they are required in the inductive proof of the endoscopic classification for quasi-split classical groups due to Arthur and Mok. In addition, we construct the co-tempered local A-packets by Aubert duality and verify their key properties by purely local means, which provide the seed cases needed as an input to the inductive proof. Together with further technical results that we establish, this makes the endoscopic classification conditional only on the validity of the twisted weighted fundamental lemma.

Contents

Introduction

The theory of automorphic forms is a profound topic in number theory with broad applications to other areas of mathematics and theoretical physics. A fundamental problem, which is central in the Langlands program, is to classify automorphic representations of connected reductive groups G over global fields and to obtain the analogous classification over local fields in terms of parameters pertaining to the Langlands L-group ${}^LG$. Such a classification should be consistent with the Langlands functoriality conjecture, whose rough form posits that a morphism of L-groups ${}^LH \to {}^LG$ should induce a functorial lifting of representations from H to G, provided that G is quasi-split. The functoriality conjecture is beyond our current technology in general, but some special cases fit in the framework of endoscopy à la Langlands further developed by Arthur, Clozel, Kottwitz, Labesse, Shelstad, and others.

When G is a quasi-split classical group, one can hope to study the classification problem for G by relating it to the general linear groups and to quasi-split classical groups of smaller rank, via (ordinary and twisted) endoscopy. This was achieved in Arthur's book [Ar2], which represents a crowning achievement in the endoscopic approach to functoriality, and builds on tremendous foundational work on the trace formula and related matters spanning multiple decades. The results of [Ar2] were later extended to quasi-split unitary groups by [Mok], following the same arguments. These results were partially extended to non-quasi-split classical groups in [KMSW] and [Ish]. Since classical groups are ubiquitous, the endoscopic classification for them has played an indispensable role in a number of arithmetic applications such as:

- new instances of the global Langlands reciprocity, see [Sc2, Section 5.1], [BCGP, Section 1.4.1], [KS1, KS2];
- the p-adic Gross-Zagier formula and the Beilinson–Bloch–Kato conjecture, see [DL], [LL], [LTXZZ];
- Euler systems, see [GS], [LSk], [LTX];
- the Gan–Gross–Prasad conjecture, the Ichino–Ikeda conjecture and their local analogues, see [W6], [BP1, BP2, BP3], [BPLZZ, Remark 1.7], [BPCZ], [BPC];
- the Sarnak–Xue density conjecture, see [DGG], [EGGG];
- an extension of the Shimura–Waldspurger correspondence, see [GI3], [Li];
- classification/counting of irreducible algebraic cuspidal automorphic representations π of GL_N or classical groups over $\mathbb{Q}$ of level one, see [CR], [Taï], [CL], [CT];
- Harder's conjecture, see [CL], [ACIKY1, ACIKY2].

The main theorems of [Ar2] and [Mok] depend on several results which were unproven but expected at the time. Some of the results have become available over the past ten years. The most notable is the stabilization of the twisted trace formula by Mœglin–Waldspurger [MW4, MW5], assuming validity of the twisted weighted fundamental lemma; the latter is as yet not available and is discussed further in Section 0.4 below. Mœglin–Waldspurger also established the local twisted trace formula in [MW6], which is one of the vital ingredients in [Ar2, Chapter 6]. The remaining issues were to be resolved in the projected papers by Arthur, which are named as [A24, A25, A26, A27] in [Ar2], at least in the symplectic and orthogonal cases. The problem to be addressed by [A24] has been solved in [MW4], whereas the other three references [A25, A26, A27] have not been treated yet. For [A24], see also Section 0.3 below. The problems to be covered by [A25] appear to be particularly challenging, and the Hecke algebra method mentioned below [Ar2, Lemma 7.1.2] leads to rather complicated calculations that are delicate even in the (supposedly simplest) case pertaining to the Iwahori Hecke algebras.

The goal of this paper is to prove all unproven assertions that [Ar2] and [Mok] rely on, apart from the twisted weighted fundamental lemma, uniformly for quasi-split symplectic, special orthogonal, and unitary groups. Our three main theorems correspond to what should be expected from [A27], [A26], and [A25], respectively, including their analogues for unitary groups. In addition we justify a few other technical results that are used in [Ar2] and [Mok] without explicit reference. It is worth noting that we develop a novel method for [A25] and [A26] based on a careful study of local intertwining operators, that is quite different from the approaches suggested by Arthur in [Ar2]. As a consequence of this paper, the main endoscopic classification for quasi-split classical groups will become unconditional as soon as the twisted weighted fundamental lemma is fully verified. Also unconditional will be the wide range of applications resting on it.

We remark that a weak global Langlands functoriality from split classical groups (as well as split general spin groups) to general linear groups was established by Cai–Friedberg–Kaplan [CFK2] via doubling constructions and the converse theorem, extending the work by Cogdell–Kim–Piatetski-Shapiro–Shahidi et al. (see [CPSS] and the references therein) in the globally generic case. Even though the trace formula method leads to more precise results that are suitable for broader applications, their theorem is unconditional and less demanding in terms of prerequisites as they avoid the trace formula. On the other hand, it is possible to deduce a weak global Langlands functoriality from (not necessarily quasi-split) classical groups to general linear groups by the trace formula in a way much softer than [Ar2] and [Mok], and conditional only on the twisted weighted fundamental lemma; see [Shi], cf. [Ar2, Proposition 9.5.2].

0.1. **Context.** Now we partially review the outline of Arthur's inductive argument in [Ar2] to put our work in context. (The same applies to [Mok] regarding unitary groups. Since the structure of [Mok] closely follows that of [Ar2], our review focuses on [Ar2].) We may organize the main theorems in [Ar2] as follows using the numbering from there.

- Local classification theorems: Theorems 1.5.1, 2.2.1, 2.2.4.
- Local intertwining relations (LIR): Theorems 2.4.1, 2.4.4.
- Global seed theorems: Theorems 1.4.1, 1.4.2.
- Global classification theorems: Theorems 1.5.2, 1.5.3.
- Global stable multiplicity formulas: Theorems 4.1.2, 4.2.2.

The local classification includes the local Langlands correspondence, construction and internal parametrization of A-packets, and the endoscopic character relations. The global seed theorems support the formalism of Arthur's global parameters for classical groups; the global classification includes Arthur's multiplicity formula as well as the dichotomy for self-dual cuspidal automorphic representations of GL_N, with N even, into symplectic and orthogonal types. The above theorems are proven all together by induction on a positive integer N for quasi-split symplectic and special orthogonal groups which are twisted endoscopic groups for GL_N.

The most crucial ingredient of the proof is the stabilization of the trace formulas for quasi-split classical groups and twisted general linear groups. Essential for its use is a precise understanding of the intertwining operators appearing in the trace formulas. This is the role of our main theorems corresponding to [A26] and [A27] (announced in [Ar2, Section 2.5]) clarifying the relationship between normalized intertwining operators and Whittaker models, as well as the part of [A25] pertaining to LIR for A-parameters.

In addition, the theorems on [A25] serve as the cornerstone for Chapter 7 of [Ar2]. Let us provide more details. Arthur obtains the local theorems for tempered representations and bounded Langlands parameters, a.k.a., tempered L-parameters, by the end of Chapter 6 of [Ar2]. Chapter 7 is devoted to the local classification and LIR for non-tempered A-parameters. Building on the local results of Chapters 6 and 7, Arthur finishes the proof of the global theorems, thereby completes the inductive argument, in Chapter 8.

Arthur's strategy in Chapter 7 broadly consists of two steps:

Step 1: Handle a certain class of A-parameters over p-adic fields by a local method.
Step 2: Prove the general case via globalization.

Step 2 propagates the results from Step 1 by carefully globalizing a given local A-parameter such that the local A-parameters at all the other places, apart from the place of interest, are essentially understood. With that said, we concentrate on Step 1 as this is what our main theorem is about.

A key input in Step 1 is Aubert duality, which is defined on local A-parameters as well as on irreducible representations of p-adic reductive groups. On A-parameters, it is induced by simply permuting the two SL_2-factors in the source group. On representations, Aubert duality is defined in terms of parabolic inductions composed with Jacquet modules; for example, the trivial representation and the Steinberg representation are Aubert-dual to each other, and each supercuspidal representation is its own Aubert dual. Aubert duality on the two sides should be compatible with each other through the local classification. This is indeed shown to be so by the end of Chapter 7 in [Ar2].

Arthur's strategy for Step 1 is to turn the tables around. Since the local theorems are known for tempered L-parameters, we can hope to construct A-packets and prove the local theorems via Aubert duality when the A-parameters are co-tempered, i.e., when they are Aubert-dual to tempered L-parameters, provided that we understand how Aubert duality interacts with the local classification and LIR. This is exactly what our third main theorem achieves.

In [Ar2] the counterpart of this theorem is stated as Lemma 7.1.2 and the penultimate paragraph on p.428, whose proof was deferred to [A25]. In fact Arthur asserted only a weaker statement for certain co-tempered A-parameters, which nevertheless suffices for Step 2 above. On the other hand, our method seems robust and optimal in that it allows us to deal with all co-tempered A-parameters.

0.2. **Results, proofs and organization.** Let us describe our results. Let F be a local field of characteristic zero with the local Langlands group L_F, and let either $G = \mathrm{GL}_N(F)$ equipped with a non-trivial pinned outer automorphism θ (which coincides with $g \mapsto {}^t g^{-1}$ up to an inner automorphism), or let G be a symplectic group $\mathrm{Sp}_{2n}(F)$ over F; for simplicity, in this introduction, we will not discuss orthogonal or unitary groups, which require more notations. See also Section 1, where we review Arthur's results and state our three main theorems in the general setting.

Fix a proper standard parabolic subgroup $P = MN_P$ of G, and an A-parameter

$$\psi_M \colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L M$$

for the Levi component M. As an induction hypothesis, we assume that we have the A-packet Π_{ψ_M}, which is a multi-set of irreducible unitary representations of M. Let w be a (twisted) Weyl element of G that preserves M, i.e., an element of $N_G(M)/M$ when G is the symplectic group, or an element of $N_{G \rtimes \theta}(M)/M$ when G is the general linear group. After fixing some auxiliary data, for $\pi_M \in \Pi_{\psi_M}$, Arthur defines a normalized intertwining operator

$$R_P(w, \pi_M, \psi_M) \colon I_P(\pi_M) \to I_P(w\pi_M),$$

where $I_P(\pi_M)$ is the normalized parabolic induction of π_M and $(w\pi_M)(m) = \pi_M(\widetilde{w}^{-1} m \widetilde{w})$ with a carefully chosen representative $\widetilde{w}$ of w (see Section 1.7 for more details). Our three main theorems concern these normalized intertwining operators.

0.2.1. *Main Theorem 1.* In the first main theorem (Theorem 1.8.1), we consider the tempered and generic case. Let $\psi_M = \phi_M$ be a tempered L-parameter, which means that $\phi_M|_{\{\mathbf{1}_{L_F}\} \times \mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, and let π_M be a generic representation lying in Π_{ϕ_M}. In this case, from a fixed Whittaker functional on π_M, one can get a Whittaker functional $\Omega(\pi_M)$ of $I_P(\pi_M)$ by the Jacquet integral (see Section 1.8). Then Theorem 1.8.1 claims that

$$\Omega(w\pi_M) \circ R_P(w, \pi_M, \phi_M) = \Omega(\pi_M)$$

if $w\pi_M \cong \pi_M$. This is [Ar2, Theorem 2.5.1 (b)], whose proof was deferred to [A27]. Note that a similar result is proven by Shahidi [Sha7], but Shahidi's definition of $R_P(w, \pi_M, \phi_M)$ is not the same as Arthur's because it uses different normalizing factors. As suggested in the proof of [Ar2, (2.5.5)], Theorem 1.8.1 will be proven by comparing Arthur's normalization factors with Shahidi's local coefficients. This is done in Section 2. In particular, if G is a classical group, we need the coincidence of Shahidi's gamma factors (constructed in terms of representations of reductive groups) with those defined by Artin, Deligne, and Langlands (constructed in terms of Galois representations). This is well-known to experts, but for completeness, we give a proof of this fact in Section A.2.

Another result, whose proof was deferred to [A27], is [Ar2, Lemma 2.5.5]. This lemma claims that the local intertwining relation (see Section 0.2.3 below for more details) for the tempered case can be reduced to a slightly weaker statement. We will show this lemma in Appendix D. Theorems 1.8.1 and D.2.1 together with the twisted endoscopic character relations for the archimedean case, which are explained in Appendix E, make the discussion of [Ar2, Chapter 6], and hence the local classification in the tempered case, conditional only on the twisted weighted fundamental lemma.

0.2.2. *Main Theorem 2.* The second main theorem concerns $G = \mathrm{GL}_N(F)$ and its automorphism θ. In this case, $\Pi_{\psi_M} = \{\pi_M\}$ is a singleton, and $I_P(\pi_M)$ is an irreducible unitary representation. Let w be a θ-twisted Weyl element such that $w\pi_M \cong \pi_M$. Then we have a normalized isomorphism $\widetilde{\pi}_M(w) \colon w\pi_M \xrightarrow{\sim} \pi_M$, whose composition with the normalized intertwining operator $R_P(w, \pi_M, \psi_M)$ discussed above produces the intertwining operator

$$\widetilde{R}_P(w, \widetilde{\pi}_M) \colon I_P(\pi_M) \to I_P(\pi_M) \circ \theta,$$

see Section 1.9. On the other hand, the assumption $w\pi_M \cong \pi_M$ implies that $I_P(\pi_M)$ is self-dual, i.e., $I_P(\pi_M) \cong I_P(\pi_M) \circ \theta$. Using a Whittaker functional on the standard module of $I_P(\pi_M)$, one can define a normalized isomorphism $\theta_A\colon I_P(\pi_M) \xrightarrow{\sim} I_P(\pi_M)\circ\theta$ (see Section 1.4). Our second main theorem (Theorem 1.9.1) asserts that

$$\widetilde{R}_P(w, \widetilde{\pi}_M) = \theta_A.$$

This is [Ar2, Theorem 2.5.3], whose proof was deferred to [A26].

The proof of Theorem 1.9.1 is given in Section 3. When π_M is tempered (and hence generic), the assertion immediately follows from the first main theorem (Theorem 1.8.1). In the non-tempered case, Arthur notes below [Ar2, Theorem 2.5.3] that it requires further techniques, based on some version of minimal K-types. While trying to follow this argument we were led to heavy calculations. Therefore, in this paper we will prove Theorem 1.9.1 in the non-tempered case with a completely different approach. One of the challenges is that the isomorphism θ_A is inexplicit, since it is defined through the Langlands quotient map from the standard module of $I_P(\pi_M)$, which is a priori known to exist only abstractly. Our novelty is to construct the standard module carefully and to realize the Langlands quotient map as a composition of normalized intertwining operators (Lemma 3.3.1). This realizes θ_A itself as an intertwining operator, and Theorem 1.9.1 is reduced to a commutativity of a certain diagram in Theorem 3.4.2, which we call the *main diagram*. However, since $I_P(\pi_M)$ is a non-tempered unitary induction, this commutativity does not follow directly from the previous results, and requires further arguments. A simple but non-trivial example for Theorem 3.4.2 is given in Example 3.4.3.

0.2.3. *Main Theorem 3.* We now discuss the results whose proofs were deferred to [A25]. They pertain to a classical group G over a non-archimedean base field F (in this introduction we are taking the example of $G = \mathrm{Sp}_{2n}(F)$), and are formulated as [Ar2, Lemma 7.1.2]. This lemma builds on the inductive assumptions that the local theorems have been proved for

- all tempered L-parameters for G; and
- all A-parameters for G' any classical group with $\mathrm{rank}(G') < \mathrm{rank}(G)$.

These inductive assumptions hold at the start of [Ar2, Chapter 7]. The statements of this lemma concern (certain) co-tempered parameters. More precisely, for an A-parameter $\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$, we define its *Aubert dual* parameter $\widehat{\psi}\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ by $\widehat{\psi}(w, g_1, g_2) = \psi(w, g_2, g_1)$. We say that ψ is *co-tempered* if its restriction to the first copy of $\mathrm{SL}_2(\mathbb{C})$ is trivial. In other words, $\phi = \widehat{\psi}$ is a tempered L-parameter.

Under the above inductive hypotheses, our third main theorem (Theorem 1.10.5) asserts that for every co-tempered A-parameter $\psi = \widehat{\phi}$ for G,

(1) we can construct an A-packet Π_ψ together with a character $\langle \cdot, \pi\rangle_\psi$ of the component group $\mathcal{S}_\psi$ assigned to each $\pi \in \Pi_\psi$ which satisfies the standard endoscopic character relations, as well as the twisted endoscopic character relations with respect to $\mathrm{GL}_N(F)$ (where $N = 2n+1$ if $G = \mathrm{Sp}_{2n}(F)$); and
(2) it also satisfies the local intertwining relation, explained further below.

Note that this is a stronger statement than what is claimed in [Ar2, Lemma 7.1.2], because it covers all co-tempered parameters, rather than the special class of tamely ramified quadratic co-tempered parameters.

Let us be a bit more precise about the statement of our theorem. The construction of the A-packet Π_ψ in (1) is actually very simple. We define

$$\Pi_\psi = \{\hat{\pi} \mid \pi \in \Pi_\phi\},$$

where $\phi = \widehat{\psi}$ is the tempered L-parameter that is the Aubert dual to ψ, Π_ϕ is the L-packet constructed in [Ar2, Chapter 6], and $\hat{\pi}$ is the Aubert dual to π (see [Au]). The problem is to prove that this definition satisfies the twisted and standard endoscopic character identities.

For the twisted character identities, we need to show that the sum of the characters of the members of the packet Π_ψ defined above is the twisted transfer of the twisted character of the representation of $\mathrm{GL}_N(F)$ associated to ψ (now viewed as a parameter for $\mathrm{GL}_N(F)$ via the standard embedding of ${}^LG$ into $\mathrm{GL}_N(\mathbb{C})$). For this we need to relate Aubert duality for the classical group G to twisted Aubert duality for $\mathrm{GL}_N(F)$. Building on the work of Hiraga [Hi], this comes down to verifying that certain signs defined in terms of Aubert duality for representations agree with corresponding signs defined in terms of Aubert duality for parameters.

For standard character identities, we need to establish a map $\pi \mapsto \langle \cdot, \pi \rangle_\psi$ between Π_ψ and the set of characters on $\mathcal{S}_\psi$ in such a way that, for each $s \in \mathcal{S}_\psi$, the virtual character $\sum_{\pi \in \Pi_\psi} \langle s \cdot s_\psi, \pi \rangle_\psi \Theta_\pi$ matches its endoscopic counterpart. By construction there is an obvious bijection $\pi \mapsto \hat{\pi}$ between Π_ψ and Π_ϕ, while at the same time we have the identity $\mathcal{S}_\psi = \mathcal{S}_\phi$. However, it is *not* true that $\langle \cdot, \pi \rangle_\psi = \langle \cdot, \hat{\pi} \rangle_\phi$, in the sense that, if we took the above identity as a definition, then the endoscopic character identities would *not* hold[1].

To see what the correct definition would be, we assume that the desired character relations hold for an arbitrary A-parameter ψ and its dual $\widehat{\psi}$, and investigate the implication of this assumption on the relationship between $\langle \cdot, \pi \rangle_\psi$ and $\langle \cdot, \hat{\pi} \rangle_{\widehat{\psi}}$ in Lemma 4.4.4 (2), where we show that the quotient of these characters can be described by certain signs $\beta(\psi_+)$, $\beta(\psi_-)$, and $\beta(\psi)$; here $\beta(\psi)$ is the sign that occurs in twisted Aubert duality on GL_N for the representation corresponding to ψ (see Section 4.1) and $\beta(\psi_+)$, $\beta(\psi_-)$ are the analogous signs for certain supplementary parameters ψ_+ and ψ_- defined in terms of ψ and the element s (see Section 1.6). While we cannot compute these signs in full generality, we do compute them for tempered representations in Proposition 4.1.3, and for a certain class of representations in Proposition 4.5.2, respectively. Hence we obtain an explicit relation between $\langle \cdot, \hat{\pi} \rangle_{\widehat{\psi}}$ and $\langle \cdot, \pi \rangle_\psi$ in Corollaries 4.4.5 and 4.5.3. Note that Proposition 4.5.2 is an application of the second main theorem (Theorem 1.9.1), and Corollary 4.5.3 will be applied in the final step of the proof of Theorem 1.10.5 (2).

[1] It was also pointed out by Liu–Lo–Shahidi [LLS] recently. Their "anti-tempered" is synonymous to "co-tempered" in this paper.

Turning the tables around (see Remark 4.4.6), we now drop the assumption that the desired character identities hold (after all, these are the identities we want to prove) and for a co-tempered A-parameter $\psi = \widehat{\phi}$, we define the character $\langle \cdot, \pi \rangle_\psi$ such that the formula obtained in Corollary 4.4.5 holds. In order to see that this is well-defined, one has to check a certain equality $(*)$ in Proposition 5.1.2. As a consequence, we can see that the A-packet Π_ψ satisfies the endoscopic character identities (Theorem 5.4.1).

Finally, we shall explain the proof of Theorem 1.10.5 (2), the local intertwining relation for co-tempered A-parameters. Our approach is entirely different from the one suggested below the statement of [Ar2, Lemma 7.1.2]. While the suggestion there was based on the theory of Hecke algebras and an expected extension of results by Morris, our initial attempts in this direction quickly led to very difficult calculations. Therefore, we developed a new approach that is based on the study of certain special representations, motivated by a result of Tadić [Tad2], as well as the study [At] of Jacquet modules for tempered L-packets by one of us, to treat the case of maximal parabolic subgroups and parameters ψ_M whose linear part is irreducible and self-dual, and then an induction procedure to generalize this to arbitrary parabolic subgroups and arbitrary co-tempered parameters.

Let $P = MN_P$ be a proper standard parabolic subgroup of $G = \mathrm{Sp}_{2n}(F)$. If ψ_M is an A-parameter for M, we denote by ψ the A-parameter for G given by ψ_M and the embedding ${}^LM \hookrightarrow {}^LG$. Then the A-packet Π_ψ is the multi-set of the irreducible components of $I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$. Moreover, if a Weyl element w of G preserves M and ψ_M, we can normalize an isomorphism $w\pi_M \xrightarrow{\sim} \pi_M$. This allows us to define for an element $u \in S_\psi = \mathrm{Cent}(\mathrm{Im}(\psi), \widehat{G})$ that normalizes $\widehat{M}$, a normalized self-intertwining operator

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) \colon I_P(\pi_M) \to I_P(\pi_M),$$

where the Weyl element w_u is determined by u. If we knew that $I_P(\pi_M)$ is multiplicity-free, then this operator, being G-equivariant, would act on each irreducible summand $\pi \subset I_P(\pi_M)$ by a scalar. The *local intertwining relation* (**LIR**) asserts that this scalar can be described by the character $\langle \cdot, \pi \rangle_\psi$ associated to π and ψ. For a more precise statement, see Section 1.10. When $\psi_M = \widehat{\phi}_M$ is co-tempered, the multiplicity-free statement follows from the corresponding statement for ϕ_M, which has been established in [Ar2, Chapter 6], and Theorem 1.10.5 (2) claims that (**LIR**) holds. We prove Theorem 1.10.5 (2) in Sections 6 and 7.

As mentioned above, the proof can be reduced by an inductive argument (Lemma 6.2.1) to the case where $P = MN_P$ is a maximal parabolic subgroup, so that $M \cong \mathrm{GL}_k(F) \times G_0$ with $G_0 = \mathrm{Sp}_{2n_0}(F)$, and the GL-part ψ_{GL} of $\psi_M = \psi_{\mathrm{GL}} \oplus \psi_0$ is irreducible and self-dual. After that, our approach to attack (**LIR**) is to extend our method for Theorem 1.9.1 from the case of $\mathrm{GL}_N(F)$ to the case of classical groups. The reason why our method works well for $\mathrm{GL}_N(F)$ is that the unitary induction $I_P(\pi_M)$ is irreducible, which implies that its Langlands data are easily described from the ones of π_M. When G is a classical group, $I_P(\pi_M)$ need not be irreducible. In Section 6.4, we isolate a certain irreducible summand of $I_P(\pi_M)$ for which our method would work, which we call a *highly non-tempered summand*. The relevance of this notion is that one can infer the Langlands data of a highly non-tempered summand of $I_P(\pi_M)$ from that of π_M, while the Langlands data for general subquotients of parabolically induced representations are mysterious. It follows from the definition that for any $\pi_M \in \Pi_{\psi_M}$, there exists a highly non-tempered summand of $I_P(\pi_M)$, and in many cases it is unique. Our method would work only for highly non-tempered summands of $I_P(\pi_M)$. Since we are assuming that P is maximal and that ψ_{GL} is irreducible, the unitary induction $I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$ is a direct sum of at most two irreducible representations. In this sense, our method has a chance to prove (**LIR**) only for "half" of the cases.

The good news is that Corollary B.3.3 tells us that "half" is enough! More precisely, if $I_P(\pi_M) = \pi_1 \oplus \pi_2$ is reducible, then this result, which says that Aubert duality commutes with the normalizing intertwining operator up to a nonzero scalar, implies that (**LIR**) for π_1 is equivalent to (**LIR**) for π_2 (see Lemma 6.3.2). Note that Corollary B.3.3 was established in an appendix in an arXiv version of [KMSW], but because it is a key input to our argument, we will move this appendix to Appendix B in this paper.

This allows us to focus on (**LIR**) for a highly non-tempered summand $\pi \subset I_P(\pi_M)$. By the definition of this summand, we have at our disposal the analogue of the *main diagram* from the $\mathrm{GL}_N(F)$ case. However, since the intertwining operator on $I_P(\pi_M)$ is normalized using the A-parameter ψ_M, whereas the other operators appearing in the main diagram use L-parameters, the main diagram is commutative only up to an explicit scalar, which is a special value of the quotient of the normalizing factors defined using the A-parameter ψ_M and the L-parameter ϕ_{π_M} of π_M. See Theorem 6.5.1. By (**LIR**) in the tempered case, we can relate this scalar with the eigenvalue of the normalized intertwining operator on π. In conclusion, (**LIR**) for the highly non-tempered summand $\pi \subset I_P(\pi_M)$ is equivalent to a certain scalar equation $(\star)$ presented in Corollary 6.5.2.

The proof is therefore reduced to establishing this scalar equation $(\star)$. Note that the left-hand side of this equation involves the L-parameter of π_M. In general, it is very difficult to completely list the L-parameters of representations in a given A-packet Π_{ψ_M}. Circumventing this problem, we give an inductive argument to show $(\star)$. The initial step is where the classical part π_0 of $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0$ is *almost supercuspidal* (see Definition 7.2.1). In this case, we will compute everything by hand in Section 7.3. For the inductive steps (Sections 7.4, 7.5 and 7.6), we use [At, Theorem 4.2] that computes the Jacquet modules of tempered representations. According to this theorem, we need to consider three cases separately. The first and second cases are amenable, whereas in the last case in Section 7.6, we have to treat an A-parameter which is beyond co-tempered, but it fits the artificial assumption in Corollary 4.5.3.

In our proof of Theorem 1.10.5, Mœglin's work on the explicit construction of A-packets (see [X2] and its references) plays a fundamental role. In particular, her results on the computation of the cuspidal support of discrete series representations and their reinterpretation in terms of enhanced L-parameters by Xu (see [X1]), as well as the extension [At] of these results to the tempered case by one of us (H.A.), are an essential tool in our argument.

However one has to be quite careful at this point. As explained in [X2, Section 8], the original arguments presented in [X2] and [At], follow a strategy that requires the validity of Arthur's results not only for the group G itself, but also for groups of higher rank. Therefore, such approach cannot be taken in the middle of a proof by induction on the rank of G, which is the situation of [Ar2, Chapter 7]. This requires us to give new proofs of some of these results, in particular [At, Theorem 4.2], which avoid appealing to groups of larger rank. We do this in Appendix C, and also extend the results to cover the case of unitary groups.

Note also that, if we could use the full extent of these results, that would easily lead to a generalization of Corollary 4.5.3 that does not use Theorem 1.9.1. But because of the inductive constraints, we are forced to prove the stated version of Corollary 4.5.3 and appeal to Theorem 1.9.1 in that proof.

0.2.4. *Supplementary results.* In addition to the main theorems, we prove a number of supplementary results in the appendices, most of which are also required for the inductive argument in [Ar2], and some of which may be of independent interest.

In Appendix A, we prove that the local Langlands correspondence for classical groups, that is established by the inductive argument of [Ar2] and [Mok], matches the Galois-theoretic local factors defined by Artin, Deligne and Langlands with the automorphic local factors defined by Shahidi. This result is used in the proof of our first main theorem (Theorem 1.8.1) and it is also of independent interest: for example, it is used in [GI1] and [GI2], where it was taken as an assumption.

In general, the matching of such local factors is a basic expectation for the local Langlands correspondence, at least in the setting in which automorphic local factors have been defined. In some constructions of the correspondence (such as those based on converse theorems), this matching is built into the construction. In the approach via endoscopy that is the subject of [Ar2] and [Mok], this matching does not follow directly from the construction. In Appendix A, we verify it for groups of the form $\mathrm{GL}_k \times G_0$, where G_0 is a classical group. In this setting, the automorphic local factors were defined by Shahidi, and we verify that they match the Galois factors defined by Artin, Deligne and Langlands.

In Appendix B, we review the Aubert involution for connected reductive groups and prove that it is compatible with intertwining operators up to a scalar. This result is used in an essential way in the proof of our third main theorem (Theorem 1.10.5). This theorem, roughly speaking, explicitly determines the scalar.

We also review a forthcoming work on the Aubert involution for disconnected reductive groups, which is also necessary in the proof of the third main theorem (Theorem 1.10.5). In order to keep this note complete and self-contained, we have included in this appendix those definitions and proofs that are required for our purposes.

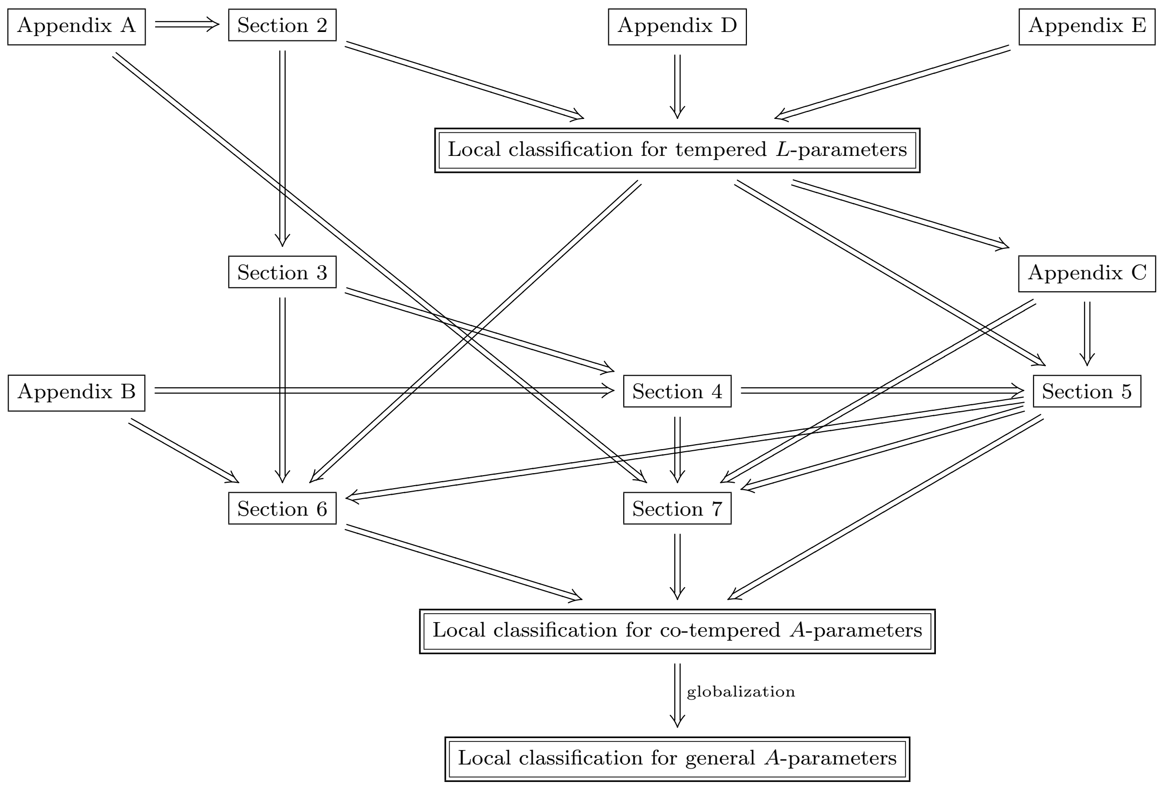
In Appendix C, we extend and reprove some results of [Mœ], [X1], and [At], to all classical groups. These results are essential in the proof of our third main theorem (Theorem 1.10.5). On the one hand, some of these results were originally formulated only for symplectic and orthogonal groups, and we take the opportunity here to formulate and prove them also for unitary groups. As already mentioned, the earlier proofs of some of these results relied on some arguments that presuppose that the endoscopic classification of representations has been established for all groups, in particular for groups whose rank is higher than the group one is interested in. Since such an assumption is problematic in the midst of a proof by induction on the rank, we present in this appendix alternative proofs that avoid this assumption.

Appendix D is devoted to the proofs of two results which are based on the same argument. The first result is the strong form of Shahidi's tempered L-packet conjecture (a strengthening of [Sha7, Conjecture 9.4]), which states that every tempered L-packet contains exactly one member that is generic with respect to a given fixed Whittaker datum $\mathfrak{w}$, and moreover that the pairing of this L-packet with the centralizer group of its parameter matches the generic constituent with the trivial character. For classical groups, the existence and uniqueness of generic constituent was proven by Varma [V2]. We prove this for general connected reductive groups under relevant assumptions, which the construction of the local Langlands correspondence in [Ar2] and [Mok] does indeed satisfy.

The second result is of a more technical nature. It is formulated in [Ar2] as Lemma 2.5.5 and is used in the inductive proof. This lemma states that a weaker version of the local intertwining relation, where a certain unknown scalar is inserted, implies the stronger version, in which this scalar equals 1. In [Ar2], this statement is formulated for classical groups and for odd residual characteristic. Building on results of Kottwitz [Kot] and Varma [V1], we are able to give a proof that works uniformly for all reductive groups and arbitrary residual characteristic.

In Appendix E, we prove certain twisted character identities over the real numbers that are assumed at various places in the inductive argument of [Ar2]. Such identities are now available in the literature, so we simply collect the required references and supply the necessary additional arguments to adapt them to the form needed in [Ar2].

0.2.5. *A roadmap.* The following is a roadmap of our paper and Arthur's results:



0.3. **On** [A24]**.** As alluded to above, the problem associated with [A24] has been resolved. The main role of [A24] is to justify [Ar2, Proposition 2.1.1], which asserts that choosing a test function on the twisted general linear group essentially amounts to choosing a family of test functions on (not necessarily elliptic) twisted endoscopic groups satisfying a certain compatibility condition. The proposition "represents part of the stabilization of the twisted trace formula" (see [Ar2, p.57, line -8]), and indeed, it has been proved in general (not only for twisted general linear groups) by Mœglin–Waldspurger [MW4, I.4.11]. Note that [A24] is used again in the proof of [Ar2, Corollary 8.4.5] to back up an implicit fact in one of Arthur's papers; this fact is covered by [MW4, I.4.11] as well.

0.4. **On the twisted weighted fundamental lemma.** As mentioned above, the stabilization of the twisted trace formula depends on the twisted weighted fundamental lemma stated as a theorem in [MW4, II.4.4], which remains conditional to the best of our knowledge; cf. the third last paragraph in the preface of [MW5]. As explained in [MW4, II.4.4], this theorem is reduced via [W4, Theorem 3.8] to

(i) the weighted fundamental lemma for Lie algebras; and
(ii) the non-standard version thereof; see [W4, Conjectures 3.6, 3.7] for the precise statements.

We should point out that (i) is already needed to stabilize the untwisted trace formula. The proof of (i) was completed for split groups by Chaudouard–Laumon [CL1, CL2]. Even though their methods are expected to generalize beyond the split case, no written account has appeared on the proof of (i) for non-split groups, or on the proof of (ii). Such a generalization is necessary for the stabilized trace formulas considered in [Ar2] and [Mok].

It is worth noting that all versions of the unweighted fundamental lemma are theorems thanks to Ngô, Waldspurger, and others; see the introduction of [N] and [LMW] for the explanation and further references, and [GWZ] and [Wa] for other proofs.

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1. Arthur's theory of the endoscopic classification

In this section, we review Arthur's theory and state our main theorems in their appropriate generality.

1.1. **Basic notation.** Fix a local field F of characteristic zero. Let E be either F or a quadratic field extension of F. We denote by $x \mapsto \overline{x}$ the generator of $\mathrm{Gal}(E/F)$. Fix a non-trivial unitary character $\psi_F \colon F \to \mathbb{C}^\times$, and put $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$. The normalized absolute value of E is denoted by $|\cdot|_E$. In particular, if E is non-archimedean, and if ϖ_E is a uniformizer of E, then $|\varpi_E|_E = q_E^{-1}$, where q_E is the cardinality of the residue field of E.

Fix an algebraic closure $\overline{F}$ of F, which contains E. The absolute Galois group of F is denoted by $\Gamma = \Gamma_F = \mathrm{Gal}(\overline{F}/F)$. Let W_E be the Weil group of E, and let

$$L_E = \begin{cases} W_E & \text{if } E \text{ is archimedean,} \\ W_E \times \mathrm{SL}_2(\mathbb{C}) & \text{if } E \text{ is non-archimedean} \end{cases}$$

be the local Langlands group of E. Let $|\cdot|_E$ be the norm map of L_E, i.e.,

$$|\cdot|_E \colon L_E \twoheadrightarrow L_E^{\mathrm{ab}} \cong E^\times \xrightarrow{|\cdot|_E} \mathbb{R}_{>0},$$

where $L_E^{\mathrm{ab}} \cong E^\times$ is the isomorphism given by the local class field theory. Here, we normalize this isomorphism such that an arithmetic Frobenius map corresponds to a uniformizer in $E^\times$.

1.2. **Groups.** In this paper, we often identify a connected reductive group over F with the group of its F-points.

Let G° be a general linear group $\mathrm{GL}_N(E)$ or one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{SO}_{2n}(F), \quad \mathrm{U}_n.$$

Here, when $G^\circ = \mathrm{U}_n$, it is an outer form of GL_n with respect to a specified quadratic extension E of F, whereas when $G^\circ = \mathrm{SO}_N(F), \mathrm{Sp}_{2n}(F)$ we simply set $E = F$. On the other hand, when $G^\circ = \mathrm{SO}_{2n}(F)$, we denote by K the splitting field of G°, which is equal to F or a quadratic extension of F. Letting η be the (possibly trivial) quadratic character of $F^\times$ associated to K/F, we sometimes write $\mathrm{SO}_{2n}(F) = \mathrm{SO}_{2n}^\eta(F)$.

Fix an F-splitting $\boldsymbol{spl} = (B^\circ, T^\circ, \{X_\alpha\}_\alpha)$ of G°. Namely, $B^\circ = T^\circ U$ is an F-rational Borel subgroup, T° is a maximal torus, U is the unipotent radical of B°, and $\{X_\alpha\}_\alpha$ is a Γ-invariant set of root vectors, where α runs over simple roots of T° with respect to B°. Then $\boldsymbol{spl}$ and ψ_F give rise to a Whittaker datum $\mathfrak{w} = (B^\circ, \chi)$, where χ is a non-degenerate character of U. For any Levi subgroup M° of G° containing T°, we take the F-splitting of M° induced by $\boldsymbol{spl}$, so that the associated Whittaker datum is given by $\mathfrak{w}_M = (B^\circ \cap M^\circ, \chi|_{U \cap M^\circ})$.

For any subgroup N of U stable under the adjoint action of T°, we take the Haar measure on N determined by $\{X_\alpha\}_\alpha$ and the self-dual Haar measure on F with respect to ψ_F.

If $G^\circ = \mathrm{SO}_{2n}^\eta(F)$, we also consider the full orthogonal group $G = \mathrm{O}_{2n}^\eta(F)$ such that G° is the connected component of $\mathbf{1} \in G$. We also abbreviate $\mathrm{O}_{2n}^\eta(F)$ as $\mathrm{O}_{2n}(F)$. Let T be the normalizer of (T°, B°) in G. Then $T/T^\circ \cong G/G^\circ$. Fix $\epsilon \in T \setminus T^\circ$ with $\epsilon^2 = \mathbf{1}$ such that ϵ preserves the splitting $\boldsymbol{spl}$. (See cf., Section A.3 below.) It gives an identification of $\mathrm{O}_{2n}(F)$ with a twisted group $\mathrm{SO}_{2n}(F) \rtimes \langle \epsilon \rangle$. If $G^\circ \neq \mathrm{SO}_{2n}(F)$, we set $G = G^\circ$ and $T = T^\circ$.

Let $P^\circ = M^\circ N$ be a parabolic subgroup of G°, where M° is a Levi component of P° and $N = N_P$ is the unipotent radical of P°. We say that P° (resp. M°) is *standard* (resp. *semi-standard*) if it contains B° (resp. T°). If P° is stable under the adjoint action of T, we set $P = P^\circ \cdot T$ and $M = M^\circ \cdot T$. Otherwise, we put $P = P^\circ$ and $M = M^\circ$. We call the subgroup of the form $P = MN$ (resp. M) a *(standard) parabolic subgroup* (resp. a *(semi-standard) Levi subgroup*) of G.

1.3. **Representations.** Let G be one of groups $\mathrm{GL}_N(E)$, $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$, $\mathrm{O}_{2n}(F)$ or U_n as in the previous subsection. We denote by $\mathrm{Rep}(G)$ the category of smooth (Fréchet) admissible complex representations of G (of moderate growth) of finite length. Here, the notions of Fréchet and moderate growth are meaningful only when F is archimedean. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible objects of $\mathrm{Rep}(G)$. The subset of $\mathrm{Irr}(G)$ consisting of irreducible unitary (resp. tempered) representations is denoted by $\mathrm{Irr}_{\mathrm{unit}}(G)$ (resp. $\mathrm{Irr}_{\mathrm{temp}}(G)$).

Let $P = MN$ be a standard parabolic subgroup of G with semi-standard Levi component M. Put $\mathfrak{a}_M = \mathrm{Hom}(\mathrm{Rat}(M), \mathbb{R})$ and $\mathfrak{a}_M^* = \mathrm{Rat}(M) \otimes \mathbb{R}$, where $\mathrm{Rat}(M)$ is the group of algebraic characters of M defined over F. Let $\mathfrak{a}_{M,\mathbb{C}}^* = \mathrm{Rat}(M) \otimes \mathbb{C}$ be the complexification of $\mathfrak{a}_M^*$. Define a homomorphism $H_M \colon M \to \mathfrak{a}_M$ by

$$|\chi(m)|_F = e^{\langle H_M(m), \chi \rangle}$$

for all $\chi \in \mathrm{Rat}(M)$ and $m \in M$, where $\langle \cdot, \cdot \rangle \colon \mathfrak{a}_M \times \mathfrak{a}_M^* \to \mathbb{R}$ is the natural pairing. For an irreducible representation π of M and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, we define a representation π_λ of M by $\pi_\lambda(m) = e^{\langle H_M(m), \lambda \rangle} \pi(m)$ realized on the space $\mathcal{V}_\pi$ of π. We denote by

$$I_P(\pi_\lambda) = \mathrm{Ind}_P^G(\pi_\lambda)$$

the associated normalized parabolically induced representation of G. As $\mathbb{C}$-vector spaces, $I_P(\pi_\lambda)$ is the space of smooth functions $f_\lambda \colon G \to \mathcal{V}_\pi$ such that

$$f_\lambda(nmg) = \delta_P^{\frac{1}{2}}(m) \pi_\lambda(m) f_\lambda(g)$$

for $n \in N$, $m \in M$ and $g \in G$. When F is archimedean, this space is a Fréchet space with some natural semi-norms. For more precision, see [Cas, Section 4].

On the other hand, if F is non-archimedean and if $(\pi, \mathcal{V}_\pi)$ is a smooth representation of G, set $\mathcal{V}_\pi(N)$ to be the subspace of $\mathcal{V}_\pi$ generated by vectors of the form $v - \pi(n)v$ for $v \in \mathcal{V}_\pi$ and $n \in N$. Define an action $\overline{\pi}$ of M on $\mathcal{V}_\pi / \mathcal{V}_\pi(N)$ by

$$\overline{\pi}(m)(v \bmod \mathcal{V}_\pi(N)) = \delta_P^{-\frac{1}{2}}(m) \pi(m) v \bmod \mathcal{V}_\pi(N).$$

The representation $(\overline{\pi}, \mathcal{V}_\pi / \mathcal{V}_\pi(N))$ of M is called the *normalized Jacquet module* of π along P, and is denoted by $\mathrm{Jac}_P(\pi)$.

1.4. **Arthur's extension of conjugate-self-dual representations.** We define an involution θ on $\mathrm{GL}_N(E)$ by

$$\theta\colon x \mapsto \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{N-1} & & \end{pmatrix} {}^t\overline{x}^{-1} \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{N-1} & & \end{pmatrix}^{-1}.$$

Set $\widetilde{\mathrm{GL}}_N(E) = \mathrm{GL}_N(E) \rtimes \langle\theta\rangle$.

Let π be an irreducible representation of $\mathrm{GL}_N(E)$. Suppose that π is *conjugate-self-dual*, i.e., $\pi \cong \pi \circ \theta$. Then there is a linear isomorphism $T\colon \pi \xrightarrow{\sim} \pi$ such that

$$T \circ \pi(g) = \pi(\theta(g)) \circ T, \quad g \in \mathrm{GL}_N(E).$$

Note that T is unique up to a nonzero scalar. We can normalize T as follows. Let $\mathcal{I}_\pi$ be the standard module of $\mathrm{GL}_N(E)$ whose Langlands quotient is π. Since $\mathcal{I}_\pi$ is $\mathfrak{w}$-generic, we can fix a nonzero $\mathfrak{w}$-Whittaker functional Ω on $\mathcal{I}_\pi$. By $\pi \cong \pi \circ \theta$, there is a unique linear isomorphism $\theta_W\colon \mathcal{I}_\pi \xrightarrow{\sim} \mathcal{I}_\pi$ satisfying the equations

$$\begin{aligned} \theta_W \circ \mathcal{I}_\pi(g) &= \mathcal{I}_\pi(\theta(g)) \circ \theta_W, \quad g \in \mathrm{GL}_N(E), \\ \Omega \circ \theta_W &= \Omega. \end{aligned}$$

Since Ω is unique up to a scalar, the definition of θ_W is independent of the choice of Ω. Then we can define a linear isomorphism $\theta_A\colon \pi \xrightarrow{\sim} \pi$ satisfying $\theta_A \circ \pi(g) = \pi(\theta(g)) \circ \theta_A$ for any $g \in \mathrm{GL}_N(E)$ and making the diagram

$$\begin{array}{ccc} \mathcal{I}_\pi & \xrightarrow{\theta_W} & \mathcal{I}_\pi \\ \downarrow & & \downarrow \\ \pi & \xrightarrow{\theta_A} & \pi \end{array}$$

commutative, where the vertical map $\mathcal{I}_\pi \to \pi$ is the (fixed) Langlands quotient map, which is unique up to a scalar. In particular, θ_A gives an extension $\widetilde{\pi} = \pi \boxtimes \theta_A$ of π to $\widetilde{\mathrm{GL}}_N(E)$. We call it *Arthur's extension* of π. Note that θ_A depends on π.

1.5. A**-parameters.** First, we consider $\mathrm{GL}_N(E)$. A *representation* of $L_E \times \mathrm{SL}_2(\mathbb{C})$ is a homomorphism

$$\psi\colon L_E \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$$

such that

- $\psi(W_E)$ consists of semisimple elements;
- $\psi|_{W_E}$ is continuous;
- $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic if E is archimedean, whereas, $\psi|_{\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})}$ is algebraic if E is non-archimedean.

An *A-parameter* for $\mathrm{GL}_N(E)$ is an equivalence class of N-dimensional representations $\psi\colon L_E \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ such that $\psi(W_E)$ is bounded. By the *local Langlands correspondence (LLC)* for $\mathrm{GL}_N(E)$ established by Langlands himself [L], Harris–Taylor

[HT], Henniart [He2], and Scholze [Sc1], we obtain an irreducible representation π_ψ of $\mathrm{GL}_N(E)$ associated to the L-parameter

$$\phi_\psi \colon L_E \ni w \mapsto \psi\left(w, \begin{pmatrix} |w|_E^{\frac{1}{2}} & 0 \\ 0 & |w|_E^{-\frac{1}{2}} \end{pmatrix}\right) \in \mathrm{GL}_N(\mathbb{C}).$$

This is a unitary representation.

If $E \neq F$, fix $s \in W_F \setminus W_E$ and set ${}^c\psi(w,\alpha) = \psi(sws^{-1},\alpha)$. We call ${}^c\psi$ the *conjugate* of ψ. When $E = F$, we simply set ${}^c\psi = \psi$. We say that ψ is *conjugate-self-dual* if $\psi \cong {}^c\psi^\vee$. In this case, π_ψ is also conjugate-self-dual, i.e., $\pi_\psi \cong \pi_\psi \circ \theta$. Let $\widetilde{\pi}_\psi = \pi_\psi \boxtimes \theta_A$ be Arthur's extension of π_ψ to $\widetilde{\mathrm{GL}}_N(E)$. We denote the character of $\widetilde{\pi}_\psi$ by $\Theta_{\widetilde{\pi}_\psi}(\widetilde{f}) = \mathrm{tr}(\widetilde{\pi}_\psi(\widetilde{f}))$ for $\widetilde{f} \in C_c^\infty(\mathrm{GL}_N(E) \rtimes \theta)$.

Next, let G be one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{O}_{2n}^\eta(F), \quad \mathrm{U}_n.$$

Set $N = 2n, 2n+1, 2n$ or n according to $G = \mathrm{SO}_{2n+1}(F), \mathrm{Sp}_{2n}(F), \mathrm{O}_{2n}^\eta(F)$ or U_n, respectively. We denote by G° the connected component of the identity of G. Then $G = G^\circ$ unless $G = \mathrm{O}_{2n}^\eta(F)$ in which case $G^\circ = \mathrm{SO}_{2n}^\eta(F)$. As explained in Section 1.2, we sometimes write $\mathrm{SO}_{2n}(F) = \mathrm{SO}_{2n}^\eta(F)$ and $\mathrm{O}_{2n}(F) = \mathrm{O}_{2n}^\eta(F)$.

Let $\Psi(G)$ be the set of equivalence classes of conjugate-self-dual representations $\psi \colon L_E \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ with

$$\mathrm{sign} = \begin{cases} -1 & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ +1 & \text{if } G = \mathrm{Sp}_{2n}(F), \mathrm{O}_{2n}(F) \\ (-1)^{n-1} & \text{if } G = \mathrm{U}_n, \end{cases}$$

$$\det(\psi) = \begin{cases} \mathbf{1} & \text{if } G = \mathrm{SO}_{2n+1}(F), \mathrm{Sp}_{2n}(F) \\ \eta & \text{if } G = \mathrm{O}_{2n}^\eta(F). \end{cases}$$

See [GGP, Section 3] for the notions of the signs of conjugate-self-dual representations.

An *A-parameter* for G° is a $\widehat{G^\circ}$-conjugacy class of L-homomorphisms

$$\psi \colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G^\circ$$

such that $\psi(W_F)$ is bounded. If $G = G^\circ$, then any A-parameter $\psi \colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ can be identified with an element of $\Psi(G)$ by $\psi \mapsto \psi|_{L_E \times \mathrm{SL}_2(\mathbb{C})}$. On the other hand, when $G = \mathrm{O}_{2n}(F)$, if we denote by $\Psi(G^\circ)$ the set A-parameters $\psi \colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G^\circ$, then there is a surjective map $\Psi(G^\circ) \to \Psi(G)$ whose fibers have order 1 or 2. We say that $\phi \in \Psi(G)$ is tempered if ϕ is trivial on the last $\mathrm{SL}_2(\mathbb{C})$, i.e., $\phi \colon L_E \to \mathrm{GL}_N(\mathbb{C})$. We denote by $\Phi_{\mathrm{temp}}(G)$ the subset of $\Psi(G)$ consisting of tempered A-parameters.

Let $\psi \in \Psi(G)$. We decompose

$$\psi = \psi_{\mathrm{bad}} \oplus \psi_{\mathrm{good}} \oplus {}^c\psi_{\mathrm{bad}}^\vee,$$

where ψ_{good} is a sum of irreducible conjugate-self-dual representations of the same type as ψ, and ψ_{bad} is a sum of irreducible representations of other types (see cf., [GGP, Section 4]). We say that ψ is *of good parity* if $\psi_{\mathrm{bad}} = 0$. In general, if we write

$$\psi_{\mathrm{good}} = \bigoplus_{i=1}^{t} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$$

with S_a the unique a-dimensional irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$, we set

$$A_\psi = \bigoplus_{i=1}^{t} \mathbb{Z}/2\mathbb{Z} e(\rho_i, a_i, b_i).$$

Namely, it is a free $\mathbb{Z}/2\mathbb{Z}$-module with a canonical basis $\{e(\rho_i, a_i, b_i)\}_{1 \leq i \leq t}$. Let A_ψ^+ be the kernel of the homomorphism

$$\det \colon A_\psi \to \mathbb{Z}/2\mathbb{Z},\ e(\rho_i, a_i, b_i) \mapsto \dim(\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}) \bmod 2.$$

Define A_ψ^0 as the subgroup of A_ψ generated by elements of the form $e(\rho_i, a_i, b_i) + e(\rho_j, a_j, b_j)$ such that $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \cong \rho_j \boxtimes S_{a_j} \boxtimes S_{b_j}$. Note that $A_\psi^0 \subset A_\psi^+$. Finally, set $z_\psi = \sum_{i=1}^{t} e(\rho_i, a_i, b_i)$.

Let $S_\psi = \mathrm{Cent}(\mathrm{Im}(\psi), \widehat{G^\circ})$ be the centralizer of $\mathrm{Im}(\psi)$ in $\widehat{G^\circ}$, and consider the component group

$$\mathcal{S}_\psi = \pi_0(S_\psi / Z(\widehat{G^\circ})^\Gamma).$$

Then we have the following.

- If $G = \mathrm{SO}_{2n+1}(F)$, then $z_\psi \in A_\psi^+ = A_\psi$. Moreover,

$$\mathcal{S}_\psi \cong A_\psi / (A_\psi^0 + \mathbb{Z}/2\mathbb{Z} z_\psi).$$

- If $G = \mathrm{Sp}_{2n}(F)$, then $z_\psi \notin A_\psi^+ \neq A_\psi$. Moreover,

$$\mathcal{S}_\psi \cong A_\psi^+ / A_\psi^0 \cong A_\psi / (A_\psi^0 + \mathbb{Z}/2\mathbb{Z} z_\psi).$$

- If $G = \mathrm{O}_{2n}(F)$, then $z_\psi \in A_\psi^+$. Moreover,

$$\mathcal{S}_\psi \cong A_\psi^+ / (A_\psi^0 + \mathbb{Z}/2\mathbb{Z} z_\psi).$$

  If we define $\widetilde{\mathcal{S}}_\psi^+$ similarly to $\mathcal{S}_\psi$ with replacing $\mathrm{Cent}(\mathrm{Im}(\psi), \widehat{G^\circ})$ with $\mathrm{Cent}(\mathrm{Im}(\psi), \mathrm{O}_{2n}(\mathbb{C}))$, then

$$\widetilde{\mathcal{S}}_\psi^+ \cong A_\psi / (A_\psi^0 + \mathbb{Z}/2\mathbb{Z} z_\psi).$$

- If $G = \mathrm{U}_n$, then

$$\mathcal{S}_\psi \cong A_\psi / (A_\psi^0 + \mathbb{Z}/2\mathbb{Z} z_\psi).$$

1.6. **A-packets via endoscopic character relations.** Set $\mathcal{A}_\psi = A_\psi/(A_\psi^0 + \mathbb{Z}/2\mathbb{Z}z_\psi)$ and denote its Pontryagin dual by $\widehat{\mathcal{A}_\psi}$. According to [Ar2, Theorems 2.2.1, 2.2.4] and [Mok, Theorem 3.2.1] for ψ, there is a multi-set Π_ψ over $\mathrm{Irr}_{\mathrm{unit}}(G)$ and a map

$$\Pi_\psi \to \widehat{\mathcal{A}_\psi},\ \pi \mapsto \langle \cdot, \pi\rangle_\psi.$$

We call Π_ψ the *A-packet* associated to ψ. By identifying $\widehat{\mathcal{A}_\psi}$ with a subgroup of $\widehat{A_\psi}$, one regards $\langle \cdot, \pi\rangle_\psi$ as a character of A_ψ which factors through the surjection $A_\psi \twoheadrightarrow \mathcal{A}_\psi$. In particular, we can consider a sign $\langle e(\rho_i, a_i, b_i), \pi\rangle_\psi \in \{\pm 1\}$ for $1 \leq i \leq t$.

The A-packet Π_ψ together with the pairing $\langle \cdot, \pi\rangle_\psi$ will be determined by the following *endoscopic character relations*.

(1) The equation

$$\textbf{(ECR1)} \qquad \Theta_{\widetilde{\pi}_\psi}(\widetilde{f}) = \frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi\rangle_\psi \Theta_\pi(f_G)$$

holds whenever $\widetilde{f} \in C_c^\infty(\mathrm{GL}_N(E) \rtimes \theta)$ and $f_G \in C_c^\infty(G^\circ)$ have matching orbital integrals, where Θ_π is the character of π, and we set

$$s_\psi = \sum_{\substack{1 \leq i \leq t \\ b_i \equiv 0 \bmod 2}} e(\rho_i, a_i, b_i).$$

(2) For $s = \sum_{i \in I_-} e(\rho_i, a_i, b_i) \in A_\psi$, set $\psi_\pm$ as

$$\psi_- = \bigoplus_{i \in I_-} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}, \quad \psi_+ = \psi - \psi_-.$$

Fix a conjugate-self-dual character $\eta_\pm$ of $E^\times$ such that there is a classical group $G_\pm$ satisfying that $\psi_\pm \otimes \eta_\pm \in \Psi(G_\pm)$. For $f_G \in C_c^\infty(G)$, taking $f_{G_\pm} \in C_c^\infty(G_\pm^\circ)$ such that
- f_G and $f_{G_+} \otimes f_{G_-}$ have matching orbital integrals;
- when $G = \mathrm{O}_{2n}(F)$ and $s \in A_\psi^+$ (resp. $s \notin A_\psi^+$), we further assume that $f_G(g) = 0$ for $g \in \mathrm{O}_{2n}(F) \setminus \mathrm{SO}_{2n}(F)$ (resp. for $g \in \mathrm{SO}_{2n}(F)$),

we define

$$f'_G(\psi, s) = \prod_{\kappa \in \{\pm\}} \frac{1}{(G_\kappa : G_\kappa^\circ)} \left(\sum_{\pi_\kappa \in \Pi_{\psi_\kappa \otimes \eta_\kappa}} \langle s_{\psi_\kappa \otimes \eta_\kappa}, \pi_\kappa\rangle_{\psi_\kappa \otimes \eta_\kappa} \Theta_{\pi_\kappa}(f_{G_\kappa}) \right).$$

Then $f'_G(\psi, s)$ is independent of the choice of $f_{G_\pm}$, and the equation

$$\textbf{(ECR2)} \qquad f'_G(\psi, s) = \frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s \cdot s_\psi, \pi\rangle_\psi \Theta_\pi(f_G)$$

holds.

Remark 1.6.1. (1) We normalize the transfer factors such that it is consistent with Arthur's Whittaker normalization. It gives a precise meaning of "matching orbital integrals".

(2) The pair of characters (η_+, η_-) determines an L-embedding

$$\xi \colon {}^L(G_+^\circ \times G_-^\circ) \hookrightarrow {}^L G^\circ,$$

and the notion of matching orbital integrals depends on (η_+, η_-) or ξ.

(3) A non-trivial pair of characters (η_+, η_-) is necessary when $G = \mathrm{Sp}_{2n}(F)$ or $G = \mathrm{U}_n$ in general. More precisely, if $G = \mathrm{Sp}_{2n}(F)$ and $\dim(\psi_\kappa) \equiv 1 \bmod 2$, we must take $\eta_\kappa = \det(\psi_\kappa)$. If $G = \mathrm{U}_n$ and $\dim(\psi_\kappa) \not\equiv n \bmod 2$, we need to choose a conjugate-symplectic character η_κ, of which there is no canonical choice.

Remark 1.6.2. In [Ar2], Arthur works exclusively with $\mathrm{SO}_{2n}(F)$, but we have elected to work with $\mathrm{O}_{2n}(F)$ in this paper. For a discussion of why this is natural and preferable, see the introduction of [AG1].

Suppose that $G = \mathrm{O}_{2n}(F)$ and $\psi \in \Psi(G)$. We shall explain how to reformulate Arthur's results in this setting.

(1) Arthur defines a packet $\widetilde{\Pi}_\psi$ over $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}_{2n}(F))/\mathrm{O}_{2n}(F)$ together with a map $\widetilde{\Pi}_\psi \to \widehat{\mathcal{S}_\psi}$, $[\pi] \mapsto \langle \cdot, [\pi] \rangle_\psi$. We set Π_ψ to be the inverse image of $\widetilde{\Pi}_\psi$ under the canonical map

$$\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}_{2n}(F)) \to \mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}_{2n}(F))/\mathrm{O}_{2n}(F)$$

obtained by taking the orbit of an irreducible component of the restriction. Then [Ar2, Theorem 2.2.4] gives a map $\Pi_\psi \to \widehat{\mathcal{A}_\psi}$ such that the diagram

$$\begin{array}{ccc} \Pi_\psi & \longrightarrow & \widehat{\mathcal{A}_\psi} \\ \big\downarrow & & \big\downarrow \\ \widetilde{\Pi}_\psi & \longrightarrow & \widehat{\mathcal{S}_\psi} \end{array}$$

is commutative. In particular, Π_ψ is stable under the determinant twist $\pi \mapsto \pi \otimes \det$.

(2) Note that Π_ψ is actually defined as a packet over $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}_{2n}(F) \rtimes \langle \mathrm{Ad}(\epsilon) \rangle)$. To fix an identification $\mathrm{O}_{2n}(F) \cong \mathrm{SO}_{2n}(F) \rtimes \langle \mathrm{Ad}(\epsilon) \rangle$, we need to choose $\epsilon \in T \setminus T^\circ$. Hence this pairing $\langle \cdot, \pi \rangle$ for $\pi \in \Pi_\psi$ depends on this choice.

(3) As in (**ECR1**), one can consider the distribution

$$\Theta_\psi(f_G) = \frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle_\psi \Theta_\pi(f_G), \quad f_G \in C_c^\infty(G).$$

On the other hand, if one denotes by $C_c^\infty(G^\circ)^\epsilon$ the subspace of $C_c^\infty(G^\circ)$ consisting of f_G° such that $f_G^\circ \circ \mathrm{Ad}(\epsilon) = f_G^\circ$, Arthur considers

$$\Theta_\psi^\circ(f_G^\circ) = \sum_{[\pi] \in \widetilde{\Pi}_\psi} \langle s_\psi, [\pi] \rangle_\psi \Theta_\pi(f_G^\circ), \quad f_G^\circ \in C_c^\infty(G^\circ)^\epsilon.$$

They are related such that the diagram

$$\begin{array}{ccc} C_c^\infty(G^\circ) & \xrightarrow{\Theta_\psi|_{C_c^\infty(G^\circ)}} & \mathbb{C} \\ \downarrow & & \| \\ C_c^\infty(G^\circ)^\epsilon & \xrightarrow{\Theta_\psi^\circ} & \mathbb{C} \end{array}$$

is commutative, where the left arrow is defined by $f_G \mapsto \frac{1}{2}(f_G + f_G \circ \mathrm{Ad}(\epsilon))$. Note that Θ_ψ is a distribution on G, but it is restricted to $C_c^\infty(G^\circ)$ in the diagram. Since $f_G \circ \mathrm{Ad}(\epsilon)$ has the same transfer as f_G, our (**ECR1**) is the same as the ECR by Arthur [Ar2, Theorem 2.2.1]. Similarly, (**ECR2**) is the same as [Ar2, Theorems 2.2.1, 2.2.4].

1.7. **Normalized intertwining operators.** For semi-standard Levi subgroups M_1° and M_2° of G°, put

$$N(M_1^\circ, M_2^\circ) = \{g \in G \,|\, gM_1^\circ g^{-1} = M_2^\circ\}.$$

The group M_1° (resp. M_2°) acts on this set by multiplication on the right (resp. the left). We consider the Weyl set

$$W(M_1^\circ, M_2^\circ) = M_2^\circ \backslash N(M_1^\circ, M_2^\circ) = N(M_1^\circ, M_2^\circ)/M_1^\circ.$$

In particular, we write $W(M_1^\circ) = W(M_1^\circ, M_1^\circ) = N_G(M_1^\circ)/M_1^\circ$. We also set $W^{G^\circ} = N_{G^\circ}(T^\circ)/T^\circ$.

For $w \in W(M_1^\circ, M_2^\circ)$, there is a unique element $w_T \in N(M_1^\circ, M_2^\circ)/T^\circ$ such that it is a lift of w, and it maps the Borel pair $(T^\circ, B^\circ \cap M_1^\circ)$ of M_1° to the Borel pair $(T^\circ, B^\circ \cap M_2^\circ)$ of M_2°. Unless $G = \mathrm{O}_{2n}(F)$ and $\det(w) = -1$, we have $w_T \in W^{G^\circ}$ and hence the the Langlands–Shelstad representative $\widetilde{w_T}$ of w_T with respect to ***spl***. See [LSh, p. 228]. We call $\widetilde{w} = \widetilde{w_T}$ the *Tits lifting* of w. If $G = \mathrm{O}_{2n}(F)$ and $\det(w) = -1$, then $w_T\epsilon^{-1} \in W^{G^\circ}$ and we have Langlands–Shelstad representative $\widetilde{w_T\epsilon^{-1}}$. Then we set $\widetilde{w} = \widetilde{w_T\epsilon^{-1}} \cdot \epsilon$ and call it the *Tits lifting* of w. Note that $\widetilde{w}$ depends on the choice of ϵ.

Let $P = MN$ and $P' = M'N'$ be standard parabolic subgroups of G with the semi-standard Levi components M and M', respectively, such that $W(M^\circ, M'^\circ) \neq \emptyset$. For $w \in W(M^\circ, M'^\circ)$, an irreducible representation π of M, and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, we define a representation $w\pi_\lambda$ of M' by $w\pi_\lambda(m') = \pi_\lambda(\widetilde{w}^{-1}m'\widetilde{w})$ realized on the space of π.

Lemma 1.7.1. *For $w \in W(M^\circ, M'^\circ)$ and $w' \in W(M'^\circ, M''^\circ)$, if we write*

$$\widetilde{w'w} = \widetilde{w}'\widetilde{w} \cdot z,$$

then z belongs to the center $Z(M)$ of M. In particular,

$$(w'w)\pi_\lambda = w'(w\pi_\lambda).$$

Proof. Since $\widetilde{w'w}$ and $\widetilde{w}'\widetilde{w}$ are two representatives of $w'w \in W(M^\circ, M''^\circ)$, we have $z \in M^\circ$. Moreover, by definition, z preserves the splitting of M°. Hence $z \in Z(M^\circ)$. This completes the proof if $Z(M^\circ) = Z(M)$. If $Z(M^\circ) \neq Z(M)$, then M is of the form

$$M = \mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_t}(F) \times \mathrm{O}_2(F).$$

In this case, by calculating

$$\widetilde{w}_\alpha = \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha)$$

for each simple root α, where w_α is the simple reflection with respect to α, one can check $z \in Z(M)$. See Section A.3 for the root vectors $\{X_\alpha\}$. □

By this lemma, for $w \in W(M^\circ, M'^\circ)$, we may write $M' = \widetilde{w}M\widetilde{w}^{-1}$ as wMw^{-1}.

Now we have an intertwining operator

$$J_P(w,\pi_\lambda)\colon I_P(\pi_\lambda) \to I_{P'}(w\pi_\lambda)$$

given by (the meromorphic continuation of) the integral

$$(J_P(w,\pi_\lambda)f_\lambda)(g) = \int_{(\widetilde{w}N\widetilde{w}^{-1}\cap N')\backslash N'} f_\lambda(\widetilde{w}^{-1}ug)du.$$

Suppose that M is a proper Levi subgroup, and we are within an inductive argument. Hence by inductive hypothesis, we have an A-packet Π_ψ for $\psi \in \Psi(M)$ (with all the desired properties) at our disposal. Let $\pi \in \Pi_\psi$. In particular, π is a unitary representation of M. Following Arthur [Ar2, Section 2.3], [Mok, Section 3.3], we define the normalized intertwining operator $R_P(w,\pi_\lambda,\psi_\lambda)$ by

$$R_P(w,\pi_\lambda,\psi_\lambda) = r_P(w,\psi_\lambda)^{-1}\cdot J_P(w,\pi_\lambda),$$

with

$$r_P(w,\psi_\lambda) = \lambda(w)\cdot\gamma_A(0,\psi_\lambda,\rho^\vee_{w^{-1}P'|P},\psi_F)^{-1}$$

where the notation is as follows.

- Put $\psi_\lambda = a_\lambda\cdot\psi$, where $a_\lambda \in Z^1(W_F, Z(\widehat{M}))$ is a 1-cocycle whose class in $H^1(W_F, Z(\widehat{M}))$ corresponds to the character $m \mapsto e^{\langle H_M(m),\lambda\rangle}$ of M (see [LMa, Lemma A.1]).
- For any finite dimensional representation ρ of ${}^LM$, we write
$$L(s,\psi,\rho) = L(s,\rho\circ\phi_\psi), \quad \varepsilon(s,\psi,\rho,\psi_F) = \varepsilon(s,\rho\circ\phi_\psi,\psi_F)$$
for the associated Artin L- and ε-factors, and
$$\gamma_A(s,\psi,\rho,\psi_F) = \varepsilon(s,\psi,\rho,\psi_F)\frac{L(1+s,\psi,\rho)}{L(s,\psi,\rho)}$$
is Arthur's modified gamma factor.
- Set $w^{-1}P' = \widetilde{w}^{-1}P'\widetilde{w}$ and write $\rho_{w^{-1}P'|P}$ for the adjoint representation of ${}^LM$ on
$$\mathrm{Ad}(\widetilde{w})^{-1}\widehat{\mathfrak{n}}'/(\mathrm{Ad}(\widetilde{w})^{-1}\widehat{\mathfrak{n}}'\cap\widehat{\mathfrak{n}})$$
with $\widehat{\mathfrak{n}} = \mathrm{Lie}(\widehat{N})$ and $\widehat{\mathfrak{n}}' = \mathrm{Lie}(\widehat{N}')$.
- Let A_{T° be the split component of T°. For a reduced root a of A_{T° in G°, we denote by G_a the associated Levi subgroup of G° of semi-simple rank 1 and by $G_{a,sc}$ the simply connected cover of the derived group of G_a. Let $\Delta_1(w)$ (resp. $\Delta_2(w)$) be the set of reduced roots a with $a>0$ and $wa<0$ (with respect to B°) such that $G_{a,sc} \cong \mathrm{Res}_{F_a/F}\mathrm{SL}_2$ (resp. $\mathrm{Res}_{F_a/F}\mathrm{SU}_{E_a/F_a}(2,1)$), where F_a is

a finite extension of F and E_a is a quadratic extension of F_a. Following [KeSh, (4.1)], we define

$$\lambda(w) = \lambda(w, \psi_F) = \prod_{a \in \Delta_1(w)} \lambda(F_a/F, \psi_F) \prod_{a \in \Delta_2(w)} \lambda(E_a/F, \psi_F)^2 \lambda(F_a/F, \psi_F)^{-1},$$

where for a finite extension F' of F, $\lambda(F'/F, \psi_F)$ is the associated Langlands λ-factor.

Recall that $R_P(w, \pi_\lambda, \psi_\lambda)$ is regular at $\lambda = 0$ ([Ar2, Proposition 2.3.1], [Mok, Proposition 3.3.1]), and hence we have a well-defined operator

$$R_P(w, \pi, \psi) \colon I_P(\pi) \to I_{P'}(w\pi).$$

When π is tempered, taking its L-parameter ϕ, we set $R_P(w, \pi_\lambda) = R_P(w, \pi_\lambda, \phi_\lambda)$.

Let $P'' = M''N''$ be another standard parabolic subgroup of G with the semi-standard Levi component M'' such that $W(M'^\circ, M''^\circ) \neq \emptyset$. Then the normalized intertwining operators satisfy the following multiplicative property.

Proposition 1.7.2. *Let π be an irreducible tempered representation of M. Then we have*

$$R_P(w'w, \pi_\lambda) = R_{P'}(w', w\pi_\lambda) \circ R_P(w, \pi_\lambda)$$

for $w \in W(M^\circ, M'^\circ)$ and $w' \in W(M'^\circ, M''^\circ)$.

Proof. When $G = \mathrm{GL}_N(E)$, the assertion was proved in [Sha3], [Ar1]. (Note that the local factors of Shahidi agree with those of Jacquet–Piatetski-Shapiro–Shalika by [Sha4] and hence with the Artin factors by the desiderata of the local Langlands correspondence.)

When G is a classical group and $M_1 = M_2 = M_3$, the assertion was proved in [Ar2, (2.3.28)], [Mok, Proposition 3.3.5] at least unless $G = \mathrm{O}_{2n}(F)$. The general case will be handled in Section A.6 below. □

The following is an important property of γ_A-factors throughout this paper.

Lemma 1.7.3. *Suppose that F is non-archimedean. Let ϕ_1 and ϕ_2 be two conjugate-self-dual representations of $W_E \times \mathrm{SL}_2(\mathbb{C})$ of dimension N. For $i = 1, 2$, define a representation λ_{ϕ_i} of W_E by*

$$\lambda_{\phi_i}(w) = \phi_i \left(w, \begin{pmatrix} |w|_E^{\frac{1}{2}} & 0 \\ 0 & |w|_E^{-\frac{1}{2}} \end{pmatrix} \right).$$

If $\lambda_{\phi_1} \cong \lambda_{\phi_2}$, then the quotient

$$\frac{\gamma_A(s, \phi_1, \psi_E)}{\gamma_A(s, \phi_2, \psi_E)}$$

is holomorphic at $s = 0$, and its special value at $s = 0$ is in $\{\pm 1\}$.

Proof. Let π_{ϕ_i} be the irreducible representation of $\mathrm{GL}_N(E)$ corresponding to ϕ_i. If $\lambda_{\phi_1} \cong \lambda_{\phi_2}$, then π_{ϕ_1} and π_{ϕ_2} share the same cuspidal support, and hence we obtain an equation of Godement–Jacquet γ-factors

$$\gamma(s, \pi_{\phi_1}, \psi_E) = \gamma(s, \pi_{\phi_2}, \psi_E).$$

Since $\gamma(s, \pi_{\phi_i}, \psi_E)$ is equal to the usual γ-factor $\gamma(s, \phi_i, \psi_E)$ in the Galois side, it is enough to consider the quotient $\gamma_A(s, \phi_i, \psi_E)/\gamma(s, \phi_i, \psi_E)$. Note that

$$\frac{\gamma_A(s, \phi_i, \psi_E)}{\gamma(s, \phi_i, \psi_E)} = \frac{L(1+s, \phi_i)}{L(1-s, \phi_i^\vee)} = \frac{L(1+s, \phi_i)}{L(1-s, {}^c\phi_i)} = \frac{L(1+s, \phi_i)}{L(1-s, \phi_i)}.$$

If we write the Laurent expansion of $L(1+s, \phi_i)$ as

$$L(1+s, \phi_i) = as^m + (\text{higher terms})$$

with $a \neq 0$, then we conclude that

$$\left.\frac{\gamma_A(s, \phi_i, \psi_E)}{\gamma(s, \phi_i, \psi_E)}\right|_{s=0} = \left.\frac{L(1+s, \phi_i)}{L(1-s, \phi_i)}\right|_{s=0} = (-1)^m.$$

This proves the lemma. □

In the rest of this section, we state three main theorems. Before doing them, let us clarify the dependence of these results.

- The first main theorem (Theorem 1.8.1) depends on results in Sections 1.6 and 1.7 for proper Levi subgroups.
- The second main theorem (Theorem 1.9.1) is for $G = \mathrm{GL}_N(E)$ so that it is independent of Section 1.6. However, it still uses results in Section 1.7, in particular, Proposition 1.7.2.
- In the third main theorem (Theorem 1.10.5), we will use results in Sections 1.6 and 1.7 not only for proper Levi subgroups, but also for other classical groups G' such that $\mathrm{rank}(G') < \mathrm{rank}(G)$. For more precision, see Hypothesis 1.10.4.

1.8. **Main Theorem 1: [A27].** We set G to be $\mathrm{GL}_N(E)$ or a (possibly disconnected) quasi-split classical group as in the previous subsection. Let $P = MN$ be a standard parabolic subgroup of G, and let $\overline{P} = M\overline{N}$ be the parabolic subgroup of G opposite to P. Recall that we obtain a Whittaker datum $\mathfrak{w}_M$ of M° from the F-splitting ***spl*** of G°.

Let π be an irreducible tempered representation of M. Suppose that π admits a non-trivial $\mathfrak{w}_M$-Whittaker functional ω. We may also regard ω as a $\mathfrak{w}_M$-Whittaker functional on π_λ for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Then ω gives rise to a $\mathfrak{w}$-Whittaker functional $\Omega(\pi_\lambda)$ on $I_P(\pi_\lambda)$ given by (the holomorphic continuation of) the Jacquet integral

$$\Omega(\pi_\lambda)f = \int_{N'} \omega(f(\widetilde{w}_0^{-1}n'))\chi(n')^{-1}dn'.$$

Here $N' = \widetilde{w}_0\overline{N}\widetilde{w}_0^{-1}$ and $w_0 = w_\ell w_\ell^M$, where w_ℓ and w_ℓ^M are the longest elements in W^{G° and W^{M°, respectively.

Let $P' = M'N$ be another standard parabolic subgroup of G, and $w \in W(M^\circ, M'^\circ)$. Then we may also regard ω as a $\mathfrak{w}_{M'}$-Whittaker functional on $w\pi_\lambda$. Hence, it gives rise to a $\mathfrak{w}$-Whittaker functional $\Omega(w\pi_\lambda)$ on $I_{P'}(w\pi_\lambda)$.

The following is our first main theorem, which was supposed to be proven in [A27].

Theorem 1.8.1 (cf. [Ar2, Theorem 2.5.1 (b)], [Mok, Proposition 3.5.3 (a)]). *Let π be an irreducible $\mathfrak{w}_M$-generic tempered representation of M.*

(1) *If $w \in W(M^\circ)$ satisfies that $w\pi \cong \pi$, then*

$$\Omega(w\pi) \circ R_P(w, \pi) = \begin{cases} \Omega(\epsilon\pi) \circ L(\epsilon) & \text{if } G = \mathrm{O}_{2n}(F),\ \det(w) = -1, \\ \Omega(\pi) & \text{otherwise.} \end{cases}$$

Here

$$L(\epsilon)\colon I_P(\pi) \to I_{\epsilon P \epsilon^{-1}}(\epsilon\pi),\ (L(\epsilon)f)(g) = f(\epsilon^{-1}g).$$

(2) *Suppose that $G = \mathrm{GL}_N(E)$. If $w \in W(M, \theta(M))$ satisfies that $w\pi \cong \pi \circ \theta$, then*

$$\Omega(w\pi) \circ R_P(w, \pi) = \Omega(\pi).$$

This theorem is regarded as the *local intertwining relation* for generic tempered representations. We will prove Theorem 1.8.1 in Section 2.

Note that the proof of [Ar2, Lemma 2.5.2] was also supposed to be included in [A27]. We will show it in Appendix D.

1.9. **Main Theorem 2: [A26].** The second main theorem concerns $G = \mathrm{GL}_N(E)$. Recall that we have an involution θ on G. For a function f on G, define a new function $\theta^*(f)$ on G by

$$\theta^*(f)(g) = f(\theta(g)).$$

Let $P = MN$ be a standard parabolic subgroup of G. Note that $\theta(P) = \theta(M)\theta(N)$ is also a standard parabolic subgroup. Let ψ be an A-parameter for M, and let π_ψ be the corresponding irreducible unitary representation of M. Suppose that the composition

$$L_E \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\psi} \widehat{M} \hookrightarrow \widehat{G}$$

is conjugate-self-dual. Then the corresponding representation is the induced representation $I_P(\pi_\psi)$, which is an irreducible unitary conjugate-self-dual representation of G. For this irreducibility, see [Ber1]. Recall from Section 1.4 that we have a specific linear isomorphism

$$\theta_A\colon I_P(\pi_\psi) \xrightarrow{\sim} I_P(\pi_\psi).$$

On the other hand, by the assumption on ψ, there is an element $w \in W(\theta(M), M)$ such that $w(\pi_\psi \circ \theta) \cong \pi_\psi$. Similar to the definition of θ_A, we have a normalized isomorphism

$$\widetilde{\pi}_\psi(w \rtimes \theta)\colon w(\pi_\psi \circ \theta) \xrightarrow{\sim} \pi_\psi$$

as representations of M by using $\mathfrak{w}_M$-Whittaker functional on the standard module $\mathcal{I}_{\pi_\psi} \cong w(\mathcal{I}_{\pi_\psi} \circ \theta)$ of M whose Langlands quotient is $\pi_\psi \cong w(\pi_\psi \circ \theta)$. Then we can define a self-intertwining operator

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi)\colon I_P(\pi_\psi) \to I_P(\pi_\psi)$$

by the composition

$$I_P(\pi_\psi) \xrightarrow{\theta^*} I_{\theta(P)}(\pi_\psi \circ \theta) \xrightarrow{R_{\theta(P)}(w,\pi_\psi\circ\theta,\psi)} I_P(w(\pi_\psi \circ \theta)) \xrightarrow{I_P(\widetilde{\pi}_\psi(w\rtimes\theta))} I_P(\pi_\psi).$$

If we write $(h \cdot f)(g) = f(gh)$ for $g, h \in G$, then we have

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi)(h \cdot f)(g) = \widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi) f(g \cdot \theta(h)).$$

Hence $\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi)$ is a constant multiple of θ_A.

The second main theorem, which was supposed to be proven in [A26], is now stated as follows.

Theorem 1.9.1 (cf. [Ar2, Theorem 2.5.3], [Mok, Proposition 3.5.1 (b)])**.** *Let $P = MN$ be a standard parabolic subgroup of $G = \mathrm{GL}_N(E)$, and let ψ be an A-parameter for M. Then for any $w \in W(\theta(M), M)$ with $w(\pi_\psi \circ \theta) \cong \pi_\psi$, we have*

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi) = \theta_A.$$

We can say that this theorem is the *twisted local intertwining relation* for $\mathrm{GL}_N(E)$. See also [Ar2, Corollary 2.5.4] for Arthur's form of local intertwining relation for twisted $\mathrm{GL}_N(E)$. We will prove Theorem 1.9.1 in Section 3.

1.10. **Main Theorem 3: [A25].** The third main theorem concerns classical groups over a non-archimedean local field.

Assume that F is a non-archimedean local field of characteristic zero. Hence $L_E = W_E \times \mathrm{SL}_2(\mathbb{C})$. For a representation $\psi\colon W_E \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$, we define its Aubert dual $\widehat{\psi}\colon W_E \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ by

$$\widehat{\psi}(w, \alpha_1, \alpha_2) = \psi(w, \alpha_2, \alpha_1).$$

We say that ψ is co-tempered if $\psi = \widehat{\phi}$ for some ϕ with $\phi|_{\{\mathbf{1}_{W_F}\}\times\{\mathbf{1}_{\mathrm{SL}_2(\mathbb{C})}\}\times\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$.

Let G be one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{O}_{2n}(F), \quad \mathrm{U}_n.$$

Fix a standard parabolic subgroup $P = MN$ of G such that $M \cong \mathrm{GL}_{k_t}(E) \times \cdots \times \mathrm{GL}_{k_1}(E) \times G_0$, where G_0 is a classical group of the same type as G. Let $\psi_M = \psi_t \oplus \cdots \oplus \psi_1 \oplus \psi_0$ be an A-parameter for M, where ψ_i (resp. ψ_0) is an A-parameter for $\mathrm{GL}_{k_i}(E)$ for $1 \leq i \leq t$ (resp. G_0). It gives the A-parameter

$$\psi = \psi_t \oplus \cdots \oplus \psi_1 \oplus \psi_0 \oplus {}^c\psi_1^\vee \oplus \cdots \oplus {}^c\psi_t^\vee$$

for G. We assume that $\psi_i = \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$ is irreducible and conjugate-self-dual.

Let $V = \mathbb{C}^N$ and decompose

$$V = V_t \oplus \cdots \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus \cdots \oplus V_{-t}$$

with a fixed isomorphism $I_i\colon V_i \xrightarrow{\sim} V_{-i}$ for $1 \leq i \leq t$. We regard ψ_i as a homomorphism $\psi_i\colon W_E \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V_i)$. Let $\mathfrak{S}_t$ be the symmetric group on $\{1, \ldots, t\}$. We identify $\sigma \in \mathfrak{S}_t$ with an element in $\mathrm{GL}(V_t \oplus \cdots \oplus V_1) \subset \widehat{M}$ with entries in $\{0, 1\}$ such that $\sigma^{-1}(\mathrm{GL}(V_t) \times \cdots \times \mathrm{GL}(V_1))\sigma = \mathrm{GL}(V_{\sigma(t)}) \times \cdots \times \mathrm{GL}(V_{\sigma(1)})$. Via $\widehat{M} \hookrightarrow \widehat{G}$, we also identify $\sigma \in \mathfrak{S}_t$ with an element of $\mathrm{GL}_N(\mathbb{C}) = \mathrm{GL}(V)$. On the other hand, for $1 \leq i \leq t$, we set

$$u_i = \begin{pmatrix} 0 & I_i \\ I_i^{-1} & 0 \end{pmatrix} \quad \text{or} \quad u_i = \begin{pmatrix} 0 & I_i \\ -I_i^{-1} & 0 \end{pmatrix}$$

in $\mathrm{GL}(V_i \oplus V_{-i})$ according to whether ψ_i is of the same type as ψ_0 or not. We regard u_i as an element in $\mathrm{GL}(V)$ by setting $u_i|_{V_j} = \mathrm{id}_{V_j}$ for $j \neq \pm i$. Then we consider

$$\mathfrak{N}_\psi = \{u_1^{\epsilon_1} \dots u_t^{\epsilon_t} \mid \epsilon_i \in \mathbb{Z}/2\mathbb{Z}\} \rtimes \{\sigma \in \mathfrak{S}_t \mid \psi_{\sigma(i)} \cong \psi_i \, (1 \leq i \leq t)\}.$$

Remark 1.10.1. When $G = \mathrm{O}_{2n}(F)$ and $k_i = \dim(V_i)$ is odd, since ψ_i is an irreducible self-dual representation of dimension k_i, it must be of orthogonal type. Hence u_i is always $\begin{pmatrix} 0 & I_i \\ I_i^{-1} & 0 \end{pmatrix}$ in $\mathrm{GL}(V_i \oplus V_{-i})$ so that $\det(u) = -1$.

For $u = u_1^{\epsilon_1} \dots u_t^{\epsilon_t} \sigma \in \mathfrak{N}_\psi$, by a similar definition, we obtain

$$w_u = w_{u_1}^{\epsilon_1} \dots w_{u_t}^{\epsilon_t} w_\sigma \in W(M^\circ).$$

It satisfies that $w_u \pi_M \cong \pi_M$ for any $\pi_M \in \Pi_\psi$. Moreover, as in [Ar2, Section 2.4] and [Mok, Section 3.4], there is a linear isomorphism

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u) \colon \pi_M \to \pi_M$$

making the diagram

$$\begin{array}{ccc} \pi_M & \xrightarrow{\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)} & \pi_M \\ {\scriptstyle \pi_M(\widetilde{w_u}^{-1} m \widetilde{w_u})} \downarrow & & \downarrow {\scriptstyle \pi_M(m)} \\ \pi_M & \xrightarrow{\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)} & \pi_M \end{array}$$

commutative for any $m \in M$. In this paper, we understand that the symbol $\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)$ denotes this map, and we do not separate it into two objects $\langle \widetilde{u}, \widetilde{\pi}_M \rangle$ and $\widetilde{\pi}_M(w_u)$.

Now for $u = u_1^{\epsilon_1} \dots u_t^{\epsilon_t} \sigma \in \mathfrak{N}_\psi$, we define the normalized self-intertwining operator

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) \colon I_P(\pi_M) \to I_P(\pi_M)$$

by

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) f(g) = \langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u) \left(R_P(w_u, \pi_M, \psi_M) f(g) \right).$$

On the other hand, let I_u be the set of $1 \leq i \leq t$ such that $\epsilon_i = 1$ and ϕ_i is of the same type as ϕ_0. Then we set

$$s_u = \sum_{i \in I_u} e(\rho_i, a_i, b_i) \in A_\psi.$$

Now we assume that the (multi-)set Π_ψ of irreducible components of $I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$ is equipped with a pairing $\langle \cdot, \pi \rangle_\psi$ satisfying (**ECR1**), (**ECR2**) and that

$$\langle \cdot, \pi \rangle_\psi |_{A_{\psi_0}} = \langle \cdot, \pi_0 \rangle_{\psi_0}$$

if $\pi \subset I_P(\pi_M)$ where $\pi_M = \pi_{\psi_t} \boxtimes \cdots \boxtimes \pi_{\psi_1} \boxtimes \pi_0$ with $\pi_0 \in \Pi_{\psi_0}$. See [Ar2, Proposition 2.4.3] and [Mok, Proposition 3.4.4]. Define a distribution $f_G(\psi, u)$ on G for $u \in \mathfrak{N}_\psi$ by

$$f_G(\psi, u) = \frac{1}{(G : G^\circ)} \sum_{\pi_M \in \Pi_{\psi_M}} \mathrm{tr}(\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) I_P(\pi_M, f))$$

for $f \in C_c^\infty(G)$. Then Arthur's *local intertwining relation* ([Ar2, Theorems 2.4.1, 2.4.4], [Mok, Theorem 3.4.3]) states that the equation

$$\textbf{(A-LIR)} \qquad f'_G(\psi, s_\psi s_u) = f_G(\psi, u)$$

holds for $u \in \mathfrak{N}_\psi$, where the left hand-side is defined in (**ECR2**) in Section 1.6. Notice that if $G = \mathrm{O}_{2n}(F)$, then $\widetilde{w}_u$ can be in $G \setminus G^\circ$, in which case, (**A-LIR**) is [Ar2, Theorem 2.4.4].

On the other hand, in this paper, we consider the following statement for our *local intertwining relation*. Fix an irreducible summand $\pi \subset I_P(\pi_M)$. Then the equation

$$\textbf{(LIR)} \qquad \langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)|_\pi = \langle s_u, \pi \rangle_\psi \cdot \mathrm{id}_\pi$$

holds for any $u \in \mathfrak{N}_\psi$. Notice that this statement is slightly different from (**A-LIR**). We clarify the relation between (**A-LIR**) and our (**LIR**).

Lemma 1.10.2. *Assume the existence of the A-packet Π_ψ together with the pairing $\langle \cdot, \pi \rangle_\psi$ satisfying* (**ECR1**) *and* (**ECR2**). *We assume further that we know that*

- *$I_P(\pi_M)$ is multiplicity-free for any $\pi_M \in \Pi_{\psi_M}$; and*
- *for $\pi_M, \pi'_M \in \Pi_{\psi_M}$, if $\pi_M \not\cong \pi'_M$, then $I_P(\pi_M)$ and $I_P(\pi'_M)$ have no common irreducible summand.*

Then (**A-LIR**) *holds if and only if our* (**LIR**) *holds for each irreducible summand $\pi \subset I_P(\pi_M)$ and $\pi_M \in \Pi_{\psi_M}$.*

Proof. Note that we do not assume whether Π_{ψ_M} is multiplicity-free. Only in this proof, we denote the canonical map $\Pi_\psi \to \mathrm{Irr}_{\mathrm{unit}}(G)$ by $\pi \mapsto [\pi]$, and the multiplicity of $\sigma \in \mathrm{Irr}_{\mathrm{unit}}(G)$ in Π_ψ, i.e, the cardinality of the fiber of σ under this map, by $m_\psi(\sigma)$. By our assumptions, for any $\sigma \in \mathrm{Irr}(G)$ with $m_\psi(\sigma) > 0$, there exists exactly one $\sigma_M \in \mathrm{Irr}(M)$ such that $\sigma \subset I_P(\sigma_M)$ and $m_{\psi_M}(\sigma_M) > 0$. Moreover, $m_\psi(\sigma) = m_{\psi_M}(\sigma_M)$.

Fix $u \in \mathfrak{N}_\psi$. By (**ECR2**), $f'_G(\psi, s_\psi s_u)$ is equal to

$$\frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s_u, \pi \rangle_\psi \Theta_\pi(f_G) = \frac{1}{(G : G^\circ)} \sum_{\substack{\sigma \in \mathrm{Irr}(G) \\ m_\psi(\sigma) > 0}} \left(\sum_{\substack{\pi \in \Pi_\psi \\ [\pi] = \sigma}} \langle s_u, \pi \rangle_\psi \right) \Theta_\sigma(f_G).$$

If $\pi \subset I_P(\pi_M)$, since we assume that π appears in $I_P(\pi_M)$ with multiplicity one, $\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)$ acts on π by a scalar $c_{[\pi]}$, which depends only on $[\pi]$ and $[\pi_M]$.

Then one can write $f_G(\psi, u)$ as

$$f_G(\psi,u) = \frac{1}{(G:G^\circ)} \sum_{\sigma\in\mathrm{Irr}(G)} \left(\sum_{\substack{\pi_M\in\Pi_{\psi_M} \\ \exists\pi\subset I_P(\pi_M),[\pi]=\sigma}} c_{[\pi]} \right) \Theta_\sigma(f_G)$$

$$= \frac{1}{(G:G^\circ)} \sum_{\substack{\sigma\in\mathrm{Irr}(G) \\ m_\psi(\sigma)>0}} \left(\sum_{\substack{\pi_M\in\Pi_{\psi_M} \\ [\pi_M]=\sigma_M}} c_\sigma \right) \Theta_\sigma(f_G)$$

$$= \frac{1}{(G:G^\circ)} \sum_{\substack{\sigma\in\mathrm{Irr}(G) \\ m_\psi(\sigma)>0}} m_\psi(\sigma) c_\sigma \Theta_\sigma(f_G).$$

By the linear independence of the characters Θ_σ, (**A-LIR**) holds if and only if

$$c_\sigma = \frac{1}{m_\psi(\sigma)} \sum_{\substack{\pi\in\Pi_\psi \\ [\pi]=\sigma}} \langle s_u, \pi\rangle_\psi.$$

Note that $\langle s_u, \pi\rangle_\psi \in \{\pm 1\}$, whereas $c_{[\pi]}$ is a root of unity since

$$\mathfrak{N}_\psi \ni u \mapsto \langle \widetilde{u}, \widetilde{\pi}_M\rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)$$

is multiplicative by Proposition 1.7.2 and [KMSW, Lemma 2.5.3] (see also the paragraph in [Ar2] containing (2.4.2)). Hence the above equation holds if and only if $\langle s_u, \pi\rangle_\psi = c_\sigma$ for all $\pi \in \Pi_\psi$ with $[\pi] = \sigma$. This is our (**LIR**). □

Remark 1.10.3. The multiplicity-free assumptions can be proven in generality using Mœglin's explicit construction of A-packets. However, when ψ_M is tempered or co-tempered, one can argue more directly as follows. The co-tempered case will be deduced from the tempered case by the construction (see Theorem 5.4.1), and for the tempered case, the multiplicity-free assumptions are included in [Ar2, Theorem 1.5.1 (b)] and [Mok, Theorem 2.5.1 (b)]. In Section 7.6 below, we will need the multiplicity-free assumptions for a little more general parameter, which we will prove in Lemma 7.6.1. In this paper, except for Section 7.6, we may identify Arthur's LIR with our (**LIR**).

In light of the inductive setting as in [Ar2, Chapter 7] in which [A25] is positioned, as explained in Sections 0.1 and 0.2.3, we are free to assume the following.

Hypothesis 1.10.4. There are A-packets satisfying (**ECR1**), (**ECR2**) and (**A-LIR**) associated to

- all tempered L-parameters for G;
- all A-parameters for G' with G' any classical group such that $\mathrm{rank}(G') < \mathrm{rank}(G)$.

In particular, we have the A-packet Π_{ψ_M} for $\psi_M \in \Psi(M)$ and for any proper Levi subgroup M of G.

We state the third main theorem, which was supposed to be proven in [A25].

Theorem 1.10.5 (cf. [Ar2, Section 7.1], [Mok, Section 8.2])**.** *Assume Hypothesis 1.10.4.*

(1) *For any co-tempered A-parameter $\psi = \widehat{\phi} \in \Psi(G)$, we can construct an A-packet Π_ψ together with a pairing $\langle \cdot, \pi \rangle_\psi$ for $\pi \in \Pi_\psi$ which satisfies* (**ECR1**) *and* (**ECR2**)*. Moreover, Π_ψ is a (multiplicity-free) subset of* $\mathrm{Irr}(G)$*.*
(2) *Let $P = MN$ be a parabolic subgroup of G with $M \cong \mathrm{GL}_{k_t}(E) \times \cdots \times \mathrm{GL}_{k_1}(E) \times G_0$, and let $\psi_M = \widehat{\phi}_M = \psi_t \oplus \cdots \oplus \psi_1 \oplus \psi_0$ be a co-tempered A-parameter for M such that ψ_i is irreducible and conjugate-self-dual for $1 \leq i \leq t$. Set $\psi = \iota \circ \psi_M \in \Psi(G)$ with $\iota \colon {}^L M \hookrightarrow {}^L G$. Then* (**LIR**) *also holds for every irreducible summand $\pi \subset I_P(\pi_M)$ for any $\pi_M \in \Pi_{\psi_M}$.*

Remark 1.10.6. The A-packet Π_ψ for a given co-tempered A-parameter $\psi = \widehat{\phi}$ is constructed by Aubert duality from the tempered L-packet Π_ϕ. To be precise, see Theorem 5.4.1 below. At this stage, it is not known that each representation $\pi \in \Pi_\psi$ is unitary. The unitarity will be proven after establishing [Ar2, Proposition 7.4.3] and [Mok, Proposition 8.4.2].

For co-tempered A-parameters we will establish (**ECR1**) and (**ECR2**) in Section 5, whereas (**LIR**) will be proven in Sections 6 and 7 using results in Section 4. As a consequence of Theorem 1.10.5 together with Lemma 1.10.2, we have the following.

Corollary 1.10.7. *Assume Hypothesis 1.10.4. For any co-tempered A-parameter $\psi \in \Psi(G)$ and for any $u \in \mathfrak{N}_\psi$, we have an identity*

$$f'_G(\psi, s_\psi s_u) = f_G(\psi, u)$$

of distributions on G.

2. Normalizations of intertwining operators

The purpose of this section is to prove Theorem 1.8.1. Notice that almost the same assertion was already proven by Shahidi [Sha7], but he used his own normalization of the intertwining operators. So what we have to do is to compare Arthur's normalization with Shahidi's.

2.1. Local coefficients.

Let $\mathfrak{w} = (B^\circ, \chi)$ be a Whittaker datum for G°. We say that an irreducible representation π° of G° is $\mathfrak{w}$-generic if

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi^\circ, \chi) \neq 0,$$

in which case

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi^\circ, \chi) = 1$$

by the uniqueness of Whittaker functionals. For $G = \mathrm{O}_{2n}(F)$, we also say that an irreducible representation π of G is $\mathfrak{w}$-generic if

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi, \chi) \neq 0,$$

but as we see in the next lemma, the uniqueness of Whittaker functionals does not necessarily hold. Recall that we have chosen an element $\epsilon \in T \setminus T^\circ \subset G \setminus G^\circ$ with $\epsilon^2 = \mathbf{1}$ such that ϵ preserves $\boldsymbol{spl}$. In particular, $\chi \circ \mathrm{Ad}(\epsilon) = \chi$. For any representation π° of G°, we define a representation $\epsilon\pi^\circ$ of G° by $\epsilon\pi^\circ(g) = \pi^\circ(\epsilon^{-1} g \epsilon)$.

Lemma 2.1.1. *Suppose that $G = \mathrm{O}_{2n}(F)$. Let π be an irreducible $\mathfrak{w}$-generic representation of G.*

(a) *If $\pi|_{G^\circ}$ is irreducible, then $\pi|_{G^\circ}$ is $\mathfrak{w}$-generic and we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi, \chi) = 1.$$

(b) *If $\pi|_{G^\circ}$ is reducible, then any irreducible component of $\pi|_{G^\circ}$ is $\mathfrak{w}$-generic and we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi, \chi) = 2.$$

Proof. Since

$$\mathrm{Hom}_U(\pi, \chi) = \mathrm{Hom}_U(\pi|_{G^\circ}, \chi),$$

the assertion (a) follows.

Assume that $\pi|_{G^\circ}$ is reducible. Let π° be an irreducible component of $\pi|_{G^\circ}$. Then we have $\pi|_{G^\circ} \cong \pi^\circ \oplus \epsilon\pi^\circ$ and

$$\mathrm{Hom}_U(\pi, \chi) \cong \mathrm{Hom}_U(\pi^\circ, \chi) \oplus \mathrm{Hom}_U(\epsilon\pi^\circ, \chi).$$

Since $\chi \circ \mathrm{Ad}(\epsilon) = \chi$, we see that π° is $\mathfrak{w}$-generic if and only if $\epsilon\pi^\circ$ is $\mathfrak{w}$-generic. This implies the assertion (b). □

Remark 2.1.2. In Lemma 2.1.1 (b), the multiplicity two statement may be a bit disconcerting to the readers. While for connected reductive groups G°, the stabilizer of a Whittaker datum is the center of G°, in the case of $G = \mathrm{O}_{2n}(F)$, the stabilizer of a Whittaker datum contains an extra group, generated by ϵ. This group $\langle \epsilon \rangle$ acts on $\mathrm{Hom}_U(\pi, \chi)$ with two possible eigenvalues and each eigenspace has dimension at most 1. In other words, the multiplicity one statement is restored if we take into account the full symmetry of the situation.

Let $P^\circ = M^\circ N$ be a standard parabolic subgroup of G°. The heredity of Whittaker functionals asserts that if π° is an irreducible essentially unitary representation of M°, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ), \chi) = \dim_{\mathbb{C}} \mathrm{Hom}_{U \cap M^\circ}(\pi^\circ, \chi).$$

(See [Rod], [CS, Corollary 1.7], [Wl1, Theorem 15.6.7], [Wl2, Theorem 40]; note that this equality holds for all admissible representations π° when F is non-archimedean.) In particular, this dimension is at most 1.

For $G = \mathrm{O}_{2n}(F)$, we modify this property as follows.

Lemma 2.1.3. *Let $G = \mathrm{O}_{2n}(F)$, and let $P = MN$ be a standard parabolic subgroup of G. Set $P^\circ = P \cap G^\circ$ and $M^\circ = M \cap G^\circ$, so that $P = P^\circ \iff M = M^\circ$. Let π be an irreducible essentially unitary representation of M.*

(a) *If $M \neq M^\circ$, and $\pi|_{M^\circ}$ is irreducible, then*

$$I_P(\pi)|_{G^\circ} \cong \mathrm{Ind}_{P^\circ}^{G^\circ}(\pi|_{M^\circ}).$$

Moreover,

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(I_P(\pi), \chi) = \dim_{\mathbb{C}} \mathrm{Hom}_{U\cap M}(\pi, \chi) \leq 1.$$

(b) *If $M = M^\circ$, or $\pi|_{M^\circ}$ is reducible, then for any irreducible component π° of $\pi|_{M^\circ}$, we have*

$$I_P(\pi) \cong I_{P^\circ}(\pi^\circ) = \mathrm{Ind}_{P^\circ}^{G}(\pi^\circ).$$

Moreover,

$$\frac{1}{(G : G^\circ)} \dim_{\mathbb{C}} \mathrm{Hom}_U(I_P(\pi), \chi) = \frac{1}{(M : M^\circ)} \dim_{\mathbb{C}} \mathrm{Hom}_{U\cap M}(\pi, \chi) \leq 1.$$

(Note that the left-hand side is an integer.)

Proof. Suppose that $P \neq P^\circ$. Then $G/G^\circ \cong M/M^\circ$, and the restriction map gives an isomorphism

$$\mathrm{Ind}_P^G(\pi)|_{G^\circ} \xrightarrow{\sim} \mathrm{Ind}_{P^\circ}^{G^\circ}(\pi|_{M^\circ}).$$

This implies the assertion (a).

On the other hand, if $P = P^\circ$, or if $\pi|_{M^\circ}$ is reducible, for any irreducible component π° of $\pi|_{M^\circ}$, we have $\pi \cong \mathrm{Ind}_{M^\circ}^M(\pi^\circ)$. Then $I_P(\pi) \cong I_{P^\circ}(\pi^\circ)$. If we denote by $I_{P^\circ}^+(\pi^\circ)$ (resp. $I_{P^\circ}^-(\pi^\circ)$) the subspace of $I_{P^\circ}(\pi^\circ)$ consisting of functions f on G whose supports are contained in G° (resp. $G \setminus G^\circ$), then $I_{P^\circ}(\pi^\circ)|_{G^\circ} = I_{P^\circ}^+(\pi^\circ) \oplus I_{P^\circ}^-(\pi^\circ)$. Moreover, we have isomorphisms

$$I_{P^\circ}^+(\pi^\circ) \xrightarrow{\sim} \mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ),\ f \mapsto f|_{G^\circ},$$
$$I_{P^\circ}^-(\pi^\circ) \xrightarrow{\sim} \epsilon\mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ),\ f \mapsto (\epsilon^{-1} f)|_{G^\circ}$$

as representations of G°, where $(\epsilon^{-1}f)(x) = f(x\epsilon^{-1})$. In particular, we have

$$I_P(\pi)|_{G^\circ} \cong \mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ) \oplus \epsilon\mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ).$$

Since $\chi \circ \mathrm{Ad}(\epsilon) = \chi$, the following are equivalent:

- $\mathrm{Hom}_{U\cap M}(\pi, \chi) \neq 0$;
- $\mathrm{Hom}_{U\cap M^\circ}(\pi^\circ, \chi) \neq 0$;
- $\dim_{\mathbb{C}} \mathrm{Hom}_U(\mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ), \chi) \neq 0$;
- $\dim_{\mathbb{C}} \mathrm{Hom}_U(\epsilon\mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ), \chi) \neq 0$.

Hence, in this case, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{U\cap M}(\pi, \chi) = (M : M^\circ), \quad \dim_{\mathbb{C}} \mathrm{Hom}_U(I_P(\pi), \chi) = 2.$$

This completes the proof. □

Fix two standard parabolic subgroups $P = MN_P$ and $P' = M'N_{P'}$ of G such that $W(M^\circ, M'^\circ) \neq \emptyset$, an element $w \in W(M^\circ, M'^\circ)$, and an irreducible unitary $\mathfrak{w}_M$-generic representation π of M. Choose a non-trivial $\mathfrak{w}_M$-Whittaker functional ω on π. For $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, define a $\mathfrak{w}$-Whittaker functional $\Omega(\pi_\lambda) = \Omega_\omega(\pi_\lambda)$ on $I_P(\pi_\lambda)$ as in Section 1.8. Then $\Omega(\pi_\lambda)$ is holomorphic and nonzero for all λ (see [CS], [Wl1, Theorem 15.6.7], [Wl2, Theorem 40]). Similarly, define a $\mathfrak{w}$-Whittaker functional $\Omega(w\pi_\lambda) = \Omega_\omega(w\pi_\lambda)$ on $I_{P'}(w\pi_\lambda)$, where we regard ω as a $\mathfrak{w}_{M'}$-Whittaker functional on $w\pi$. When $G = G^\circ$, following Shahidi [Sha2, p. 333, Theorem 3.1], we define a meromorphic function $C_P(w, \pi_\lambda)$ of λ, called the *local coefficient*, such that

$$C_P(w, \pi_\lambda) \cdot \Omega_\omega(w\pi_\lambda) \circ J_P(w, \pi_\lambda) = \Omega_\omega(\pi_\lambda).$$

Note that such a function exists and does not depend on the choice of ω by the uniqueness of Whittaker functionals.

When $G = \mathrm{O}_{2n}(F)$, this uniqueness may fail, but we can define an analogous function as follows.

Lemma 2.1.4. *Suppose that $G = \mathrm{O}_{2n}(F)$. Then there exists a meromorphic function $C_P(w, \pi_\lambda)$ of λ such that*

$$\begin{aligned} &C_P(w, \pi_\lambda) \cdot \Omega_\omega(w\pi_\lambda) \circ J_P(w, \pi_\lambda) \\ &= \begin{cases} \Omega_\omega(\pi_\lambda) & \text{if } \det(w) = 1, \\ \Omega_\omega(\epsilon\pi_\lambda) \circ L(\epsilon) & \text{if } \det(w) = -1, \end{cases} \end{aligned}$$

where $L(\epsilon)\colon I_P(\pi_\lambda) \to I_{\epsilon P \epsilon^{-1}}(\epsilon\pi_\lambda)$ is given by $(L(\epsilon)f)(g) = f(\epsilon^{-1}g)$. Moreover, $C_P(w, \pi_\lambda)$ does not depend on the choice of ω.

To prove this, we need the following.

Lemma 2.1.5. *Suppose that $G^\circ = \mathrm{SO}_{2n}(F)$, $P \neq P^\circ$ (and hence $\epsilon P^\circ \epsilon^{-1} = P^\circ$), and* $\det(w) = 1$. *Let π° be an irreducible unitary $\mathfrak{w}_M$-generic representation of M°. Then we have*

$$C_{P^\circ}(w, \epsilon\pi^\circ_\lambda) = C_{P^\circ}(w, \pi^\circ_\lambda).$$

Proof. Since $P \neq P^\circ$, we have

$$\epsilon N_P \epsilon^{-1} = N_P, \quad \epsilon N_{P'} \epsilon^{-1} = N_{P'}, \quad \epsilon w \epsilon^{-1} = w, \quad \epsilon w_0 \epsilon^{-1} = w_0$$

with $w_0 = w_\ell w_\ell^M$, where w_ℓ and w_ℓ^M are the longest elements in W^{G° and W^{M°, respectively. Fix a non-trivial $\mathfrak{w}_M$-Whittaker functional ω° on π°. Since $\chi \circ \mathrm{Ad}(\epsilon) = \chi$, we may regard ω° as a $\mathfrak{w}_M$-Whittaker functional on $\epsilon\pi^\circ$. Then ω° induces $\mathfrak{w}$-Whittaker functionals $\Omega(\pi^\circ_\lambda)$ and $\Omega(\epsilon\pi^\circ_\lambda)$ on $\mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ_\lambda)$ and $\mathrm{Ind}_{P^\circ}^{G^\circ}(\epsilon\pi^\circ_\lambda)$, respectively. By definition, we have

$$\Omega(\epsilon\pi^\circ_\lambda) = \Omega(\pi^\circ_\lambda) \circ \mathrm{Ad}(\epsilon)^*,$$

where $\mathrm{Ad}(\epsilon)^* : \mathrm{Ind}_{P^\circ}^{G^\circ}(\epsilon\pi^\circ_\lambda) \to \mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ_\lambda)$ is the linear isomorphism given by $\mathrm{Ad}(\epsilon)^* f(g) = f(\epsilon g \epsilon^{-1})$. Similarly, we have

$$J(w, \pi^\circ_\lambda) \circ \mathrm{Ad}(\epsilon)^* = \mathrm{Ad}(\epsilon)^* \circ J(w, \epsilon\pi^\circ_\lambda).$$

Hence we have

$$\begin{aligned}\Omega(\epsilon\pi_\lambda^\circ) &= \Omega(\pi_\lambda^\circ) \circ \mathrm{Ad}(\epsilon)^* \\ &= C_{P^\circ}(w,\pi_\lambda^\circ)\cdot\Omega(w\pi_\lambda^\circ)\circ J_{P^\circ}(w,\pi_\lambda^\circ)\circ\mathrm{Ad}(\epsilon)^* \\ &= C_{P^\circ}(w,\pi_\lambda^\circ)\cdot\Omega(w\pi_\lambda^\circ)\circ\mathrm{Ad}(\epsilon)^*\circ J_{P^\circ}(w,\epsilon\pi_\lambda^\circ) \\ &= C_{P^\circ}(w,\pi_\lambda^\circ)\cdot\Omega(\epsilon w\pi_\lambda^\circ)\circ J_{P^\circ}(w,\epsilon\pi_\lambda^\circ) \\ &= C_{P^\circ}(w,\pi_\lambda^\circ)\cdot\Omega(w\epsilon\pi_\lambda^\circ)\circ J_{P^\circ}(w,\epsilon\pi_\lambda^\circ).\end{aligned}$$

This implies the lemma. □

Now we prove Lemma 2.1.4.

Proof of Lemma 2.1.4. First, we assume that $M \neq M^\circ$, and $\pi|_{M^\circ}$ is irreducible. Then the existence of $C_P(w,\pi_\lambda)$ follows from Lemma 2.1.3 (a). Moreover, since ω is unique up to a scalar, $C_P(w,\pi_\lambda)$ does not depend on the choice of ω.

Next, we assume that $M = M^\circ$, or $\pi|_{M^\circ}$ is reducible. Fix an irreducible component π° of $\pi|_{M^\circ}$. Note that π° is $\mathfrak{w}_M$-generic. If $M = M^\circ$, then ω is unique up to a scalar. If $M \neq M^\circ$ (so that $\epsilon \in M \setminus M^\circ$ and $\pi|_{M^\circ} = \pi^\circ \oplus \epsilon\pi^\circ$), then by Lemma 2.1.1 (b), we may take a basis ω, ω' of $\mathrm{Hom}_{U\cap M}(\pi,\chi)$ such that $\omega|_{\epsilon\pi^\circ} = 0$, $\omega'|_{\pi^\circ} = 0$. We identify π with $\mathrm{Ind}_{M^\circ}^M(\pi^\circ)$, so that π° (resp. $\epsilon\pi^\circ$) is the subspace of $\mathrm{Ind}_{M^\circ}^M(\pi^\circ)$ consisting of functions f on M whose supports are contained in M° (resp. $M \setminus M^\circ$). Then ω can be realized by $\omega(f) = \omega^\circ(f(\mathbf{1}))$ for $f \in \mathrm{Ind}_{M^\circ}^M(\pi^\circ)$, where ω° is a non-trivial $\mathfrak{w}_M$-Whittaker functional on π°. In both cases, we have $I_P(\pi) = I_{P^\circ}(\pi^\circ)$ and

$$I_P(\pi)|_{G^\circ} = I_{P^\circ}^+(\pi^\circ) \oplus I_{P^\circ}^-(\pi^\circ)$$

as in the proof of Lemma 2.1.3. Then $\Omega_\omega(\pi_\lambda)$ (resp. $\Omega_\omega(\epsilon\pi_\lambda)\circ L(\epsilon)$) is a nonzero element in $\mathrm{Hom}_U(I_P(\pi_\lambda),\chi)$ which is identically zero on $I_{P^\circ}^-(\pi^\circ)$ (resp. $I_{P^\circ}^+(\pi^\circ)$). In particular, $\Omega_\omega(\pi_\lambda)$ and $\Omega_\omega(\epsilon\pi_\lambda)\circ L(\epsilon)$ are linearly independent, and hence form a basis of $\mathrm{Hom}_U(I_P(\pi_\lambda),\chi)$ by Lemma 2.1.3 (b). On the other hand, $\Omega_\omega(w\pi_\lambda)\circ J_P(w,\pi_\lambda)$ is also an element in $\mathrm{Hom}_U(I_P(\pi_\lambda),\chi)$ (provided that $J_P(w,\pi_\lambda)$ is holomorphic at λ) which is identically zero on $I_{P^\circ}^-(\pi^\circ)$ (resp. $I_{P^\circ}^+(\pi^\circ)$) if $\det(w) = 1$ (resp. $\det(w) = -1$). This proves the existence of the desired function $C_{P,\omega}(w,\pi_\lambda)$ with respect to ω. When $M = M^\circ$, $C_{P,\omega}(w,\pi_\lambda)$ does not depend on the choice of ω by the uniqueness of Whittaker functionals.

Finally, we assume that $M \neq M^\circ$, and $\pi|_{M^\circ}$ is reducible. In particular, we have $\epsilon P\epsilon^{-1} = P$. Recall the isomorphisms

$$I_{P^\circ}^+(\pi^\circ) \xrightarrow{\sim} \mathrm{Ind}_{P^\circ}^{G^\circ}(\pi^\circ),\ f \mapsto f|_{G^\circ},$$
$$I_{P^\circ}^-(\pi^\circ) \xrightarrow{\sim} \mathrm{Ind}_{P^\circ}^{G^\circ}(\epsilon\pi^\circ),\ f \mapsto (L(\epsilon)f)|_{G^\circ}.$$

If $\det(w) = 1$, then the restriction to $I_{P^\circ}^+(\pi^\circ)$ of the equality in the statement of the lemma yields

$$C_{P,\omega}(w,\pi_\lambda) = C_{P^\circ}(w,\pi_\lambda^\circ).$$

Similarly, if $\det(w) = -1$, then the restriction to $I_{P^\circ}^-(\pi^\circ)$ yields

$$C_{P,\omega}(w, \pi_\lambda) = C_{P^\circ}(w\epsilon^{-1}, \epsilon\pi_\lambda^\circ),$$

noting that $J_P(w, \pi_\lambda) = J_P(w\epsilon^{-1}, \epsilon\pi_\lambda) \circ L(\epsilon)$.

Now we switch the roles of π° and $\epsilon\pi^\circ$, i.e., we identify π with $\mathrm{Ind}_{M^\circ}^M(\epsilon\pi^\circ)$, so that $\epsilon\pi^\circ$ (resp. π°) is the subspace of $\mathrm{Ind}_{M^\circ}^M(\epsilon\pi^\circ)$ consisting of functions f on M whose supports are contained in M° (resp. $M \setminus M^\circ$). Then ω' can be realized by $\omega'(f) = \omega^\circ(f(\mathbf{1}))$ for $f \in \mathrm{Ind}_{M^\circ}^M(\epsilon\pi^\circ)$, where ω° is now a non-trivial $\mathfrak{w}_M$-Whittaker functional on $\epsilon\pi^\circ$. The same argument proves the existence of the desired function $C_{P,\omega'}(w, \pi_\lambda)$ with respect to ω' and shows that

$$C_{P,\omega'}(w, \pi_\lambda) = \begin{cases} C_{P^\circ}(w, \epsilon\pi_\lambda^\circ) & \text{if } \det(w) = 1, \\ C_{P^\circ}(w\epsilon^{-1}, \pi_\lambda^\circ) & \text{if } \det(w) = -1. \end{cases}$$

By Lemma 2.1.5, we have

$$C_{P,\omega'}(w, \pi_\lambda) = C_{P,\omega}(w, \pi_\lambda).$$

This shows that $C_{P,\omega}(w, \pi_\lambda)$ does not depend on the choice of ω and completes the proof. □

In addition, from the proofs of Lemmas 2.1.3 and 2.1.4, we can deduce the following relation between local coefficients for $\mathrm{O}_{2n}(F)$ and those for $\mathrm{SO}_{2n}(F)$.

Lemma 2.1.6. *Suppose that $G = \mathrm{O}_{2n}(F)$ and $\det(w) = 1$. Then we have*

$$C_P(w, \pi_\lambda) = C_{P^\circ}(w, \pi_\lambda^\circ),$$

where π° is an arbitrary irreducible component of $\pi|_{M^\circ}$.

Now we assume that π is tempered. Let ϕ be the tempered L-parameter of π. To show Theorem 1.8.1, it suffices to prove the following.

Proposition 2.1.7. *In each situation in Theorem 1.8.1, the function $C_P(w, \pi_\lambda) \cdot r_P(w, \phi_\lambda)$ is holomorphic and equal to 1 at $\lambda = 0$.*

We will prove Proposition 2.1.7 in Sections 2.4 and 2.5. By the next lemma, which follows from the definitions, it suffices to consider Proposition 2.1.7 in the following two situations.

(1) $G = G^\circ$, or $G = \mathrm{O}_{2n}(F)$ and $\det(w) = 1$, and $w\pi \cong \pi$;
(2) $G = \mathrm{O}_{2n}(F)$, $\det(w) = 1$ and $w\epsilon\pi \cong \pi$.

Lemma 2.1.8. *Suppose that $G = \mathrm{O}_{2n}(F)$ and $\det(w) = -1$.*

- *We have*
$$J_P(w, \pi_\lambda) = J_{\epsilon P \epsilon^{-1}}(w\epsilon^{-1}, \epsilon\pi_\lambda) \circ L(\epsilon)$$
so that $C_P(w, \pi_\lambda) = C_{\epsilon P \epsilon^{-1}}(w\epsilon^{-1}, \epsilon\pi_\lambda)$.
- *We have $r_P(w, \phi_\lambda) = r_{\epsilon P \epsilon^{-1}}(w\epsilon^{-1}, \phi_\lambda)$.*

2.2. **The maximal case.** Let $P = MN_P$ and $P' = M'N_{P'}$ be standard maximal parabolic subgroups of G such that $W(M^\circ, M'^\circ) \neq \emptyset$. From Lemma 2.1.8, we may assume that

- if $G = \mathrm{O}_{2k}(F)$, $M = \mathrm{GL}_k(F)$, and $k > 1$ is odd, then $P' = \epsilon P \epsilon^{-1}$ and $M' = \epsilon M \epsilon^{-1}$ so that $W(M^\circ, M'^\circ) \neq \emptyset$;
- otherwise, $G = \mathrm{GL}_N(E)$, or $M = M'$ so that $W(M^\circ, M'^\circ) \neq \{\mathbf{1}\}$.

In either case, let w be the unique non-trivial element in $W(M^\circ, M'^\circ)$.

First suppose that $G = \mathrm{GL}_N(E)$ and $M = \mathrm{GL}_{N_1}(E) \times \mathrm{GL}_{N_2}(E)$ so that $M' = \mathrm{GL}_{N_2}(E) \times \mathrm{GL}_{N_1}(E)$. We write $\pi = \pi_1 \otimes \pi_2$ with irreducible tempered representations π_1 and π_2 of $\mathrm{GL}_{N_1}(E)$ and $\mathrm{GL}_{N_2}(E)$, respectively. If we denote the L-parameters of π_1 and π_2 by ϕ_{π_1} and ϕ_{π_2}, respectively, then we denote by $L(s, \phi_{\pi_1} \otimes \phi_{\pi_2})$ and $\varepsilon(s, \phi_{\pi_1} \otimes \phi_{\pi_2}, \psi_E)$ the Artin factors associated to the tensor product of the standard representations.

Lemma 2.2.1. *The function $C_P(w, \pi_\lambda) \cdot r_P(w, \phi_\lambda)$ is holomorphic and equal to*

$$\frac{L(1, \phi_{\pi_1}^\vee \otimes \phi_{\pi_2})}{L(1, \phi_{\pi_1} \otimes \phi_{\pi_2}^\vee)}$$

at $\lambda = 0$.

Proof. If we write $\pi_\lambda = \pi_1 |\det|_E^{s_1} \boxtimes \pi_2 |\det|_E^{s_2}$ with $s_1, s_2 \in \mathbb{C}$ and put $s = s_1 - s_2$, then we have, by definition,

$$r_P(w, \phi_\lambda) = \frac{L(s, \phi_{\pi_1} \otimes \phi_{\pi_2}^\vee)}{\varepsilon(s, \phi_{\pi_1} \otimes \phi_{\pi_2}^\vee, \psi_E) L(1+s, \phi_{\pi_1} \otimes \phi_{\pi_2}^\vee)}.$$

On the other hand, by [Sha7, Theorem 3.5] (see also Section 2.6 below), we have

$$C_P(w, \pi_\lambda) = \frac{\varepsilon^{\mathrm{Sh}}(s, \pi_1 \times \pi_2^\vee, \psi_E) L^{\mathrm{Sh}}(1-s, \pi_1^\vee \times \pi_2)}{L^{\mathrm{Sh}}(s, \pi_1 \times \pi_2^\vee)},$$

where the superscript Sh indicates Shahidi's local factors. We know that

$$L^{\mathrm{Sh}}(s, \pi_1 \times \pi_2^\vee) = L(s, \phi_{\pi_1} \otimes \phi_{\pi_2}^\vee), \quad \varepsilon^{\mathrm{Sh}}(s, \pi_1 \times \pi_2^\vee, \psi_E) = \varepsilon(s, \phi_{\pi_1} \otimes \phi_{\pi_2}^\vee, \psi_E)$$

since these local factors agree with those of Jacquet–Piatetski-Shapiro–Shalika by [Sha4] and the desiderata of the local Langlands correspondence. This implies the lemma. □

Next suppose that G is a classical group, and write $M = \mathrm{GL}_k(E) \times G_0$. Note that $M' = M$ unless $G = \mathrm{O}_{2k}(F)$, $M = \mathrm{GL}_k(F)$, and $k > 1$ is odd, in which case $M' = \epsilon M \epsilon^{-1}$. We write $\pi = \tau \boxtimes \pi_0$ with irreducible tempered representations τ and π_0 of $\mathrm{GL}_k(E)$ and G_0, respectively. If we denote the L-parameters of τ and π_0 by ϕ_τ and ϕ_{π_0}, respectively, we denote by $L(s, \phi_\tau \otimes \phi_{\pi_0})$ and $\varepsilon(s, \phi_\tau \otimes \phi_{\pi_0}, \psi_E)$ the Artin factors over E associated to the tensor product of the standard representations. We also denote by $L(s, \phi_\tau, R)$ and $\varepsilon(s, \phi_\tau, R, \psi_F)$ the Artin factors over F associated to the representation R of ${}^L\mathrm{GL}_k(E)$ given by

$$R = \begin{cases} \mathrm{Sym}^2 & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \wedge^2 & \text{if } G = \mathrm{Sp}_{2n}(F),\ \mathrm{O}_{2n}(F), \\ \mathrm{Asai}^+ & \text{if } G = \mathrm{U}_n,\ n \equiv 0 \bmod 2, \\ \mathrm{Asai}^- & \text{if } G = \mathrm{U}_n,\ n \equiv 1 \bmod 2. \end{cases}$$

(See [GGP, Section 7] for the definition of the Asai representations Asai^+ and Asai^-.)

Lemma 2.2.2. *The function $C_P(w, \pi_\lambda) \cdot r_P(w, \phi_\lambda)$ is holomorphic and equal to*

$$\frac{L(1, \phi_\tau^\vee \otimes \phi_{\pi_0})}{L(1, \phi_\tau \otimes \phi_{\pi_0}^\vee)} \frac{L(1, \phi_\tau^\vee, R)}{L(1, \phi_\tau, R)}$$

at $\lambda = 0$.

Proof. If we write $\pi_\lambda = \tau |\det|_E^s \boxtimes \pi_0$ with $s \in \mathbb{C}$, then we have by definition

$$\begin{aligned} r_P(w, \pi_\lambda) = \lambda(w) &\times \frac{L(s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee)}{\varepsilon(s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F) L(1+s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee)} \\ &\times \frac{L(2s, \phi_\tau, R)}{\varepsilon(2s, \phi_\tau, R, \psi_F) L(1+2s, \phi_\tau, R)}. \end{aligned}$$

Here $L(s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee)$ and $\varepsilon(s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F)$ is the Artin factors associated to the L-parameter ϕ_π of π and the tensor product representation $\mathrm{St} \otimes \mathrm{St}^\vee$, where St is the standard representation of ${}^L\mathrm{GL}_N(E)$ or ${}^L G_0$. Note that

$$\begin{aligned} L(s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee) &= L(s, \phi_\tau \otimes \phi_{\pi_0}^\vee), \\ \varepsilon(s, \phi_\pi, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F) &= \lambda_0 \cdot \varepsilon(s, \phi_\tau \otimes \phi_{\pi_0}^\vee, \psi_E), \end{aligned}$$

where $\lambda_0 = 1$ unless $[E : F] = 2$ (so that $G = \mathrm{U}_n$ and $M = \mathrm{GL}_k(E) \times \mathrm{U}_{n_0}$), in which case,

$$\lambda_0 = \lambda(E/F, \psi_F)^{kn_0}.$$

See e.g., [D, Section 5.6]. On the other hand, by [Sha7, Theorem 3.5] (see also Section 2.6 below), we have

$$\begin{aligned} C_P(w, \pi_\lambda) = \lambda(w)^{-1} &\times \frac{\varepsilon^{\mathrm{Sh}}(s, \pi, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F) L^{\mathrm{Sh}}(1-s, \pi^\vee, \mathrm{St} \otimes \mathrm{St}^\vee)}{L^{\mathrm{Sh}}(s, \pi, \mathrm{St} \otimes \mathrm{St}^\vee)} \\ &\times \frac{\varepsilon^{\mathrm{Sh}}(2s, \tau, R, \psi_F) L^{\mathrm{Sh}}(1-2s, \tau^\vee, R)}{L^{\mathrm{Sh}}(2s, \tau, R)}, \end{aligned}$$

where the superscript Sh indicates Shahidi's local factors. Here, when $G = \mathrm{O}_{2n}(F)$, we have $\det(w) = 1$ by the assumption at the beginning of this subsection, and by Lemma 2.1.6, the above equality holds if we set

$$\begin{aligned} L^{\mathrm{Sh}}(s, \pi, \mathrm{St} \otimes \mathrm{St}^\vee) &= L^{\mathrm{Sh}}(s, \pi^\circ, \mathrm{St} \otimes \mathrm{St}^\vee), \\ \varepsilon^{\mathrm{Sh}}(s, \pi, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F) &= \varepsilon^{\mathrm{Sh}}(s, \pi^\circ, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F), \end{aligned}$$

where π° is an arbitrary irreducible component of $\pi|_{M^\circ}$. Putting

$$L^{\mathrm{Sh}}(s, \tau \times \pi_0^\vee) = L^{\mathrm{Sh}}(s, \pi, \mathrm{St} \otimes \mathrm{St}^\vee),$$
$$\varepsilon^{\mathrm{Sh}}(s, \tau \times \pi_0^\vee, \psi_E) = \lambda_0^{-1} \cdot \varepsilon^{\mathrm{Sh}}(s, \pi, \mathrm{St} \otimes \mathrm{St}^\vee, \psi_F),$$

by Proposition A.2.1 below, we have

$$L^{\mathrm{Sh}}(s, \tau \times \pi_0^\vee) = L(s, \phi_\tau \otimes \phi_{\pi_0}^\vee), \quad \varepsilon^{\mathrm{Sh}}(s, \tau \times \pi_0^\vee, \psi_E) = \varepsilon(s, \phi_\tau \otimes \phi_{\pi_0}^\vee, \psi_E).$$

Also, by [He3], [CST], [Shan], [He4], we have

$$L^{\mathrm{Sh}}(s, \tau, R) = L(s, \phi_\tau, R), \quad \varepsilon^{\mathrm{Sh}}(s, \tau, R, \psi_F) = \varepsilon(s, \phi_\tau, R, \psi_F).$$

This implies the lemma. □

2.3. **Preliminary to the general case.** Now we consider the general case. Recall that U is the unipotent radical of the Borel subgroup B°. Let $P = MN_P$ and $P' = M'N_{P'}$ be standard parabolic subgroups of G such that $W(M^\circ, M'^\circ)$ has an element w whose representatives lie in G°. Then by [Sha2, Lemma 2.1.2], we may take standard parabolic subgroups $P_i = M_i N_i$ of G for $1 \leq i \leq n+1$ with the following properties:

- $P = P_1$ and $P' = P_{n+1}$;
- for each $1 \leq i \leq n$, there exists a semi-standard Levi subgroup G_i of G containing M_i such that $P_i \cap G_i$ is a maximal parabolic subgroup of G_i and such that the element $w_i = w_\ell^{G_i^\circ} w_\ell^{M_i^\circ}$ belongs to $W(M_i^\circ, M_{i+1}^\circ)$, where $w_\ell^\bullet$ is the longest element in $W^\bullet$;
- $w = w_n \cdots w_1$;
- for each $1 \leq i \leq n$, we have
$$\overline{\mathfrak{n}}_{w_i'} = \overline{\mathfrak{n}}_{w_i} \oplus \mathrm{Ad}(\widetilde{w}_i)^{-1} \overline{\mathfrak{n}}_{w_{i+1}'},$$
where $\overline{\mathfrak{n}}_w$ is the Lie algebra of $\overline{N}_w = \overline{N}_P \cap \widetilde{w}^{-1} U \widetilde{w}$ for $w \in W^{G^\circ}$, and $w_i' = w_n \cdots w_i$ (with interpreting $w_{n+1}' = \mathbf{1}$).

This gives rise to a factorization of the intertwining operator

$$J_P(w, \pi_\lambda) = J_{P_n}(w_n, w_{n-1} \cdots w_1 \pi_\lambda) \circ \cdots \circ J_{P_2}(w_2, w_1 \pi_\lambda) \circ J_{P_1}(w_1, \pi_\lambda)$$

(see [Sha2, Theorem 2.1.1]) and hence of the local coefficient

$$C_P(w, \pi_\lambda) = \prod_{i=1}^n C_{P_i}(w_i, w_{i-1} \cdots w_1 \pi_\lambda)$$

(see [Sha2, Proposition 3.2.1]). This also gives rise to a factorization of the λ-factor

$$\lambda(w) = \prod_{i=1}^n \lambda(w_i).$$

Moreover, since

$$\widehat{\overline{\mathfrak{n}}}_w = \widehat{\overline{\mathfrak{n}}}_{w_1} \oplus \mathrm{Ad}(\widetilde{w}_1)^{-1} \widehat{\overline{\mathfrak{n}}}_{w_2} \oplus \cdots \oplus \mathrm{Ad}(\widetilde{w}_{n-1} \cdots \widetilde{w}_1)^{-1} \widehat{\overline{\mathfrak{n}}}_{w_n},$$

we have

$$L(s,\pi_\lambda,\rho_{w^{-1}P'|P}) = \prod_{i=1}^{n} L(s, w_{i-1}\cdots w_1\pi_\lambda, \rho^\vee_{w_i^{-1}P_{i+1}|P_i}),$$

$$\varepsilon(s,\pi_\lambda,\rho_{w^{-1}P'|P},\psi_F) = \prod_{i=1}^{n} \varepsilon(s, w_{i-1}\cdots w_1\pi_\lambda, \rho^\vee_{w_i^{-1}P_{i+1}|P_i},\psi_F).$$

Hence we have

$$r_P(w,\pi_\lambda) = \prod_{i=1}^{n} r_{P_i}(w_i, w_{i-1}\cdots w_1\pi_\lambda).$$

Since the local coefficient $C_{P_i}(w_i, w_{i-1}\cdots w_1\pi_\lambda)$ and the normalizing factor $r_{P_i}(w_i, w_{i-1}\cdots w_1\pi_\lambda)$ agree with those for the intertwining operator

$$\mathrm{Ind}^{G_i}_{P_i}(w_{i-1}\cdots w_1\pi_\lambda) \to \mathrm{Ind}^{G_i}_{P_{i+1}}(w_i\cdots w_1\pi_\lambda),$$

and G_i is the product of general linear groups and a (possibly trivial) classical group, this allows us to use these computations to attack Proposition 2.1.7 in general. We will see it in the next two subsections.

2.4. **The case of general linear groups.** Suppose that $G = \mathrm{GL}_N(E)$ and $M = \mathrm{GL}_{n_1}(E)\times\cdots\times\mathrm{GL}_{n_m}(E)$. Put $I=\{1,\dots,m\}$. Write $\pi = \pi_1\boxtimes\cdots\boxtimes\pi_m$ with irreducible tempered representations $\pi_1,\dots,\pi_m$ of $\mathrm{GL}_{n_1}(E),\dots,\mathrm{GL}_{n_m}(E)$, respectively. We denote by ϕ_{π_i} the L-parameter of π_i. We regard w as an automorphism of I such that $w\pi = \pi_{w^{-1}(1)}\boxtimes\cdots\boxtimes\pi_{w^{-1}(m)}$. Then by Lemma 2.2.1 and Section 2.3, the function $C_P(w,\pi_\lambda)\cdot r_P(w,\phi_\lambda)$ is holomorphic and equal to

$$\prod_{(i,j)\in\mathrm{inv}(w)} A(\pi_i,\pi_j)$$

at $\lambda=0$, where we set

$$\mathrm{inv}(w) = \{(i,j)\in I\times I \mid i<j,\ w(i)>w(j)\}$$

and

$$A(\pi_i,\pi_j) = \frac{L(1,\phi^\vee_{\pi_i}\otimes\phi_{\pi_j})}{L(1,\phi_{\pi_i}\otimes\phi^\vee_{\pi_j})}.$$

We may write

$$\prod_{(i,j)\in\mathrm{inv}(w)} A(\pi_i,\pi_j) = \prod_{(\sigma_1,\sigma_2)} A(\sigma_1,\sigma_2)^{n(\sigma_1,\sigma_2)},$$

where (σ_1,σ_2) runs over ordered pairs of irreducible tempered representations of general linear groups and

$$n(\sigma_1,\sigma_2) = |(I(\sigma_1)\times I(\sigma_2))\cap\mathrm{inv}(w)|$$

with $I(\sigma_k)=\{i\in I\mid \pi_i\cong\sigma_k\}$. Since $A(\sigma_1,\sigma_2)=1$ if $\sigma_1\cong\sigma_2$ and $A(\sigma_1,\sigma_2)A(\sigma_2,\sigma_1)=1$, we have

$$\prod_{(\sigma_1,\sigma_2)} A(\sigma_1,\sigma_2)^{n(\sigma_1,\sigma_2)} = \prod_{\{\sigma_1,\sigma_2\}} A(\sigma_1,\sigma_2)^{n(\sigma_1,\sigma_2)-n(\sigma_2,\sigma_1)},$$

where $\{\sigma_1, \sigma_2\}$ runs over unordered pairs.

Proof of Theorem 1.8.1 (1) for $G = \mathrm{GL}_N(E)$. Assume that $P = P'$ and $w\pi \cong \pi$. It suffices to show that

$$n(\sigma_1, \sigma_2) = n(\sigma_2, \sigma_1).$$

We consider the set $I_0(\sigma_1, \sigma_2) = I_1(\sigma_1, \sigma_2) \sqcup I_2(\sigma_1, \sigma_2)$ with

$$\begin{aligned} I_0(\sigma_1, \sigma_2) &= \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i < j\}, \\ I_1(\sigma_1, \sigma_2) &= \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i < j,\, w(i) < w(j)\}, \\ I_2(\sigma_1, \sigma_2) &= \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i < j,\, w(i) > w(j)\}. \end{aligned}$$

Since $w\pi \cong \pi$, we have $\pi_{w^{-1}(i)} \cong \pi_i$ for $1 \leq i \leq m$. Hence the map $(i,j) \mapsto (w^{-1}(i), w^{-1}(j))$ gives a bijection

$$I_0(\sigma_1, \sigma_2) \xrightarrow{1:1} I_0'(\sigma_1, \sigma_2) = \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, w(i) < w(j)\}.$$

Note that $I_0'(\sigma_1, \sigma_2) = I_1(\sigma_1, \sigma_2) \sqcup I_2'(\sigma_1, \sigma_2)$ with

$$I_2'(\sigma_1, \sigma_2) = \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i > j,\, w(i) < w(j)\}.$$

Since the map $(i,j) \mapsto (j,i)$ gives a bijection $I_2'(\sigma_1, \sigma_2) \xrightarrow{1:1} I_2(\sigma_2, \sigma_1)$, we have

$$n(\sigma_1, \sigma_2) = |I_2(\sigma_1, \sigma_2)| = |I_2'(\sigma_1, \sigma_2)| = n(\sigma_2, \sigma_1).$$

This completes the proof of Proposition 2.1.7 in this case, and hence Theorem 1.8.1 (1) for $G = \mathrm{GL}_N(E)$. □

Next, we consider Theorem 1.8.1 (2). For a representation σ of $\mathrm{GL}_N(E)$, define its conjugate ${}^c\sigma$ by ${}^c\sigma(x) = \sigma(\overline{x})$. Hence $\sigma \circ \theta \cong {}^c\sigma^\vee$ if σ is irreducible. Since $L(s, {}^c\phi_{\sigma_1} \otimes {}^c\phi_{\sigma_2}^\vee) = L(s, \phi_{\sigma_1} \otimes \phi_{\sigma_2}^\vee)$, we have $A({}^c\sigma_2^\vee, {}^c\sigma_1^\vee) = A(\sigma_1, \sigma_2)$, so that

$$\prod_{(\sigma_1,\sigma_2)} A(\sigma_1, \sigma_2)^{n(\sigma_1,\sigma_2)} = \prod_{[\sigma_1,\sigma_2]} A(\sigma_1, \sigma_2)^{n'(\sigma_1,\sigma_2)/e(\sigma_1,\sigma_2)},$$

where $[\sigma_1, \sigma_2]$ runs over orbits under the action $(\sigma_1, \sigma_2) \mapsto ({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)$ and

$$\begin{aligned} n'(\sigma_1, \sigma_2) &= n(\sigma_1, \sigma_2) + n({}^c\sigma_2^\vee, {}^c\sigma_1^\vee), \\ e(\sigma_1, \sigma_2) &= \frac{2}{|[\sigma_1, \sigma_2]|}. \end{aligned}$$

Moreover, since

- $A(\sigma_1, \sigma_2) = 1$ if $[\sigma_1, \sigma_2] = [\sigma_2, \sigma_1]$;
- $A(\sigma_1, \sigma_2)A(\sigma_2, \sigma_1) = 1$;
- $e(\sigma_1, \sigma_2) = e(\sigma_2, \sigma_1)$,

we have

$$\prod_{[\sigma_1,\sigma_2]} A(\sigma_1, \sigma_2)^{n'(\sigma_1,\sigma_2)/e(\sigma_1,\sigma_2)} = \prod_{[[\sigma_1,\sigma_2]]} A(\sigma_1, \sigma_2)^{(n'(\sigma_1,\sigma_2)-n'(\sigma_2,\sigma_1))/e(\sigma_1,\sigma_2)},$$

where $[[\sigma_1, \sigma_2]]$ runs over orbits under the action $[\sigma_1, \sigma_2] \mapsto [\sigma_2, \sigma_1]$.

Proof of Theorem 1.8.1 (2). Assume that $w\pi \cong \pi \circ \theta$. This is equivalent to saying that $\pi_{w^{-1}(i)} \cong {}^c\pi_{m+1-i}^\vee$ for $1 \leq i \leq m$. It suffices to show that

$$n'(\sigma_1, \sigma_2) = n'(\sigma_2, \sigma_1).$$

Put

$$\begin{aligned} I_1(\sigma_1, \sigma_2) &= \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i < j,\, w(i) < w(j)\}, \\ I_2(\sigma_1, \sigma_2) &= \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i < j,\, w(i) > w(j)\}, \\ I_3(\sigma_1, \sigma_2) &= \{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i > j,\, w(i) < w(j)\}. \end{aligned}$$

Then the map $(i,j) \mapsto (i',j') = (w^{-1}(m+1-j), w^{-1}(m+1-i))$ gives a bijection

$$\{(i,j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i < j\} \xrightarrow{1:1} \{(i',j') \in I({}^c\sigma_2^\vee) \times I({}^c\sigma_1^\vee) \,|\, w(i') < w(j')\}.$$

Note that the left-hand side is $I_1(\sigma_1, \sigma_2) \sqcup I_2(\sigma_1, \sigma_2)$, whereas the right-hand side is $I_1({}^c\sigma_2^\vee, {}^c\sigma_1^\vee) \sqcup I_3({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)$. Hence

$$|I_1(\sigma_1, \sigma_2)| + |I_2(\sigma_1, \sigma_2)| = |I_1({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| + |I_3({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)|.$$

This implies that

$$\begin{aligned} &|I_1(\sigma_1, \sigma_2)| + |I_2(\sigma_1, \sigma_2)| + |I_1({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| + |I_2({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| \\ &= |I_1({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| + |I_3({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| + |I_1(\sigma_1, \sigma_2)| + |I_3(\sigma_1, \sigma_2)| \end{aligned}$$

so that

$$\begin{aligned} n'(\sigma_1, \sigma_2) &= |I_2(\sigma_1, \sigma_2)| + |I_2({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| \\ &= |I_3(\sigma_1, \sigma_2)| + |I_3({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)| = n'(\sigma_2, \sigma_1), \end{aligned}$$

where the last equation follows from the map $(i,j) \mapsto (j,i)$. This completes the proof of Proposition 2.1.7 in this case, and hence Theorem 1.8.1 (2). □

2.5. **The case of classical groups.** We now consider Theorem 1.8.1 (1) for classical groups. Suppose that G is a classical group and $M = \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_m}(E) \times G_0$. As explained after Proposition 2.1.7, by Lemma 2.1.8, it suffices to consider the following two situations.

(1) $G = G^\circ$, or $G = \mathrm{O}_{2n}(F)$ and $\det(w) = 1$, and $w\pi \cong \pi$;
(2) $G = \mathrm{O}_{2n}(F)$, $\det(w) = 1$ and $w\epsilon\pi \cong \pi$.

We consider the general situation for a moment. Put $I^+ = \{1, \dots, m\}$, $I^- = \{-1, \dots, -m\}$, and $I = I^+ \sqcup I^-$. Write $\pi = \tau_1 \otimes \cdots \otimes \tau_m \otimes \pi_0$ with irreducible tempered representations $\tau_1, \dots, \tau_m, \pi_0$ of $\mathrm{GL}_{n_1}(E), \dots, \mathrm{GL}_{n_m}(E), G_0$, respectively, and put $\tau_{-i} = {}^c\tau_i^\vee$ for $i \in I^+$. We denote by ϕ_{τ_i} (resp. ϕ_{π_0}) the L-parameter of τ_i (resp. π_0). We regard w as an automorphism of I such that $w(-i) = -w(i)$ for all $i \in I$ and such that

$w\pi = \tau_{w^{-1}(1)} \otimes \cdots \otimes \tau_{w^{-1}(m)} \otimes \pi_0$. Then by Lemmas 2.2.1, 2.2.2 and Section 2.3, the function $C_P(w, \pi_\lambda) \cdot r_P(w, \phi_\lambda)$ is holomorphic and equal to the product

$$\left(\prod_{(i,j) \in \mathrm{inv}_1(w)} A(\tau_i, \tau_j) \right) \left(\prod_{i \in \mathrm{inv}_2(w)} B(\tau_i, \pi_0) \right)$$

at $\lambda = 0$, where

$$\begin{aligned} \mathrm{inv}_1(w) = & \{(i,j) \in I^+ \times I \mid i < |j|,\ w(i) > 0,\ w(j) > 0,\ w(i) > w(j)\} \\ & \sqcup \{(i,j) \in I^+ \times I \mid i < |j|,\ w(i) < 0,\ w(j) > 0\} \\ & \sqcup \{(i,j) \in I^+ \times I \mid i < |j|,\ w(i) < 0,\ w(j) < 0,\ |w(i)| < |w(j)|\}, \\ \mathrm{inv}_2(w) = & \{i \in I^+ \mid w(i) < 0\} \end{aligned}$$

and

$$A(\tau_i, \tau_j) = \frac{L(1, \phi_{\tau_i}^\vee \otimes \phi_{\tau_j})}{L(1, \phi_{\tau_i} \otimes \phi_{\tau_j}^\vee)}, \quad B(\tau_i, \pi_0) = \frac{L(1, \phi_{\tau_i}^\vee \otimes \phi_{\pi_0})}{L(1, \phi_{\tau_i} \otimes \phi_{\pi_0}^\vee)} \frac{L(1, \phi_{\tau_i}^\vee, R)}{L(1, \phi_{\tau_i}, R)}.$$

First, we may write

$$\prod_{(i,j) \in \mathrm{inv}_1(w)} A(\tau_i, \tau_j) = \prod_{(\sigma_1, \sigma_2)} A(\sigma_1, \sigma_2)^{n_1(\sigma_1, \sigma_2)},$$

where (σ_1, σ_2) runs over pairs of irreducible tempered representations of general linear groups and

$$n_1(\sigma_1, \sigma_2) = |(I(\sigma_1) \times I(\sigma_2)) \cap \mathrm{inv}_1(w)|$$

with $I(\sigma) = \{i \in I \mid \tau_i \cong \sigma\}$. Since $L(s, {}^c\phi_{\sigma_1} \otimes {}^c\phi_{\sigma_2}^\vee) = L(s, \phi_{\sigma_1} \otimes \phi_{\sigma_2}^\vee)$, we have $A({}^c\sigma_2^\vee, {}^c\sigma_1^\vee) = A(\sigma_1, \sigma_2)$, so that

$$\prod_{(\sigma_1, \sigma_2)} A(\sigma_1, \sigma_2)^{n_1(\sigma_1, \sigma_2)} = \prod_{[\sigma_1, \sigma_2]} A(\sigma_1, \sigma_2)^{n_1'(\sigma_1, \sigma_2)/e(\sigma_1, \sigma_2)},$$

where $[\sigma_1, \sigma_2]$ runs over orbits under the action $(\sigma_1, \sigma_2) \mapsto ({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)$ and

$$\begin{aligned} n_1'(\sigma_1, \sigma_2) &= n_1(\sigma_1, \sigma_2) + n_1({}^c\sigma_2^\vee, {}^c\sigma_1^\vee), \\ e(\sigma_1, \sigma_2) &= \frac{2}{|[\sigma_1, \sigma_2]|}. \end{aligned}$$

Moreover, since

- $A(\sigma_1, \sigma_2) = 1$ if $[\sigma_1, \sigma_2] = [\sigma_2, \sigma_1]$;
- $A(\sigma_1, \sigma_2) A(\sigma_2, \sigma_1) = 1$;
- $e(\sigma_1, \sigma_2) = e(\sigma_2, \sigma_1)$,

we have

$$\prod_{[\sigma_1, \sigma_2]} A(\sigma_1, \sigma_2)^{n_1'(\sigma_1, \sigma_2)/e(\sigma_1, \sigma_2)} = \prod_{[[\sigma_1, \sigma_2]]} A(\sigma_1, \sigma_2)^{(n_1'(\sigma_1, \sigma_2) - n_1'(\sigma_2, \sigma_1))/e(\sigma_1, \sigma_2)},$$

where $[[\sigma_1, \sigma_2]]$ runs over orbits under the action $[\sigma_1, \sigma_2] \mapsto [\sigma_2, \sigma_1]$. Similarly, we may write

$$\prod_{i \in \mathrm{inv}_2(w)} B(\tau_i, \pi_0) = \prod_{\sigma} B(\sigma, \pi_0)^{n_2(\sigma)},$$

where σ runs over irreducible tempered representations of general linear groups and

$$n_2(\sigma) = |I(\sigma) \cap \mathrm{inv}_2(w)|.$$

Since $L(s, {}^c\phi_\sigma \otimes \phi_{\pi_0}) = L(s, \phi_\sigma \otimes \phi_{\pi_0}^\vee)$ and $L(s, {}^c\phi_\sigma, R) = L(s, \phi_\sigma, R)$, we have $B(\sigma, \pi_0) = 1$ if ${}^c\sigma^\vee \cong \sigma$, and $B(\sigma, \pi_0) B({}^c\sigma^\vee, \pi_0) = 1$, so that

$$\prod_{\sigma} B(\sigma, \pi_0)^{n_2(\sigma)} = \prod_{[\sigma]} B(\sigma, \pi_0)^{n_2(\sigma) - n_2({}^c\sigma^\vee)},$$

where $[\sigma]$ runs over orbits under the action $\sigma \mapsto {}^c\sigma^\vee$.

Proof of Theorem 1.8.1 (1) for classical groups G. Assume that $w\pi \cong \pi$, or $w\epsilon\pi \cong \pi$ (in which case, $G = \mathrm{O}_{2n}(F)$). It suffices to show that

$$n_1'(\sigma_1, \sigma_2) = n_1'(\sigma_2, \sigma_1), \quad n_2(\sigma) = n_2({}^c\sigma^\vee).$$

First we consider the set

$$I^\delta(\sigma_1, \sigma_2) = \{(i, j) \in I(\sigma_1) \times I(\sigma_2) \,|\, i > 0,\, \delta j > 0,\, i < |j|\} = \bigsqcup_{k=1}^{8} I_k^\delta(\sigma_1, \sigma_2)$$

for $\delta = \pm$, where

$$\begin{aligned}
I_1^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) > 0,\, w(j) > 0,\, |w(i)| < |w(j)|\}, \\
I_2^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) > 0,\, w(j) > 0,\, |w(i)| > |w(j)|\}, \\
I_3^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) > 0,\, w(j) < 0,\, |w(i)| < |w(j)|\}, \\
I_4^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) > 0,\, w(j) < 0,\, |w(i)| > |w(j)|\}, \\
I_5^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) < 0,\, w(j) > 0,\, |w(i)| < |w(j)|\}, \\
I_6^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) < 0,\, w(j) > 0,\, |w(i)| > |w(j)|\}, \\
I_7^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) < 0,\, w(j) < 0,\, |w(i)| < |w(j)|\}, \\
I_8^\delta(\sigma_1, \sigma_2) &= \{(i, j) \in I^+(\sigma_1) \times I^\delta(\sigma_2) \,|\, i < |j|,\, w(i) < 0,\, w(j) < 0,\, |w(i)| > |w(j)|\}
\end{aligned}$$

with $I^\delta(\sigma) = I^\delta \cap I(\sigma)$. Since $w\pi \cong \pi$ or $w\epsilon\pi \cong \pi$, the map $(i, j) \mapsto (i', j') = (w^{-1}(i), w^{-1}(j))$ gives a bijection $I^\delta(\sigma_1, \sigma_2) \xrightarrow{1:1} J^\delta(\sigma_1, \sigma_2)$, where

$$\begin{aligned}
J^\delta(\sigma_1, \sigma_2) &= \{(i', j') \in I(\sigma_1) \times I(\sigma_2) \,|\, w(i') > 0,\, \delta w(j') > 0,\, w(i') < |w(j')|\} \\
&= \bigsqcup_{k=1}^{8} J_k^\delta(\sigma_1, \sigma_2)
\end{aligned}$$

with

$$J_1^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^+(\sigma_1)\times I^+(\sigma_2) \,|\, |i|<|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_2^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^+(\sigma_1)\times I^+(\sigma_2) \,|\, |i|>|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_3^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^+(\sigma_1)\times I^-(\sigma_2) \,|\, |i|<|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_4^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^+(\sigma_1)\times I^-(\sigma_2) \,|\, |i|>|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_5^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^-(\sigma_1)\times I^+(\sigma_2) \,|\, |i|<|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_6^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^-(\sigma_1)\times I^+(\sigma_2) \,|\, |i|>|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_7^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^-(\sigma_1)\times I^-(\sigma_2) \,|\, |i|<|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\},$$
$$J_8^\delta(\sigma_1,\sigma_2) = \{(i,j) \in I^-(\sigma_1)\times I^-(\sigma_2) \,|\, |i|>|j|,\, w(i)>0,\, \delta w(j)>0,\, w(i)<|w(j)|\}.$$

Hence, noting that

$$J_1^+(\sigma_1,\sigma_2) = I_1^+(\sigma_1,\sigma_2), \qquad J_1^-(\sigma_1,\sigma_2) = I_3^+(\sigma_1,\sigma_2),$$
$$J_3^+(\sigma_1,\sigma_2) = I_1^-(\sigma_1,\sigma_2), \qquad J_3^-(\sigma_1,\sigma_2) = I_3^-(\sigma_1,\sigma_2),$$

we have

$$\sum_{k\in\{2,4,5,6,7,8\}} |I_k(\sigma_1,\sigma_2)| = \sum_{k\in\{2,4,5,6,7,8\}} |J_k(\sigma_1,\sigma_2)|,$$

where $I_k(\sigma_1,\sigma_2) = I_k^+(\sigma_1,\sigma_2) \sqcup I_k^-(\sigma_1,\sigma_2)$ and $J_k(\sigma_1,\sigma_2) = J_k^+(\sigma_1,\sigma_2) \sqcup J_k^-(\sigma_1,\sigma_2)$. Moreover, since the map $(i,j)\mapsto(-j,-i)$ gives bijections

$$J_4^+(\sigma_1,\sigma_2) \xrightarrow{1:1} I_8^-({}^c\sigma_2^\vee, {}^c\sigma_1^\vee), \qquad J_4^-(\sigma_1,\sigma_2) \xrightarrow{1:1} I_4^-({}^c\sigma_2^\vee, {}^c\sigma_1^\vee),$$
$$J_8^+(\sigma_1,\sigma_2) \xrightarrow{1:1} I_8^+({}^c\sigma_2^\vee, {}^c\sigma_1^\vee), \qquad J_8^-(\sigma_1,\sigma_2) \xrightarrow{1:1} I_4^+({}^c\sigma_2^\vee, {}^c\sigma_1^\vee),$$

we have

$$(\sharp) \qquad \sum_{k\in\{2,5,6,7\}} \left(|I_k(\sigma_1,\sigma_2)| + |I_k({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)|\right) = \sum_{k\in\{2,5,6,7\}} \left(|J_k(\sigma_1,\sigma_2)| + |J_k({}^c\sigma_2^\vee, {}^c\sigma_1^\vee)|\right).$$

Since

$$n_1(\sigma_1,\sigma_2) = \sum_{k\in\{2,5,6,7\}} |I_k(\sigma_1,\sigma_2)|,$$

the left-hand side of $(\sharp)$ is equal to $n_1'(\sigma_1,\sigma_2)$. On the other hand, noting that the map $(i,j)\mapsto(j,i)$ gives bijections

$$J_2^+(\sigma_1,\sigma_2) \xrightarrow{1:1} I_2^+(\sigma_2,\sigma_1), \qquad J_2^-(\sigma_1,\sigma_2) \xrightarrow{1:1} I_6^+(\sigma_2,\sigma_1),$$
$$J_6^+(\sigma_1,\sigma_2) \xrightarrow{1:1} I_2^-(\sigma_2,\sigma_1), \qquad J_6^-(\sigma_1,\sigma_2) \xrightarrow{1:1} I_6^-(\sigma_2,\sigma_1),$$

and the map $(i,j)\mapsto(-i,-j)$ gives bijections

$$J_5^+(\sigma_1,\sigma_2) \xrightarrow{1:1} I_7^-({}^c\sigma_1^\vee, {}^c\sigma_2^\vee), \qquad J_5^-(\sigma_1,\sigma_2) \xrightarrow{1:1} I_5^-({}^c\sigma_1^\vee, {}^c\sigma_2^\vee),$$
$$J_7^+(\sigma_1,\sigma_2) \xrightarrow{1:1} I_7^+({}^c\sigma_1^\vee, {}^c\sigma_2^\vee), \qquad J_7^-(\sigma_1,\sigma_2) \xrightarrow{1:1} I_5^+({}^c\sigma_1^\vee, {}^c\sigma_2^\vee),$$

the right-hand side of ($\sharp$) is equal to $n_1'(\sigma_2, \sigma_1)$. This proves

$$n_1'(\sigma_1, \sigma_2) = n_1'(\sigma_2, \sigma_1).$$

Next we consider the set

$$\{i \in I(\sigma) \,|\, i > 0\} = \{i \in I^+(\sigma) \,|\, w(i) > 0\} \sqcup \{i \in I^+(\sigma) \,|\, w(i) < 0\}.$$

The map $i \mapsto i' = w^{-1}(i)$ gives a bijection from this set to

$$\{i' \in I(\sigma) \,|\, w(i') > 0\} = \{i' \in I^+(\sigma) \,|\, w(i') > 0\} \sqcup \{i' \in I^-(\sigma) \,|\, w(i') > 0\}.$$

Hence we have

$$n_2(\sigma) = |\{i \in I^+(\sigma) \,|\, w(i) < 0\}| = |\{i \in I^-(\sigma) \,|\, w(i) > 0\}| = n_2({}^c\sigma^\vee),$$

where the last equation follows by considering the map $i \mapsto -i$. This completes the proof of Proposition 2.1.7 in this case, and hence Theorem 1.8.1 (1) for the classical group G. □

2.6. **Shahidi's formula for local coefficients.** In a series of influential papers [Sha1, Sha2, Sha3, Sha4, Sha5, KeSh, Sha6, Sha7] stretching over 12 years, Shahidi made a deep study of local coefficients and developed a theory of γ-factors for generic representations of connected reductive groups. In particular, as a culmination of this deep study, he showed that local coefficients can be expressed in terms of these γ-factors. Now the theory of γ-factors is very delicate because it requires a careful normalization of various quantities (such as Weyl group representatives) and a careful evaluation of pertinent integrals in the rank 1 case (i.e., for SL_2 and $\mathrm{SU}(2,1)$). As the theory evolved over a period of 12 years, and it was not a priori clear what the precise shape of the final product should be, it is understandable that different normalizations may have been preferred at different points in time, for different reasons. As a consequence, various formulas (of the same quantity) which appeared in [Sha5, KeSh, Sha7] are not consistent with each other. In this subsection, we shall explain and resolve these discrepancies in [Sha5, KeSh, Sha7], so that one has a consistent story.

Let F be a local field of characteristic zero. Fix a non-trivial unitary character ψ_F of F. In this subsection, we consider an arbitrary quasi-split connected reductive algebraic group G over F. Fix an F-splitting $\boldsymbol{spl} = (B, T, \{X_\alpha\})$ of G. Let $\mathfrak{w}$ be the Whittaker datum for G determined by $\boldsymbol{spl}$ and ψ_F.

Let $P = MN$ be a standard maximal parabolic subgroup of G. We denote by a the simple root of A_T corresponding to M, where A_T is the split component of T. Following [Sha6, p. 552], we define $\widetilde{a}$ as the restriction of $\langle \rho, \alpha^\vee \rangle^{-1} \rho$ to A_T, where ρ is half the sum of the absolute roots of T in N, α is an absolute root which restricts to a, and $\alpha^\vee$ is the coroot associated to α. (Note that $\widetilde{a}$ is the corresponding fundamental weight when G is semisimple and split over F.) We may regard $\widetilde{a}$ as an element in $\mathfrak{a}_M^*$. Let r be the adjoint representation of ${}^L M$ on $\widehat{\mathfrak{n}}$. As in [Sha6, p. 554], we decompose it as $r = \bigoplus_{i=1}^m r_i$. Let $P' = M'N'$ be another standard maximal parabolic subgroup of G and assume that $W(M, M') \neq \{\mathbf{1}\}$ (resp. $W(M, M') \neq \emptyset$) if $M = M'$ (resp. $M \neq M'$).

Take the unique non-trivial element w in $W(M, M')$. Let $\lambda(w, \psi_F)$ be the λ-factor as in [KeSh, (4.1)] (see also Section 1.7).

Let π be an irreducible $\mathfrak{w}_M$-generic representation of M, where $\mathfrak{w}_M$ is the Whittaker datum for M induced by $\mathfrak{w}$. Recall that the local coefficient $C_P(w, \pi_{s\widetilde{a}}, \psi_F)$ for $s \in \mathbb{C}$ depends on the choice of the representative of w. We take the Langlands–Shelstad representative $\widetilde{w}$ of w with respect to $\boldsymbol{spl}$. For example, if $G = \mathrm{SL}_2$ and $\boldsymbol{spl}$ is the standard splitting, then we have

$$\widetilde{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We write $C_P(w, \pi_{s\widetilde{a}}, \psi_F) = C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F)$ to indicate this dependence. In [Sha7, Theorem 3.5], Shahidi expressed local coefficients in terms of γ-factors, but the formula needs to be corrected and its correct formulation should be:

$$\text{(S1)} \qquad C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F) = \lambda(w, \psi_F)^{-1} \prod_{i=1}^{m} \gamma^{\mathrm{Sh}}(is, \pi, r_i, \psi_F),$$

where the superscript Sh indicates Shahidi's γ-factors. We shall give several remarks in which we will explain why such a correction is needed in more detail in the rest of this subsection.

Remark 2.6.1. The first remark is for [Sha7]. As explained at the middle of [Sha7, p. 281], Shahidi took the representative $\widetilde{w}^{\mathrm{Sh}}$ of w as in [Sha5, p. 979] and [KeSh, p. 74], which is the Langlands–Shelstad representative with respect to the splitting $\boldsymbol{spl}^- = (B, T, \{-X_\alpha\})$. For example, if $G = \mathrm{SL}_2$ and $\boldsymbol{spl}$ is the standard splitting, then we have

$$\widetilde{w}^{\mathrm{Sh}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We write $C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F)$ for the associated local coefficient. Since $\boldsymbol{spl}^-$ and $\overline{\psi}_F$ give rise to the Whittaker datum $\mathfrak{w}$, the equality (S1) is equivalent to

$$C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F) = \lambda(w, \overline{\psi}_F)^{-1} \prod_{i=1}^{m} \gamma^{\mathrm{Sh}}(is, \pi, r_i, \overline{\psi}_F).$$

However, [Sha7, Theorem 3.5, (3.11)] states that

$$\text{(S2)} \qquad C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F) = \lambda(w, \psi_F)^{-1} \prod_{i=1}^{m} \gamma^{\mathrm{Sh}}(is, \pi, r_i^\vee, \overline{\psi}_F).$$

First we point out that the formula (S2) is not consistent with the restriction of scalars. For simplicity, we assume that G is split over F. Let F_0 be a subfield of F such that F/F_0 has finite degree and assume that $\psi_F = \psi_{F_0} \circ \mathrm{tr}_{F/F_0}$ for some non-trivial additive character ψ_{F_0} of F_0. Put $G_0 = \mathrm{Res}_{F/F_0} G$, $P_0 = \mathrm{Res}_{F/F_0} P$, and $M_0 = \mathrm{Res}_{F/F_0} M$. Denote by a_0 the simple root corresponding to M_0. Let $\boldsymbol{spl}_0$ be the F_0-splitting of G_0 induced by $\boldsymbol{spl}$. Then $\widetilde{w}^{\mathrm{Sh}}$ (regarded as an element in $G_0(F_0)$) is the representative of w with respect to $\boldsymbol{spl}_0^-$, and we can define the local coefficient $C_{P_0}(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a_0}}, \psi_{F_0})$, where

π is regarded as a representation of $M_0(F_0)$. Since $\pi_{s\widetilde{a_0}} = \pi_{s\widetilde{a}}$ under the identification $M_0(F_0) = M(F)$, and $\boldsymbol{spl}_0$ and ψ_{F_0} give rise to the Whittaker datum $\mathfrak{w}$, it follows from the definition that $C_{P_0}(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a_0}}, \psi_{F_0}) = C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F)$. From this and (S2), we can deduce that

$$\lambda(w, \psi_{F_0})^{-1} \prod_{i=1}^{m} \gamma^{\mathrm{Sh}}(is, \pi, \mathrm{Ind}_{{}^LM}^{{}^LM_0} r_i^\vee, \overline{\psi}_{F_0}) = \prod_{i=1}^{m} \gamma^{\mathrm{Sh}}(is, \pi, r_i^\vee, \overline{\psi}_F),$$

noting that $\lambda(w, \psi_F) = 1$. By the property of λ-factors (see [D, Section 5.6]), we should have

$$\gamma^{\mathrm{Sh}}(is, \pi, \mathrm{Ind}_{{}^LM}^{{}^LM_0} r_i^\vee, \overline{\psi}_{F_0}) = \lambda(F/F_0, \overline{\psi}_{F_0})^{\dim r_i^\vee} \gamma^{\mathrm{Sh}}(is, \pi, r_i^\vee, \overline{\psi}_F)$$

and hence

$$\lambda(w, \psi_{F_0})^{-1} \lambda(F/F_0, \overline{\psi}_{F_0})^{\dim N} = 1.$$

However, this is not consistent with the definition of $\lambda(w, \psi_{F_0})$:

$$\lambda(w, \psi_{F_0}) = \lambda(F/F_0, \psi_{F_0})^{\dim N}.$$

As such, the formula (S2) cannot be accurate as stated.

In [Sha7], Shahidi used a global argument to derive the formula for local coefficients from his previous results for arbitrary generic representations in the archimedean case in [Sha5] and principal series representations in the non-archimedean case in [KeSh]. More precisely, he cited [Sha5, KeSh] in the proof of [Sha7, Propositions 3.2 and 3.4], but the formulas stated in these propositions do not agree with the ones in [Sha5, KeSh]. (Compare (S2) with (S3) and (S4) below.) Moreover, the formulas stated in [Sha5, KeSh] must also be appropriately corrected, as we discuss below.

Remark 2.6.2. The second remark is for [Sha5]. Suppose that F is archimedean. In [Sha5], Shahidi realized the induced representation $I_P(\pi)$ on the space of $\mathcal{V}_\pi$-valued smooth functions f on G such that

$$f(gmn) = \delta_P(m)^{-\frac{1}{2}} \pi(m)^{-1} f(g)$$

for $g \in G$, $m \in M$, and $n \in N$, and then defined intertwining operators, Whittaker functionals, and local coefficients. By the isomorphism $f \mapsto [g \mapsto f(g^{-1})]$ from this realization to our realization, we see that the local coefficients defined in [Sha5] agree with the ones defined in [Sha7]. Then [Sha5, Theorem 3.1] states that

$$C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F) = \lambda(w, \psi_F) \prod_{i=1}^{m} \gamma(is, \pi, r_i, \psi_F). \tag{S3}$$

(Note that $-2s\rho_\theta$ on the left-hand side of [Sha5, (3.1.1)] is a typo and should be $2s\rho_\theta$.) But this is not consistent with (S2) or (S1), and must be corrected as follows.

- When $G = \mathrm{SL}_2$, the equality (S3) does not hold by Proposition 2.6.6 below. Indeed, in the equality stated in [Sha5, Lemma 1.4 (a)], one has to use the local coefficient associated to $\widetilde{w}$. Thus one has to replace $C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F)$ in (S3) by $C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F)$. Note that $\lambda(w, \psi_F) = 1$ in this case.

- When $F = \mathbb{R}$ and $G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{SL}_2$, one uses [Sha5, (3.10)] in the proof of [Sha5, Theorem 3.1], but the inverse is missing from $\lambda(\mathbb{C}/\mathbb{R}, \psi_\mathbb{R})$ on the right-hand side of [Sha5, (3.1.1)]. Thus one has to replace $\lambda(w, \psi_F)$ in (S3) by $\lambda(w, \psi_F)^{-1}$.
- When $F = \mathbb{R}$ and $G = \mathrm{SU}(2,1)$, the equality (S3) (and the one stated in [Sha5, Lemma 1.4 (b)]) indeed holds. In this case, we have $C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F) = C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F)$ and $\lambda(w, \psi_F) = \lambda(w, \psi_F)^{-1}$ since
$$\widetilde{w}^{\mathrm{Sh}} = \widetilde{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
and $\lambda(w, \psi_F) = \lambda(\mathbb{C}/\mathbb{R}, \psi_\mathbb{R})^2 = -1$.

Consequently, (S1) is the correct formulation of [Sha5, Theorem 3.1].

Remark 2.6.3. The third remark is for [KeSh]. Suppose that F is non-archimedean and π is a principal series representation. Then [KeSh, Proposition 3.4] states that

$$\text{(S4)} \qquad C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F) = \lambda(w, \psi_F)^{-1} \prod_{i=1}^{m} \gamma(is, \pi, r_i^\vee, \psi_F).$$

But this is not consistent with (S2) or (S1), and must be corrected as follows.

- When $G = \mathrm{SL}_2$, the equality (S4) does not hold by Proposition 2.6.6 below. Again, one has to replace $C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F)$ in (S4) by $C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F)$ as in the archimedean case.
- When $G = \mathrm{SU}(2,1)$, the equality stated in [KeSh, Corollary 3.3] indeed holds. In this case, we have $C_P(\widetilde{w}^{\mathrm{Sh}}, \pi_{s\widetilde{a}}, \psi_F) = C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F)$ as in the archimedean case.
- In addition, one has to replace $\gamma(is, \pi, r_i^\vee, \psi_F)$ in (S4) by $\gamma(is, \pi, r_i, \psi_F)$ for the following reason. For simplicity, we assume that G is split over F.
  - We denote by $\mathrm{Art}\colon F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ the local reciprocity map. Recall that there are two normalizations of this isomorphism. To define Artin L-factors, we use a Frobenius element given by the image of a uniformizer of F under Art. Then the formula for $C_P(\widetilde{w}, \pi_{s\widetilde{a}}, \psi_F)$ does not depend on the choice of Art.
  - For a character χ of T, let ϕ_χ be the L-parameter of χ and regard it as a homomorphism $\phi_\chi\colon W_F^{\mathrm{ab}} \to \widehat{T}$. Recall that ϕ_χ is given by
  $$\phi_\chi = \rho_\chi \circ \mathrm{Art}^{-1},$$
  where $\rho_\chi\colon F^\times \to \widehat{T} = X^*(T) \otimes \mathbb{C}^\times$ is the homomorphism corresponding to $\chi\colon T = X_*(T) \otimes F^\times \to \mathbb{C}^\times$ by the natural isomorphism
  $$\mathrm{Hom}(X_*(T) \otimes F^\times, \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times, X^*(T) \otimes \mathbb{C}^\times).$$

Let $\alpha^\vee$ be a coroot of T, which is a root of $\widehat{T}$. Here we regard $\mu \in X_*(T)$ as a character of $\widehat{T}$ by

$$\mu(t) = z^{\langle \lambda, \mu \rangle}$$

for $t = \lambda \otimes z \in \widehat{T} = X^*(T) \otimes \mathbb{C}^\times$. Since the diagram

$$\begin{array}{ccc} \mathrm{Hom}(X_*(T) \otimes F^\times, \mathbb{C}^\times) & \xrightarrow{\sim} & \mathrm{Hom}(F^\times, X^*(T) \otimes \mathbb{C}^\times) \\ \downarrow {\scriptstyle (\alpha^\vee)^*} & & \downarrow {\scriptstyle (\alpha^\vee)_*} \\ \mathrm{Hom}(F^\times, \mathbb{C}^\times) & = & \mathrm{Hom}(F^\times, \mathbb{C}^\times) \end{array}$$

is commutative, we have $\chi \circ \alpha^\vee = \alpha^\vee \circ \rho_\chi$. Hence [KeSh, (2.1)] should be read:

$$\chi \circ \alpha^\vee = \alpha^\vee \circ \phi_\chi \circ \mathrm{Art},$$

so that one has to replace $\widetilde{r}_{\widetilde{w}}$ on the left-hand sides of [KeSh, (2.9), (2.10), (2.13), (2.14)] by $r_{\widetilde{w}}$.

– At the bottom of [KeSh, p. 74], the authors mention that the equality [KeSh, (2.2)] justifies the inverse on the left-hand side of [KeSh, (2.1)]. But in fact, this equality does not make sense since the pairing between $X^*(T) \otimes \mathbb{C}^\times$ and $X_*(T) \otimes \mathbb{C}^\times$ on the right-hand side of [KeSh, (2.2)] does not exist.

Consequently, (S1) is the correct formulation of [KeSh, Proposition 3.4].

Finally, we reconsider the rank 1 case. In Shahidi's theory of γ-factors, it is important to fix the choice of Weyl group representatives and compute local coefficients (with respect to this choice) in the rank 1 case explicitly. In the case of $\mathrm{SU}(2,1)$, our choice is the same as Shahidi's one and we refer the reader to [Sha5, Lemma 1.4 (b)], [KeSh, Corollary 3.3] for the explicit formula. On the other hand, in the case of SL_2, we take a different representative and the situation is more subtle. Thus, for the sake of completeness, we include the computation of the local coefficient in this case, following [J, Section 1], [Sha1, Lemma 4.4].

Suppose that $G = \mathrm{SL}_2$. Let $\mathcal{S}(F)$ be the space of Schwartz–Bruhat functions on F. For $\phi \in \mathcal{S}(F)$, we define its Fourier transform $\widehat{\phi} \in \mathcal{S}(F)$ by

$$\widehat{\phi}(x) = \int_F \phi(y)\psi_F(xy)dy,$$

where dy is the self-dual Haar measure on F with respect to ψ_F. For $s \in \mathbb{C}$, $\phi \in \mathcal{S}(F)$, and a character χ of $F^\times$, put

$$Z(s, \chi, \phi) = \int_{F^\times} \phi(t)\chi(t)|t|_F^s d^\times t,$$

where $d^\times t = |t|_F^{-1} dt$. This integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbb{C}$. Moreover, the functional equation

$$Z(1-s, \chi^{-1}, \widehat{\phi}) = \gamma(s, \chi, \psi_F) Z(s, \chi, \phi)$$

holds, where

$$\gamma(s,\chi,\psi_F)=\varepsilon(s,\chi,\psi_F)\frac{L(1-s,\chi^{-1})}{L(s,\chi)}.$$

We consider the normalized parabolically induced representation $I(s,\chi)=\mathrm{Ind}_B^G(\chi|\cdot|_F^s)$ of G on the space of smooth functions f on G such that

$$f\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} g\right)=\chi(a)|a|_F^{s+1}f(g)$$

for all $a\in F^\times$, $b\in F$, and $g\in G$. For $\varphi\in\mathcal{S}(F^2)$ (where we regard F^2 as the space of row vectors), we define $f_\varphi\in I(s,\chi)$ by (the meromorphic continuation of)

$$f_{\varphi,s,\chi}(g)=Z(s+1,\chi,\phi_{r(g)\varphi})=\int_{F^\times} r(g)\varphi(0,t)\chi(t)|t|_F^{s+1}\,d^\times t,$$

where $\phi_\varphi(x)=\varphi(0,x)$ and $r(g)\varphi(x_1,x_2)=\varphi((x_1,x_2)g)$. For $\varphi\in\mathcal{S}(F^2)$, we define its Fourier transform $\mathcal{F}\varphi\in\mathcal{S}(F^2)$ by

$$\mathcal{F}\varphi(x_1,x_2)=\int_{F^2}\varphi(y_1,y_2)\psi_F(x_2y_1-x_1y_2)\,dy_1\,dy_2.$$

Note that $\mathcal{F}\circ r(g)=r(g)\circ\mathcal{F}$ for all $g\in G$. Recall the intertwining operator

$$J(s,\chi)=J_B(w,\chi|\cdot|_F^s):I(s,\chi)\to I(-s,\chi^{-1})$$

given by (the meromorphic continuation of)

$$J(s,\chi)f(g)=\int_F f(\widetilde{w}^{-1}n(y)g)dy,$$

where

$$\widetilde{w}=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\quad n(y)=\begin{pmatrix} 1 & y\\ 0 & 1\end{pmatrix}.$$

Lemma 2.6.4. *For $\varphi\in\mathcal{S}(F^2)$, we have*

$$J(s,\chi)f_{\varphi,s,\chi}(g)=\gamma(s,\chi,\psi_F)^{-1}f_{\mathcal{F}\varphi,-s,\chi^{-1}}.$$

Proof. For $\mathrm{Re}(s)\gg 0$, we have

$$\begin{aligned}J(s,\chi)f_{\varphi,s,\chi}(g)&=\int_F\int_{F^\times} r(\widetilde{w}^{-1}n(y)g)\varphi(0,t)\chi(t)|t|_F^{s+1}d^\times t\,dy\\&=\int_F\int_{F^\times} r(g)\varphi(t,ty)\chi(t)|t|_F^{s+1}\,d^\times t dy\\&=\int_{F^\times}\int_F r(g)\varphi(t,y)\chi(t)|t|_F^{s}\,dy\,d^\times t=Z(s,\chi,\phi),\end{aligned}$$

where

$$\phi(x)=\int_F r(g)\varphi(x,y)dy.$$

Hence, by the functional equation, we have

$$J(s,\chi)f_\varphi(g) = \gamma(s,\chi,\psi_F)^{-1} Z(1-s,\chi^{-1},\widehat{\phi}).$$

However, we have

$$\widehat{\phi}(x) = (\mathcal{F}\circ r(g)\varphi)(0,x) = (r(g)\circ \mathcal{F}\varphi)(0,x) = \phi_{r(g)\mathcal{F}\varphi}(x).$$

Since $f_{\mathcal{F}\varphi,-s,\chi^{-1}} = Z(1-s,\chi^{-1},\phi_{r(g)\mathcal{F}\varphi})$, this completes the proof. □

For $\varphi \in \mathcal{S}(F^2)$, we also define its partial Fourier transform $\mathcal{F}'\varphi \in \mathcal{S}(F^2)$ by

$$\mathcal{F}'\varphi(x_1,x_2) = \int_F \varphi(x_1,y_2)\psi_F(-x_2y_2)dy_2.$$

Recall the Whittaker functional $\Omega(s,\chi) = \Omega(\chi|\cdot|_F^s)$ on $I(s,\chi)$ given by (the holomorphic continuation of)

$$\Omega(s,\chi)f = \int_F f(\widetilde{w}^{-1}n(y))\psi_F(-y)dy.$$

Lemma 2.6.5. *For $\varphi \in \mathcal{S}(F^2)$, we have*

$$\Omega(s,\chi)f_{\varphi,s,\chi} = \int_{F^\times} \mathcal{F}'\varphi(t,t^{-1})\chi(t)|t|_F^s d^\times t.$$

(Note that the right-hand side is absolutely convergent for all s.)

Proof. For $\mathrm{Re}(s) \gg 0$, we have

$$\begin{aligned}\Omega(s,\chi)f_{\varphi,s,\chi} &= \int_F\int_{F^\times} r(\widetilde{w}^{-1}n(y))\varphi(0,t)\chi(t)|t|_F^{s+1}\psi_F(-y)d^\times t dy\\ &= \int_F\int_{F^\times} \varphi(t,ty)\chi(t)|t|_F^{s+1}\psi_F(-y)d^\times t dy\\ &= \int_{F^\times}\int_F \varphi(t,y)\chi(t)|t|_F^{s}\psi_F(-t^{-1}y)dy d^\times t\\ &= \int_{F^\times} \mathcal{F}'\varphi(t,t^{-1})\chi(t)|t|_F^s d^\times t.\end{aligned}$$

This completes the proof. □

Finally, recall the local coefficient $C(s,\chi,\psi_F) = C_B(\widetilde{w},\chi|\cdot|_F^s,\psi_F)$ given by

$$\Omega(s,\chi) = C(s,\chi,\psi_F)\cdot\Omega(-s,\chi^{-1})\circ J(s,\chi)$$

(see [Sha2, p. 333, Theorem 3.1]).

Proposition 2.6.6. *We have*

$$C(s,\chi,\psi_F) = \gamma(s,\chi,\psi_F).$$

Proof. By Lemmas 2.6.4 and 2.6.5, we have

$$\begin{aligned}\gamma(s,\chi,\psi_F)\cdot\Omega(-s,\chi^{-1})J(s,\chi)f_{\varphi,s,\chi} &= \Omega(-s,\chi^{-1})f_{\mathcal{F}\varphi,-s,\chi^{-1}} \\ &= \int_{F^\times}\mathcal{F}'\mathcal{F}\varphi(t,t^{-1})\chi(t)^{-1}|t|_F^{-s}d^\times t \\ &= \int_{F^\times}\mathcal{F}'\mathcal{F}\varphi(t^{-1},t)\chi(t)|t|_F^{s}d^\times t\end{aligned}$$

for $\varphi\in\mathcal{S}(F^2)$. Since $\mathcal{F}'\mathcal{F}\varphi(x_1,x_2)=\mathcal{F}'\varphi(x_2,x_1)$, the right-hand side is equal to

$$\int_{F^\times}\mathcal{F}'\varphi(t,t^{-1})\chi(t)|t|_F^s d^\times t=\Omega(s,\chi)f_{\varphi,s,\chi}$$

by Lemma 2.6.5. This completes the proof. □

3. The twisted local intertwining relation

The purpose of this section is to prove Theorem 1.9.1. This theorem is stated in [Ar2, Theorem 2.5.3] and [Mok, Proposition 3.5.1 (b)]. Arthur expected that this theorem (for non-tempered representations) would be proven by an argument "based on some version of minimal K-types". However, this idea might require a huge amount of computation even if F is non-archimedean and π_ψ is unramified.

To show Theorem 1.9.1, we shall use a new approach. The difficulty of this theorem is that the linear isomorphism $\theta_A\colon\pi_\psi\xrightarrow{\sim}\pi_\psi$ is defined through the Langlands quotient map from the standard module of π_ψ. Our idea is to realize this Langlands quotient map as a composition of normalized intertwining operators (see Lemma 3.3.1 below). Then we can show Theorem 1.9.1 by the multiplicativity of normalized intertwining operators (Proposition 1.7.2).

3.1. **Representations of general linear groups.**

Through this section, we write $G=\mathrm{GL}_N(E)$. For $\tau\in\mathrm{Rep}(G)$ and for a character χ of $E^\times$, we define $\tau\chi$ by $(\tau\chi)(g)=\tau(g)\chi(\det(g))$.

Let $P=MN$ be a standard parabolic subgroup of G with Levi subgroup $M\cong\mathrm{GL}_{k_1}(E)\times\cdots\times\mathrm{GL}_{k_t}(E)$. For $\tau_i\in\mathrm{Rep}(\mathrm{GL}_{k_i}(E))$, we denote the normalized parabolic induction by

$$\tau_1\times\cdots\times\tau_t=\mathrm{Ind}_P^G(\tau_1\boxtimes\cdots\boxtimes\tau_t).$$

It is known by Bernstein [Ber1] that if $\pi_M\in\mathrm{Irr}(M)$ is unitary, then $I_P(\pi_M)=\mathrm{Ind}_P^G(\pi_M)$ is an irreducible unitary representation of G.

Recall that a *standard module* of G is an induced representation of the form

$$\tau_1|\cdot|_E^{e_1}\times\cdots\times\tau_t|\cdot|_E^{e_t},$$

where τ_i is an irreducible tempered representation of $\mathrm{GL}_{k_i}(E)$ and $e_i\in\mathbb{R}$ such that $e_1>\cdots>e_t$. It has a unique irreducible quotient, called the *Langlands quotient*. For example, if $\psi=\phi\boxtimes S_a$ is an A-parameter for G with ϕ an irreducible representation of

L_E with $\phi(W_E)$ bounded, then π_ϕ is discrete series, and π_ψ is the Langlands quotient of the standard module

$$\mathcal{I}_\psi^G = \pi_\phi|\cdot|_E^{\frac{a-1}{2}} \times \pi_\phi|\cdot|_E^{\frac{a-3}{2}} \times \cdots \times \pi_\phi|\cdot|_E^{-\frac{a-1}{2}}.$$

In this situation, we write

$$\pi_\psi = \mathrm{Speh}(\pi_\phi, a)$$

and call it a *Speh representation*. More generally, if $\psi = \oplus_{i=1}^t \phi_i \boxtimes S_{a_i}$ is an irreducible decomposition of an A-parameter for G, then

$$\pi_\psi = \bigtimes_{i=1}^{t} \mathrm{Speh}(\pi_{\phi_i}, a_i)$$

and its standard module is

$$\mathcal{I}_\psi^G = \bigtimes_{i=1}^{t} \bigtimes_{e_i=1}^{a_i} \pi_{\phi_i}|\cdot|_E^{\frac{a_i+1}{2}-e_i},$$

where the product is taken in decreasing order of the exponents.

Lemma 3.1.1. *Let* Π *be a standard module of* G*. Then*

$$\dim_{\mathbb{C}}(\mathrm{End}_G(\Pi)) = 1.$$

Proof. This is a consequence of the famous fact that the Langlands quotient π of Π appears in Π as subquotients with multiplicity one (See e.g., [BW, Chapter XI, Lemma 2.13]). This fact induces an injective linear map

$$\mathrm{End}_G(\Pi) \to \mathrm{End}_G(\pi).$$

Since $\dim_{\mathbb{C}}(\mathrm{End}_G(\pi)) = 1$ by Schur's lemma, we obtain the lemma. □

For the rest of this subsection, we assume that E is non-archimedean. We will identify an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(E)$ with the irreducible d-dimensional bounded representation of W_E by the LLC for $\mathrm{GL}_d(E)$.

Recall that a *segment* is a set of the form

$$[x, y]_\rho = \{\rho|\cdot|_E^x, \rho|\cdot|_E^{x+1}, \dots, \rho|\cdot|_E^y\},$$

where ρ is an irreducible unitary supercuspidal representation of $\mathrm{GL}_d(E)$ and $x \leq y$ are real numbers such that $x \equiv y \bmod \mathbb{Z}$. One can attach to it two irreducible representations $\Delta([x, y]_\rho)$ and $Z([x, y]_\rho)$ of $\mathrm{GL}_{d(y-x+1)}(E)$, which are the unique irreducible subrepresentation and the unique irreducible quotient of the standard module

$$\rho|\cdot|_E^y \times \cdots \times \rho|\cdot|_E^{x+1} \times \rho|\cdot|_E^x,$$

respectively. We call $\Delta([x, y]_\rho)$ a *(generalized) Steinberg representation*. Note that $\Delta([x, y]_\rho)$ is an essentially discrete series representation, and all essentially discrete series representations are of this form (see [Z, Theorem 9.3]). Similarly, any irreducible tempered representation is a product of representations of the form $\Delta([-x, x]_\rho)$. On the other hand, by definition, we have $Z([-x, x]_\rho) = \pi_{\rho\boxtimes S_1 \boxtimes S_{2x+1}}$ if $2x \in \mathbb{Z}$.

A *multi-segment* is a formal finite sum of segments. For a multi-segment $\mathfrak{m}$, writing $\mathfrak{m} = [x_1, y_1]_{\rho_1} + \cdots + [x_r, y_r]_{\rho_r}$ with $x_1 + y_1 \geq \cdots \geq x_r + y_r$, we set

$$\mathcal{I}(\mathfrak{m}) = \Delta([x_1, y_1]_{\rho_1}) \times \cdots \times \Delta([x_r, y_r]_{\rho_r}).$$

This is a standard module. For example, let $\psi = \rho \boxtimes S_{2\alpha+1} \boxtimes S_{2\beta+1}$ be an A-parameter for $G = \mathrm{GL}_N(E)$. If we set

$$\mathfrak{m} = [-\alpha+\beta, \alpha+\beta]_\rho + [-\alpha+\beta-1, \alpha+\beta-1]_\rho + \cdots + [-\alpha-\beta, \alpha-\beta]_\rho,$$

then the standard module $\mathcal{I}_\psi^G$ of π_ψ is equal to $\mathcal{I}(\mathfrak{m})$.

Lemma 3.1.2. *Let ψ be an A-parameter for $G = \mathrm{GL}_N(E)$. Then the standard module $\mathcal{I}_\psi^G$ of π_ψ contains an irreducible tempered representation π_{ψ^D} as a subrepresentation, where $\psi^D \colon W_E \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ is given by $\psi^D(w, \alpha) = \psi(w, \alpha, \alpha)$. Moreover, π_{ψ^D} appears in $\mathcal{I}_\psi^G$ as a subquotient with multiplicity one.*

Proof. By [JS], the unique irreducible subrepresentation of $\mathcal{I}_\psi^G$ is generic. By [Z, Theorem 9.7], it is an irreducible product of generalized Steinberg representations. As it has the same cuspidal support as π_ψ, we deduce that the unique irreducible subrepresentation of $\mathcal{I}_\psi^G$ is π_{ψ^D}. Finally, the multiplicity one statement follows from [Z, Proposition 8.4]. □

3.2. **The tempered case.** Let $G = \mathrm{GL}_N(E)$ with an involution θ defined in Section 1.4. Fix a standard parabolic subgroup $P = MN$. In this subsection, we will prove Theorem 1.9.1 for tempered representations, i.e., for generic (or tempered) ψ. Thus we consider an irreducible tempered representation π of M, and $w \in W(\theta(M), M)$ such that $w(\pi \circ \theta) \cong \pi$.

Since π is tempered, it is $\mathfrak{w}_M$-generic. Fix a non-trivial $\mathfrak{w}_M$-Whittaker functional ω on π. Then $I_P(\pi)$ is an irreducible $\mathfrak{w}$-generic representation of G with the $\mathfrak{w}$-Whittaker functional $\Omega(\pi)$ induced by ω as in Section 1.8. By definition (see Section 1.4), $\theta_A = \theta_W$ is the unique linear isomorphism $\theta_A \colon I_P(\pi) \xrightarrow{\sim} I_P(\pi)$ such that

$$\begin{aligned} &\theta_A \circ I_P(\pi)(h) = I_P(\pi)(\theta(h)) \circ \theta_A, \quad h \in \mathrm{GL}_N(E), \\ &\Omega(\pi) \circ \theta_A = \Omega(\pi). \end{aligned}$$

In Section 1.9, we already argued that $\widetilde{R}_P(\theta \circ w, \widetilde{\pi})$ is a constant multiple of θ_A, so the equation $\widetilde{R}_P(\theta \circ w, \widetilde{\pi}) = \theta_A$ would follow from $\Omega(\pi) \circ \widetilde{R}_P(\theta \circ w, \widetilde{\pi}) = \Omega(\pi)$.

Recall from Section 1.9 that

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}) = I_P(\widetilde{\pi}(w \rtimes \theta)) \circ R_{\theta(P)}(w, \pi \circ \theta) \circ \theta^*.$$

So it suffices to check that the three squares in the diagram

$$\begin{array}{ccccccc} I_P(\pi) & \xrightarrow{\theta^*} & I_{\theta(P)}(\pi \circ \theta) & \xrightarrow{R_{\theta(P)}(w, \pi\circ\theta)} & I_P(w(\pi \circ \theta)) & \xrightarrow{I_P(\widetilde{\pi}(w \rtimes \theta))} & I_P(\pi) \\ {\scriptstyle \Omega(\pi)} \downarrow & & {\scriptstyle \Omega(\pi\circ\theta)} \downarrow & & \downarrow {\scriptstyle \Omega(w(\pi\circ\theta))} & & \downarrow {\scriptstyle \Omega(\pi)} \\ \mathbb{C} & = & \mathbb{C} & = & \mathbb{C} & = & \mathbb{C} \end{array}$$

are all commutative. Here, we note that all these Ω are induced by the same linear functional ω. The commutativity of the middle square is nothing but Theorem 1.8.1 (2), whereas the one for the right square follows from $\omega \circ \widetilde{\pi}(w \rtimes \theta) = \omega$, which is the definition of the normalization of $\widetilde{\pi}(w \rtimes \theta)$.

We show the commutativity of the left square.

Lemma 3.2.1. *For $w \in W^G$, we have*

$$\theta(\widetilde{w}) = \widetilde{\theta(w)}.$$

Proof. We recall the definition of the Tits lifting $\widetilde{w} \in G$ of $w \in W^G$. If $w = w_{\alpha_1} \cdots w_{\alpha_k}$ is a reduced decomposition relative to the simple roots of (G,T), where w_α is a simple reflection with respect to a simple root α, then $\widetilde{w}$ is defined by $\widetilde{w} = \widetilde{w}_{\alpha_1} \cdots \widetilde{w}_{\alpha_k}$. Hence we may assume that $w = w_\alpha$ for some simple root α. Then $\widetilde{w}_\alpha$ is defined by

$$\widetilde{w}_\alpha = \exp(X_\alpha)\exp(-X_{-\alpha})\exp(X_\alpha),$$

where X_α is already given as we fix a splitting $\boldsymbol{spl}$, and $X_{-\alpha}$ is the root vector for $-\alpha$ such that $H_\alpha = [X_\alpha, X_{-\alpha}]$ is the coroot for α. Since $\boldsymbol{spl}$ is θ-stable, the claim follows. □

Recall that $\Omega(\pi)$ is defined by the holomorphic continuation of the Jacquet integral

$$\Omega(\pi_\lambda)f = \int_{N'} \omega(f(\widetilde{w}_0^{-1}n'))\chi(n')^{-1}dn'.$$

Here $N' = \widetilde{w}_0\overline{N}\widetilde{w}_0^{-1}$ with $w_0 = w_\ell w_\ell^M$, where w_ℓ and w_ℓ^M are the longest elements in W^G and W^M, respectively, and χ is (the restriction of) the non-degenerate character of the unipotent radical U of the Borel subgroup B given by $\boldsymbol{spl}$ and ψ_F. Note that $\theta(w_\ell) = w_\ell$, $\theta(w_\ell^M) = w_\ell^{\theta(M)}$ and $\chi \circ \theta = \chi$. Hence, Lemma 3.2.1 implies that $\Omega(\pi \circ \theta) \circ \theta^* = \Omega(\pi)$. This completes the proof of Theorem 1.9.1 for tempered representations.

3.3. **Construction of the Langlands quotient map.** Fix a standard parabolic subgroup $P = MN_P$ of $G = \mathrm{GL}_N(E)$. Let ψ be an A-parameter for M, and let π_ψ be the associated irreducible unitary representation of M. Assume that there is $w \in W(\theta(M), M)$ such that $w(\pi_\psi \circ \theta) \cong \pi_\psi$, and we fix such an element w in this and next subsections.

Note that $I_P(\pi_\psi)$ is irreducible. Let $\mathcal{I}_\psi^G$ be a standard module of G whose Langlands quotient is $I_P(\pi_\psi)$. Recall from Section 1.4 that a Whittaker functional Ω on $\mathcal{I}_\psi^G$ defines a linear isomorphism $\theta_W \colon \mathcal{I}_\psi^G \xrightarrow{\sim} \mathcal{I}_\psi^G$, which induces $\theta_A \colon \pi_\psi \xrightarrow{\sim} \pi_\psi$. To show Theorem 1.9.1 for π_ψ, we shall carefully construct $\mathcal{I}_\psi^G$.

Set $V = E^N$ so that $G = \mathrm{GL}(V)$. Decompose V into a direct sum

$$V = V^{(1)} \oplus \cdots \oplus V^{(t)}$$

such that M is the subgroup of G stabilizing $V^{(i)}$ for $i = 1, \dots, t$. Hence $M \cong G^{(1)} \times \cdots \times G^{(t)}$, where $G^{(i)} = \mathrm{GL}(V^{(i)})$.

Recall that ψ is an A-parameter for M. It can be decomposed as

$$\psi = \psi^{(1)} \oplus \cdots \oplus \psi^{(t)},$$

where $\psi^{(i)}$ is an A-parameter for $G^{(i)}$. Consider the decomposition of $\psi^{(i)}$ into irreducible representations:

$$\psi^{(i)} = \bigoplus_{j=1}^{m_i} \psi_j^{(i)}.$$

Corresponding to this decomposition, we can also decompose $V^{(i)}$ as

$$V^{(i)} = \bigoplus_{j=1}^{m_i} V_j^{(i)}$$

such that $\dim_{\mathbb{C}}(\psi_j^{(i)}) = \dim_E(V_j^{(i)})$.

Define a representation $\phi_{\psi_j^{(i)}}$ of L_E by

$$\phi_{\psi_j^{(i)}}(w) = \psi_j^{(i)}\left(w, \begin{pmatrix} |w|_E^{\frac{1}{2}} & 0 \\ 0 & |w|_E^{-\frac{1}{2}} \end{pmatrix}\right).$$

If we write $\psi_j^{(i)} = \phi_j^{(i)} \boxtimes S_d$, where $\phi_j^{(i)}$ is an irreducible representation of L_E with $\phi_j^{(i)}(W_E)$ bounded, and $d = d_j^{(i)} \geq 1$, then we have

$$\phi_{\psi_j^{(i)}} = \bigoplus_{k=1}^{d} \phi_j^{(i)} |\cdot|_E^{\frac{d+1}{2}-k}.$$

Corresponding to this decomposition, we can decompose $V_j^{(i)}$ as

$$V_j^{(i)} = \bigoplus_{k=1}^{d} V_j^{(i)}\left(\frac{d+1}{2} - k\right)$$

such that $\dim_E(V_j^{(i)}(\frac{d+1}{2} - k)) = \dim_{\mathbb{C}}(\phi_j^{(i)}|\cdot|_E^{\frac{d+1}{2}-k}) = \dim_{\mathbb{C}}(\phi_j^{(i)})$. Fix a linear isomorphism $V_j^{(i)}(\frac{d+1}{2} - k) \cong V_j^{(i)}(\frac{d+1}{2} - k')$ for $1 \leq k, k' \leq d$. When $\alpha \in (1/2)\mathbb{Z}$ satisfies $\alpha \not\equiv \frac{d+1}{2} \bmod \mathbb{Z}$ or $|\alpha| \geq \frac{d+1}{2}$, we formally set $V_j^{(i)}(\alpha) = 0$.

Finally, we define

$$V^{(i)}(\alpha) = \bigoplus_{j=1}^{m_i} V_j^{(i)}(\alpha).$$

After all, we obtain a decomposition

$$V = \bigoplus_{i=1}^{t} \bigoplus_{\alpha \in (1/2)\mathbb{Z}} V^{(i)}(\alpha).$$

Consider the finite set

$$\mathcal{V} = \left\{ V^{(i)}(\alpha) \,\middle|\, 1 \leq i \leq t,\ \alpha \in (1/2)\mathbb{Z} \text{ such that } V^{(i)}(\alpha) \neq 0 \right\}.$$

We can define two total orders $\prec_1$ and $\prec_2$ on $\mathcal{V}$ as follows. Firstly, $V^{(i)}(\alpha) \prec_1 V^{(i')}(\alpha')$ if and only if

- $i < i'$; or
- $i = i'$ and $\alpha > \alpha'$.

Secondly, $V^{(i)}(\alpha) \prec_2 V^{(i')}(\alpha')$ if and only if

- $\alpha > \alpha'$; or
- $\alpha = \alpha'$ and $i < i'$.

If we write $\mathcal{V} = \{V_1, \dots, V_r\}$ with $V_1 \prec_1 \cdots \prec_1 V_r$ and $V_1' \prec_2 \cdots \prec_2 V_r'$, we define two parabolic subgroups $P_1 = M_1 N_{P_1}$ and $P_1' = M_1 N_{P_1'}$ as the stabilizers of the flags

$$V_1 \subset V_1 \oplus V_2 \subset \cdots \subset V_1 \oplus \cdots \oplus V_r$$

and

$$V_1' \subset V_1' \oplus V_2' \subset \cdots \subset V_1' \oplus \cdots \oplus V_r',$$

respectively. Here, M_1 is the stabilizer of elements $V_i \in \mathcal{V}$, which is a common Levi subgroup of P_1 and P_1'. Note that P_1 is contained in P. We may assume that P_1 is standard, but P_1' is not standard in general. Let $w_2 \in W^G$ be such that $\widetilde{w}_2^{-1} P_1' \widetilde{w}_2 = P_2 = M_2 N_{P_2}$ is a standard parabolic subgroup, where $\widetilde{w}_2 M_2 \widetilde{w}_2^{-1} = M_1$. We regard w_2 as an element in $W(M_2, M_1)$.

Let $\tau_j^{(i)}$ be the irreducible discrete series representation corresponding to $\phi_j^{(i)}$. We regard $\tau_j^{(i)} |\cdot|_E^{\alpha}$ as a representation of $\mathrm{GL}(V_j^{(i)}(\alpha))$ if $V_j^{(i)}(\alpha) \neq 0$. By induction, we obtain an essentially tempered representation

$$\tau^{(i)} |\cdot|_E^{\alpha} = \bigtimes_{j=1}^{m_i} \tau_j^{(i)} |\cdot|_E^{\alpha}$$

of $\mathrm{GL}(V^{(i)}(\alpha))$. Using $\prec_1$, we obtain an essentially tempered representation

$$\tau_1 = \bigotimes_{i,\alpha} \tau^{(i)} |\cdot|_E^{\alpha}$$

of M_1. Set

$$\tau_2 = w_2^{-1} \tau_1,$$

which is an essentially tempered representation of M_2.

Note that $P_1 \subset P$ and $M_1 \subset M$. Recall that $\pi_\psi \in \mathrm{Irr}(M)$ is the representation corresponding to ϕ_ψ via the LLC for M. It is the Langlands quotient of the standard module

$$\mathcal{I}_\psi^M = \mathrm{Ind}_{P_1 \cap M}^M (\tau_1)$$

of M. In particular, $I_P(\pi_\psi)$ is a quotient of $I_P(\mathcal{I}_\psi^M) = I_{P_1}(\tau_1)$. Moreover, there is $w_1 \in W(M_1)$ with $\widetilde{w}_1 \in M$ such that the image of the normalized intertwining operator

$$R_{P_1}(w_1, \tau_1) \colon I_{P_1}(\tau_1) \to I_{P_1}(w_1 \tau_1)$$

is isomorphic to $I_P(\pi_\psi)$. Hereafter, we identify $I_P(\pi_\psi)$ with this subspace, i.e., $I_P(\pi_\psi)$ is realized as the image of $R_{P_1}(w_1,\tau_1)$. Hence we obtain a surjection

$$R_{P_1}(w_1,\tau_1)\colon I_{P_1}(\tau_1) \twoheadrightarrow I_P(\pi_\psi) \subset I_{P_1}(w_1\tau_1).$$

On the other hand, as the unitary induction preserves the irreducibility for general linear groups, we see that $I_P(\pi_\psi)$ is the irreducible representation of G, which corresponds to ϕ_ψ, regarded as an L-parameter of G, via the LLC for G. It is the Langlands quotient of the standard module $\mathcal{I}_\psi^G = I_{P_2}(\tau_2)$. Since $\tau_1 = w_2\tau_2$, we have a normalized intertwining operator

$$R_{P_2}(w_2,\tau_2)\colon I_{P_2}(\tau_2) \to I_{P_1}(\tau_1).$$

Lemma 3.3.1. *The composition*

$$I_{P_2}(\tau_2) \xrightarrow{R_{P_2}(w_2,\tau_2)} I_{P_1}(\tau_1) \xrightarrow{R_{P_1}(w_1,\tau_1)} I_P(\pi_\psi)$$

is well-defined and nonzero. In particular, this composition realizes the Langlands quotient map. Namely, $I_{P_2}(\tau_2)$ is the standard module of $I_P(\pi_\psi)$, and the above composition is surjective.

Proof. Recall that the normalized intertwining operators are defined by the meromorphic continuation of certain (normalized) integrals. Note that the operators $R_{P_1}(w_1,\tau_1)$ and $R_{P_2}(w_2,\tau_2)$ are compositions of normalized intertwining operators of the form

$$\tau_j^{(i)}|\cdot|_E^{\alpha} \times \tau_{j'}^{(i')}|\cdot|_E^{\alpha'} \to \tau_{j'}^{(i')}|\cdot|_E^{\alpha'} \times \tau_j^{(i)}|\cdot|_E^{\alpha}$$

with $\alpha \geq \alpha'$. These displayed operators are regular and nonzero at the relevant points. Hence, $R_{P_1}(w_1,\tau_1)$ and $R_{P_2}(w_2,\tau_2)$ are well-defined.

We now verify that the composite $R_{P_1}(w_1,\tau_1) \circ R_{P_2}(w_2,\tau_2)$ is nonzero. Since $I_P(\pi_\psi)$ is the unique irreducible quotient of $I_{P_2}(\tau_2)$, and since $R_{P_2}(w_2,\tau_2)$ is nonzero, $I_P(\pi_\psi)$ appears in the image of $R_{P_2}(w_2,\tau_2)$. If $R_{P_1}(w_1,\tau_1) \circ R_{P_2}(w_2,\tau_2) = 0$, then we would conclude that $I_P(\pi_\psi)$ appears in $I_{P_1}(\tau_1)$ as subquotients with multiplicity greater than one. Since the semisimplification of $I_{P_1}(\tau_1)$ is the same as the one of the standard module $I_{P_2}(\tau_2)$, this contradicts the fact that the Langlands quotient $I_P(\pi_\psi)$ appears in $I_{P_2}(\tau_2)$ with multiplicity one. □

3.4. **The main diagram.** Recall that we have fixed an element $w \in W(\theta(M), M)$ such that $w(\pi_\psi \circ \theta) \cong \pi_\psi$. As in Section 1.7, we regard w as an element of W^G. By construction, $w\theta(M_1)w^{-1} = M_1$ and $w(w_1\tau_1 \circ \theta) \cong w_1\tau_1$. Set

$$w' = w_2^{-1}w_1^{-1}w\theta(w_1)\theta(w_2).$$

Lemma 3.4.1. (1) *The canonical inclusion $N_P \hookrightarrow N_{P_1}$ induces a homeomorphism*

$$N_P \cap \widetilde{w}N_{\theta(P)}\widetilde{w}^{-1}\backslash N_P \cong N_{P_1} \cap \widetilde{w}N_{\theta(P_1)}\widetilde{w}^{-1}\backslash N_{P_1}.$$

(2) *We have $w'\theta(M_2)w'^{-1} = M_2$ and $w'(\tau_2 \circ \theta) \cong \tau_2$. Let $\widetilde{\tau}_2(w' \rtimes \theta)\colon w'(\tau_2\circ\theta) \xrightarrow{\sim} \tau_2$ be the isomorphism normalized by using Whittaker functional on τ_2.*

(3) *The normalized intertwining operator* $\widetilde{R}_{P_2}(\theta \circ w', \widetilde{\tau}_2)\colon I_{P_2}(\tau_2) \to I_{P_2}(\tau_2)$ *defined by the composition*

$$I_{P_2}(\tau_2) \xrightarrow{\theta^*} I_{\theta(P_2)}(\tau_2 \circ \theta) \xrightarrow{R_{\theta(P_2)}(w', \tau_2 \circ \theta)} I_{P_2}(w'(\tau_2 \circ \theta)) \xrightarrow{I_{P_2}(\widetilde{\tau}_2(w' \rtimes \theta))} I_{P_2}(\tau_2)$$

is bijective.

Proof. Assertion (1) follows from the equation $\widetilde{w} N_{\theta(P_1)} \widetilde{w}^{-1} \cap M = N_{P_1} \cap M$.

For (2), the first assertion follows by direct computation

$$\begin{aligned} w'\theta(M_2)w'^{-1} &= w_2^{-1}w_1^{-1}w\theta(w_1)\theta(w_2)\theta(M_2)\theta(w_2)^{-1}\theta(w_1)^{-1}w^{-1}w_1w_2 \\ &= w_2^{-1}w_1^{-1}w\theta(w_1)\theta(M_1)\theta(w_1)^{-1}w^{-1}w_1w_2 \\ &= w_2^{-1}w_1^{-1}w\theta(M_1)w^{-1}w_1w_2 \\ &= w_2^{-1}w_1^{-1}M_1w_1w_2 = w_2^{-1}M_1w_2 = M_2. \end{aligned}$$

The second assertion is proven similarly.

For (3), it is obvious that $\theta^*\colon I_{P_2}(\tau_2) \to I_{\theta(P_2)}(\tau_2 \circ \theta)$ is bijective. Note that $I_{\theta(P_2)}(\tau_2 \circ \theta)$ is a standard module of G whose Langlands quotient is $I_{\theta(P)}(\pi_\psi \circ \theta) \cong I_P(\pi_\psi)$. Hence $I_{\theta(P_2)}(\tau_2 \circ \theta)$ and $I_{P_2}(\tau_2)$ are standard modules whose Langlands quotients are isomorphic to each other.

Since $\tau_2 \circ \theta$ and $w'(\tau_2 \circ \theta) \cong \tau_2$ is essentially tempered representations of $M_2 = \theta(M_2)$ such that two inductions $I_{P_2}(\tau_2 \circ \theta)$ and $I_{P_2}(w'(\tau_2 \circ \theta))$ are both standard modules, we see that $R_{\theta(P_2)}(w', \tau_2 \circ \theta)$ is nonzero. Since $I_{P_2}(\widetilde{\tau}_2(w' \rtimes \theta)) \circ R_{\theta(P_2)}(w', \tau_2 \circ \theta)$ is a nonzero element in $\mathrm{Hom}_G(I_{\theta(P_2)}(\tau_2 \circ \theta), I_{P_2}(\tau_2))$ which is one dimensional by Lemma 3.1.1, it must be bijective. □

Now we will prove a key result:

Theorem 3.4.2. *The* "main diagram"

$$\begin{array}{ccc} I_{P_2}(\tau_2) & \xrightarrow{\widetilde{R}_{P_2}(\theta \circ w', \widetilde{\tau}_2)} & I_{P_2}(\tau_2) \\ {\scriptstyle R_{P_2}(w_2, \tau_2)} \downarrow & & \downarrow {\scriptstyle R_{P_2}(w_2, \tau_2)} \\ I_{P_1}(\tau_1) & & I_{P_1}(\tau_1) \\ {\scriptstyle R_{P_1}(w_1, \tau_1)} \downarrow & & \downarrow {\scriptstyle R_{P_1}(w_1, \tau_1)} \\ I_P(\pi_\psi) & \xrightarrow{\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi)} & I_P(\pi_\psi) \end{array}$$

is commutative.

Admitting this result, we can complete the proof of Theorem 1.9.1 as follows. Recall that $\mathcal{I}_\psi^G = I_{P_2}(\tau_2)$ is the standard module of $I_P(\pi_\psi)$. Moreover, by Theorem 1.9.1 for the tempered case together with analytic continuation, we have

$$\widetilde{R}_{P_2}(\theta \circ w', \widetilde{\tau}_2) = \theta_W.$$

By the definition of θ_A (see Section 1.4), Theorem 3.4.2 together with Lemma 3.3.1 implies that

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_\psi) = \theta_A.$$

This completes the proof of Theorem 1.9.1 in general.

Now we show Theorem 3.4.2.

Proof of Theorem 3.4.2. Recall that the top and bottom maps of the main diagram are composites of three maps:

$$\widetilde{R}_{P_2}(\theta \circ w', \widetilde{\tau}_2) = I_{P_2}(\widetilde{\tau}_2(w' \rtimes \theta)) \circ R_{\theta(P_2)}(w', \tau_2 \circ \theta) \circ \theta^*$$

and

$$\widetilde{R_P}(\theta \circ w, \widetilde{\pi}_\psi) = I_P(\widetilde{\pi}_\psi(w \rtimes \theta)) \circ R_{\theta(P)}(w, \pi_\psi \circ \theta, \psi) \circ \theta^*.$$

Hence the main diagram written in its full glory has the form

$$\begin{array}{ccccccc}
I_{P_2}(\tau_2) & \xrightarrow{\theta^*} & I_{\theta(P_2)}(\tau_2 \circ \theta) & \xrightarrow{R_{\theta(P_2)}(w', \tau_2 \circ \theta)} & I_{P_2}(w'(\tau_2 \circ \theta)) & \xrightarrow{I_{P_2}(\widetilde{\tau}_2(w' \rtimes \theta))} & I_{P_2}(\tau_2) \\
\downarrow{\scriptstyle R_{P_2}(w_2,\tau_2)} & & & & & & {\scriptstyle R_{P_2}(w_2,\tau_2)}\downarrow \\
I_{P_1}(\tau_1) & & & & & & I_{P_1}(\tau_1) \\
\downarrow{\scriptstyle R_{P_1}(w_1,\tau_1)} & & & & & & {\scriptstyle R_{P_1}(w_1,\tau_1)}\downarrow \\
I_P(\pi_\psi) & \xrightarrow{\theta^*} & I_{\theta(P)}(\pi_\psi \circ \theta) & \xrightarrow{R_{\theta(P)}(w, \pi_\psi \circ \theta, \psi)} & I_P(w(\pi_\psi \circ \theta)) & \xrightarrow{I_P(\widetilde{\pi}_\psi(w \rtimes \theta))} & I_P(\pi_\psi).
\end{array}$$

We shall enhance this diagram by introducing additional stepping stones in the second row, namely by introducing the representations

$$I_{\theta(P_1)}(\tau_1 \circ \theta) \quad \text{and} \quad I_{P_1}(w''(\tau_1 \circ \theta))$$

with $w'' = w_1^{-1} w \theta(w_1)$, and additional maps connecting them to their neighbors. Hence the enhanced diagram has the form:

$$\begin{array}{ccccccc}
I_{P_2}(\tau_2) & \xrightarrow{\theta^*} & I_{\theta(P_2)}(\tau_2 \circ \theta) & \xrightarrow{R_{\theta(P_2)}(w', \tau_2 \circ \theta)} & I_{P_2}(w'(\tau_2 \circ \theta)) & \xrightarrow{I_{P_2}(\widetilde{\tau}_2(w' \rtimes \theta))} & I_{P_2}(\tau_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I_{P_1}(\tau_1) & \xrightarrow{\theta^*} & I_{\theta(P_1)}(\tau_1 \circ \theta) & & I_{P_1}(w''(\tau_1 \circ \theta)) & \xrightarrow{I_{P_1}(\widetilde{\tau}_1(w'' \rtimes \theta))} & I_{P_1}(\tau_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I_P(\pi_\psi) & \xrightarrow{\theta^*} & I_{\theta(P)}(\pi_\psi \circ \theta) & \xrightarrow{R_{\theta(P)}(w, \pi_\psi \circ \theta, \psi)} & I_P(w(\pi_\psi \circ \theta)) & \xrightarrow{I_P(\widetilde{\pi}_\psi(w \rtimes \theta))} & I_P(\pi_\psi).
\end{array}$$

Here, the vertical maps are of the form $R_{P_*}(w_*, \tau_*)$, see below for the details. To prove the commutativity of the main diagram, we shall show that the three vertical rectangles are commutative.

The commutativity of the left rectangle

$$\begin{array}{ccc} I_{P_2}(\tau_2) & \xrightarrow{\theta^*} & I_{\theta(P_2)}(\tau_2 \circ \theta) \\ {\scriptstyle R_{P_2}(w_2,\tau_2)}\downarrow & & \downarrow {\scriptstyle R_{\theta(P_2)}(\theta(w_2),\tau_2\circ\theta)} \\ I_{P_1}(\tau_1) & \xrightarrow{\theta^*} & I_{\theta(P_1)}(\tau_1 \circ \theta) \\ {\scriptstyle R_{P_1}(w_1,\tau_1)}\downarrow & & \downarrow {\scriptstyle R_{\theta(P_1)}(\theta(w_1),\tau_1\circ\theta)} \\ I_P(\pi_\psi) & \xrightarrow{\theta^*} & I_{\theta(P)}(\pi_\psi \circ \theta) \end{array}$$

follows easily from Lemma 3.2.1.

Now let us consider the right rectangle:

$$\begin{array}{ccc} I_{P_2}(w'(\tau_2 \circ \theta)) & \xrightarrow{I_{P_2}(\widetilde{\tau}_2(w' \rtimes \theta))} & I_{P_2}(\tau_2) \\ {\scriptstyle R_{P_2}(w_2,w'(\tau_2\circ\theta))}\downarrow & & \downarrow {\scriptstyle R_{P_2}(w_2,\tau_2)} \\ I_{P_1}(w''(\tau_1 \circ \theta)) & \xrightarrow{I_{P_1}(\widetilde{\tau}_2(w'' \rtimes \theta))} & I_{P_1}(\tau_1) \\ {\scriptstyle R_{P_1}(w_1,w''(\tau_1\circ\theta))}\downarrow & & \downarrow {\scriptstyle R_{P_1}(w_1,\tau_1)} \\ I_P(w(\pi_\psi \circ \theta)) & \xrightarrow{I_P(\widetilde{\pi}_\psi(w \rtimes \theta))} & I_P(\pi_\psi). \end{array}$$

If we realize $w'(\tau_2 \circ \theta)$, τ_2, $w''(\tau_1 \circ \theta)$ and τ_1 on the same vector space, say $\mathcal{V}$, then $\widetilde{\tau}_2(w' \rtimes \theta)$ is a linear isomorphism $\Phi\colon \mathcal{V} \to \mathcal{V}$ satisfying that

$$\Phi \circ \tau_2(\theta(\widetilde{w}'^{-1} m_2 \widetilde{w}')) = \tau_2(m_2) \circ \Phi, \quad m_2 \in M_2.$$

By Lemma 1.7.1, it can be rewrite as

$$\Phi \circ \tau_1(\theta(\widetilde{w}''^{-1} m_1 \widetilde{w}'')) = \tau_1(m_1) \circ \Phi, \quad m_1 \in M_1.$$

Since $\widetilde{\tau}_2(w' \rtimes \theta)$ and $\widetilde{\tau}_1(w'' \rtimes \theta)$ are both normalized using a Whittaker functional, we see that $\widetilde{\tau}_2(w' \rtimes \theta) = \widetilde{\tau}_1(w'' \rtimes \theta)$ as linear isomorphisms on $\mathcal{V}$. It implies the commutativity of the top square. To see the commutativity of the bottom square, we note that

$$I_{P_1}(\tau_1) = I_P(\mathcal{I}_\psi^M)$$

where $\mathcal{I}_\psi^M$ is the standard module of π_ψ. Likewise,

$$I_{P_1}(w''(\tau_1 \circ \theta)) = I_P(\mathrm{Ind}_{P_1 \cap M}^M(w''(\tau_1 \circ \theta))).$$

Recall that $\widetilde{w_1} \in M$ so that we can regard $w_1 \in W^M$. Hence the commutativity of the bottom square follows by the functoriality of I_P if we were to prove the diagram

$$\begin{array}{ccc} \mathrm{Ind}_{P_1\cap M}^M(w''(\tau_1 \circ \theta)) & \xrightarrow{\mathrm{Ind}_{P_1\cap M}^M(\widetilde{\tau}_1(w'' \rtimes \theta))} & \mathrm{Ind}_{P_1\cap M}^M(\tau_1) \\ {\scriptstyle R_{P_1\cap M}(w_1,w''(\tau_1\circ\theta))}\downarrow & & \downarrow {\scriptstyle R_{P_1\cap M}(w_1,\tau_1)} \\ w(\pi_\psi \circ \theta) & \xrightarrow{\widetilde{\pi}_\psi(w \rtimes \theta)} & \pi_\psi \end{array}$$

is commutative. But this follows from the definition of $\widetilde{\pi}_\psi(w \rtimes \theta)$. Hence (the bottom square of) the right rectangle in the main diagram is commutative.

To prove Theorem 3.4.2, it remains to show the commutativity of the middle rectangle:

$$\begin{array}{ccc}
I_{\theta(P_2)}(\tau_2\circ\theta) & \xrightarrow{R_{\theta(P_2)}(w',\tau_2\circ\theta)} & I_{P_2}(w'(\tau_2\circ\theta)) \\
{\scriptstyle R_{\theta(P_2)}(\theta(w_2),\tau_2\circ\theta)}\downarrow & & \downarrow{\scriptstyle R_{P_2}(w_2,w'(\tau_2\circ\theta))} \\
I_{\theta(P_1)}(\tau_1\circ\theta) & & I_{P_1}(w''(\tau_1\circ\theta)) \\
{\scriptstyle R_{\theta(P_1)}(\theta(w_1),\tau_1\circ\theta)}\downarrow & & \downarrow{\scriptstyle R_{P_1}(w_1,w''(\tau_1\circ\theta))} \\
I_{\theta(P)}(\pi_\psi\circ\theta) & \xrightarrow{R_{\theta(P)}(w,\pi_\psi\circ\theta,\psi)} & I_P(w(\pi_\psi\circ\theta)).
\end{array}$$

The proof of this commutativity will be carried out in four steps below. Roughly speaking, we will embed this diagram into a meromorphic family of diagrams. Working in the context of this meromorphic family of diagrams, we will complete the diagram by extending the last row of the diagram as

$$\begin{array}{ccc}
I_{\theta(P)}(\pi_\psi\circ\theta) & \xrightarrow{R_{\theta(P)}(w,\pi_\psi\circ\theta,\psi)} & I_P(w(\pi_\psi\circ\theta)) \\
\big\downarrow & & \big\downarrow \\
I_{\theta(P_1)}(w_1\tau_1\circ\theta) & \xrightarrow{R_{\theta(P_1)}(w,w_1\tau_1\circ\theta)} & I_{P_1}(w(w_1\tau_1\circ\theta)).
\end{array}$$

Then the commutativity of the meromorphic family of diagrams is a consequence of the multiplicativity property in Proposition 1.7.2. We will then deduce the commutativity of the desired diagram by specializing at the point of interest. Note however that the additional normalized intertwining operator will generally have a singularity at the point of interest (which is the reason why we use the dotted arrow), so its purpose is only to help prove the commutativity of the whole meromorphic family, before specialization.

Let us start the proof of the commutativity of the middle diagram.

Step 1: Recall that

$$\tau_2 = \bigotimes_{\alpha\in(1/2)\mathbb{Z}} \bigotimes_{i=1}^{t} \tau^{(i)}|\cdot|_E^{\alpha},$$

where the tensor products are taken with the order $\prec_2$. For tuples of complex numbers

$$\lambda = (\lambda^{(i)})_i \in \bigoplus_{i=1}^{t} \mathbb{C},$$

$$\mu = (\mu^{(i)}(\alpha))_{\alpha,i} \in \bigoplus_{\alpha\in(1/2)\mathbb{Z}} \bigoplus_{i=1}^{t} \mathbb{C},$$

we set

$$\tau_{2,(\lambda,\mu)} = \bigotimes_{\alpha \in (1/2)\mathbb{Z}} \bigotimes_{i=1}^{t} \tau^{(i)} |\cdot|_E^{\alpha+\lambda^{(i)}+\mu^{(i)}(\alpha)}.$$

We define $\tau_{1,(\lambda,\mu)}$ similarly. Hence

$$\tau_{1,(\lambda,\mu)} = w_2 \tau_{2,(\lambda,\mu)}.$$

Note that

$$\begin{aligned} w_2 w'(\tau_{2,(\lambda,\mu)} \circ \theta) &= w_1^{-1} w \theta(w_1) \theta(w_2) (\tau_{2,(\lambda,\mu)} \circ \theta) \\ &= w_1^{-1} w (w_1 w_2 \tau_{2,(\lambda,\mu)} \circ \theta) \\ &= w_1^{-1} w (w_1 \tau_{1,(\lambda,\mu)} \circ \theta). \end{aligned}$$

We can consider the following diagram of meromorphic families of operators:

$$\begin{array}{ccc} I_{\theta(P_2)}(\tau_{2,(\lambda,\mu)} \circ \theta) & \xrightarrow{R_{\theta(P_2)}(w', \tau_{2,(\lambda,\mu)} \circ \theta)} & I_{P_2}(w'(\tau_{2,(\lambda,\mu)} \circ \theta)) \\ {\scriptstyle R_{\theta(P_2)}(\theta(w_2), \tau_{2,(\lambda,\mu)} \circ \theta)} \downarrow & & \downarrow {\scriptstyle R_{P_2}(w_2, w'(\tau_{2,(\lambda,\mu)} \circ \theta))} \\ I_{\theta(P_1)}(\tau_{1,(\lambda,\mu)} \circ \theta) & & I_{P_1}(w''(\tau_{1,(\lambda,\mu)} \circ \theta)) \\ {\scriptstyle R_{\theta(P_1)}(\theta(w_1), \tau_{1,(\lambda,\mu)} \circ \theta)} \downarrow & & \downarrow {\scriptstyle R_{P_1}(w_1, w''(\tau_{1,(\lambda,\mu)} \circ \theta))} \\ I_{\theta(P_1)}(w_1 \tau_{1,(\lambda,\mu)} \circ \theta) & \xrightarrow{R_{\theta(P_1)}(w, w_1 \tau_{1,(\lambda,\mu)} \circ \theta)} & I_{P_1}(w(w_1 \tau_{1,(\lambda,\mu)} \circ \theta)). \end{array}$$

By Proposition 1.7.2, this diagram is commutative when $\alpha + \lambda^{(i)} + \mu^{(i)}(\alpha) \in \sqrt{-1}\mathbb{R}$ for all α, i. Hence, by analytic continuation, we see that this diagram is commutative for all (λ, μ) at which all operators are regular.

Step 2: In the diagram in Step 1, we will specialize at $\mu = 0$. We claim that all of six intertwining operators are well-defined as meromorphic families of operators in λ.

In fact, the bottom arrow $R_{\theta(P_1)}(w, w_1 \tau_{1,(\lambda,0)} \circ \theta)$ is a composition of intertwining operators of the form

$$\tau^{(i)} |\cdot|_E^{\alpha+\lambda^{(i)}} \times \tau^{(i')} |\cdot|_E^{\alpha'+\lambda^{(i')}} \to \tau^{(i')} |\cdot|_E^{\alpha'+\lambda^{(i')}} \times \tau^{(i)} |\cdot|_E^{\alpha+\lambda^{(i)}}$$

for some $i \neq i'$. Hence the subset $\{(\lambda, 0)\} \subset \{(\lambda, \mu)\}$ is not contained in the subset consisting of (λ, μ) at which $R_{\theta(P_1)}(w, w_1 \tau_{1,(\lambda,\mu)} \circ \theta)$ is singular. Here, we notice that $R_{\theta(P_1)}(w, w_1 \tau_{1,(\lambda,\mu)} \circ \theta)$ can have a singularity at $(\lambda, \mu) = (0, 0)$. On the other hand, as we have seen in the proofs of Lemmas 3.3.1 and 3.4.1 (3), the other five operators are indeed regular even at $(\lambda, \mu) = (0, 0)$.

Hence we can evaluate at $\mu = 0$ in the diagram in Step 1, and get the following commutative diagram:

$$\begin{array}{ccc}
I_{\theta(P_2)}(\tau_{2,(\lambda,0)} \circ \theta) & \xrightarrow{R_{\theta(P_2)}(w', \tau_{2,(\lambda,0)} \circ \theta)} & I_{P_2}(w'(\tau_{2,(\lambda,0)} \circ \theta)) \\
{\scriptstyle R_{\theta(P_2)}(\theta(w_2), \tau_{2,(\lambda,0)} \circ \theta)}\downarrow & & \downarrow{\scriptstyle R_{P_2}(w_2, w'(\tau_{2,(\lambda,0)} \circ \theta))} \\
I_{\theta(P_1)}(\tau_{1,(\lambda,0)} \circ \theta) & & I_{P_1}(w''(\tau_{1,(\lambda,0)} \circ \theta)) \\
{\scriptstyle R_{\theta(P_1)}(\theta(w_1), \tau_{1,(\lambda,0)} \circ \theta)}\downarrow & & \downarrow{\scriptstyle R_{P_1}(w_1, w''(\tau_{1,(\lambda,0)} \circ \theta))} \\
I_{\theta(P_1)}(w_1\tau_{1,(\lambda,0)} \circ \theta) & \xrightarrow{R_{\theta(P_1)}(w, w_1\tau_{1,(\lambda,0)} \circ \theta)} & I_{P_1}(w(w_1\tau_{1,(\lambda,0)} \circ \theta)).
\end{array}$$

Step 3: Recall that $\mathrm{Speh}(\tau_j^{(i)}, d_j^{(i)})$ is the unique irreducible quotient of

$$\tau_j^{(i)}|\cdot|_E^{\frac{d-1}{2}} \times \tau_j^{(i)}|\cdot|_E^{\frac{d-3}{2}} \times \cdots \times \tau_j^{(i)}|\cdot|_E^{-\frac{d-1}{2}}$$

with $d = d_j^{(i)}$. Note that

$$I_P(\pi_\psi) = \bigtimes_{i=1}^{t} \bigtimes_{j=1}^{m_i} \mathrm{Speh}(\tau_j^{(i)}, d_j^{(i)}), \quad \psi = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{m_i} \psi_j^{(i)}.$$

Set

$$I_P(\pi_{\psi,\lambda}) = \bigtimes_{i=1}^{t} \bigtimes_{j=1}^{m_i} \mathrm{Speh}(\tau_j^{(i)}, d_j^{(i)})|\cdot|_E^{\lambda^{(i)}},$$

and

$$\psi_\lambda = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{m_i} \psi_j^{(i)}|\cdot|_E^{\lambda^{(i)}}.$$

Note that $I_P(\pi_{\psi,\lambda})$ is an irreducible subrepresentation of $I_{P_1}(w_1\tau_{1,(\lambda,0)})$. Moreover, it is equal to the image of $R_{P_1}(w_1, \tau_{1,(\lambda,0)}) \circ R_{P_2}(w_2, \tau_{2,(\lambda,0)})$ by Lemma 3.3.1. We have now the diagram

$$\begin{array}{ccc}
I_{\theta(P)}(\pi_{\psi,\lambda} \circ \theta) & \xrightarrow{R_{\theta(P)}(w, \pi_{\psi,\lambda} \circ \theta, \psi_\lambda)} & I_P(w(\pi_{\psi,\lambda} \circ \theta)) \\
\downarrow & & \downarrow \\
I_{\theta(P_1)}(w_1\tau_{1,(\lambda,0)} \circ \theta) & \xrightarrow{R_{\theta(P_1)}(w, w_1\tau_{1,(\lambda,0)} \circ \theta)} & I_{P_1}(w(w_1\tau_{1,(\lambda,0)} \circ \theta))
\end{array}$$

where the vertical maps are the canonical inclusions. We claim that this diagram is commutative. Indeed, the defining integrals are the same by Lemma 3.4.1 (1), and the normalizing factors also agree by definition. In conclusion, we obtain the following commutative diagram of meromorphic families of operators:

$$\begin{array}{ccc} I_{\theta(P_2)}(\tau_{2,(\lambda,0)}\circ\theta) & \xrightarrow{R_{\theta(P_2)}(w',\tau_{2,(\lambda,0)}\circ\theta)} & I_{P_2}(w'(\tau_{2,(\lambda,0)}\circ\theta)) \\ {\scriptstyle R_{\theta(P_2)}(\theta(w_2),\tau_{2,(\lambda,0)}\circ\theta)}\big\downarrow & & \big\downarrow{\scriptstyle R_{P_2}(w_2,w'(\tau_{2,(\lambda,0)}\circ\theta))} \\ I_{\theta(P_1)}(\tau_{1,(\lambda,0)}\circ\theta) & & I_{P_1}(w''(\tau_{1,(\lambda,0)}\circ\theta)) \\ {\scriptstyle R_{\theta(P_1)}(\theta(w_1),\tau_{1,(\lambda,0)}\circ\theta)}\big\downarrow & & \big\downarrow{\scriptstyle R_{P_1}(w_1,w''(\tau_{1,(\lambda,0)}\circ\theta))} \\ I_{\theta(P)}(\pi_{\psi,\lambda}\circ\theta) & \xrightarrow{R_{\theta(P)}(w,\pi_{\psi,\lambda}\circ\theta,\psi_\lambda)} & I_P(w(\pi_{\psi,\lambda}\circ\theta)). \end{array}$$

Step 4: In this last step, we would like to specialize the commutative diagram above at $\lambda = 0$. As we have seen in Step 2, the five operators that appear in the top, left and right of the last diagram are regular at $\lambda = 0$. Especially, the composition

$$R_{P_1}(w_1, w''(\tau_{1,(\lambda,0)}\circ\theta)) \circ R_{P_2}(w_2, w'(\tau_{2,(\lambda,0)}\circ\theta)) \circ R_{\theta(P_2)}(w', \tau_{2,(\lambda,0)}\circ\theta)$$

is regular at $\lambda = 0$, and so is

$$R_{\theta(P)}(w, \pi_{\psi,\lambda}\circ\theta, \psi_\lambda) \circ R_{\theta(P_1)}(\theta(w_1), \tau_{1,(\lambda,0)}\circ\theta) \circ R_{\theta(P_2)}(\theta(w_2), \tau_{2,(\lambda,0)}\circ\theta).$$

Moreover, since the composition $R_{\theta(P_1)}(\theta(w_1), \tau_{1,(\lambda,0)}\circ\theta)\circ R_{\theta(P_2)}(\theta(w_2), \tau_{2,(\lambda,0)}\circ\theta)$ is surjective if λ is sufficiently close to 0 by Lemma 3.3.1, the operator

$$R_{\theta(P)}(w, \pi_{\psi,\lambda}\circ\theta, \psi_\lambda)\colon I_{\theta(P)}(\pi_{\psi,\lambda}\circ\theta) \to I_P(w(\pi_{\psi,\lambda}\circ\theta))$$

is also regular at $\lambda = 0$.

Therefore, we can specialize the last diagram in Step 3 at $\lambda = 0$, and obtain the following commutative diagram.

$$\begin{array}{ccc} I_{\theta(P_2)}(\tau_2\circ\theta) & \xrightarrow{R_{\theta(P_2)}(w',\tau_2\circ\theta)} & I_{P_2}(w'(\tau_2\circ\theta)) \\ {\scriptstyle R_{\theta(P_2)}(\theta(w_2),\tau_2\circ\theta)}\big\downarrow & & \big\downarrow{\scriptstyle R_{P_2}(w_2,w'(\tau_2\circ\theta))} \\ I_{\theta(P_1)}(\tau_1\circ\theta) & & I_{P_1}(w''(\tau_1\circ\theta)) \\ {\scriptstyle R_{\theta(P_1)}(\theta(w_1),\tau_1\circ\theta)}\big\downarrow & & \big\downarrow{\scriptstyle R_{P_1}(w_1,w''(\tau_1\circ\theta))} \\ I_{\theta(P)}(\pi_\psi\circ\theta) & \xrightarrow{R_{\theta(P)}(w,\pi_\psi\circ\theta,\psi)} & I_P(w(\pi_\psi\circ\theta)). \end{array}$$

Hence we obtain the claim.

This completes the proof of Theorem 3.4.2. □

The following example may help the reader to understand the argument.

Example 3.4.3. Let us suppose that $M = \mathrm{GL}_3(E) \times \mathrm{GL}_1(E) \subset P = MN_P \subset G = \mathrm{GL}_4(E)$, and let us consider

$$\psi = \mathbf{1}_{L_E} \boxtimes S_3 \oplus \mathbf{1}_{L_E} \boxtimes S_1 \in \Psi(M).$$

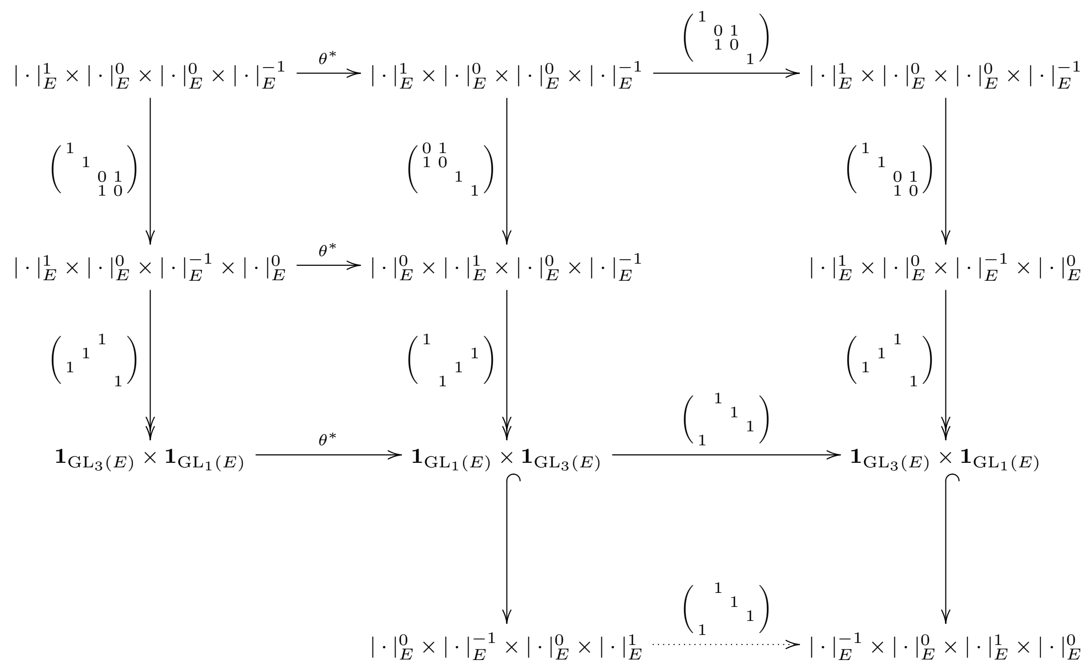
Then

$$\pi_\psi = \mathbf{1}_{\mathrm{GL}_3(E)} \boxtimes \mathbf{1}_{\mathrm{GL}_1(E)} \in \mathrm{Irr}(M)$$

is the corresponding representation. Note that $I_P(\pi_\psi) = \mathbf{1}_{\mathrm{GL}_3(E)} \times \mathbf{1}_{\mathrm{GL}_1(E)} \in \mathrm{Irr}(G)$. We realize it as a subrepresentation of

$$|\cdot|_E^{-1} \times |\cdot|_E^{0} \times |\cdot|_E^{1} \times |\cdot|_E^{0}.$$

In what follows, for simplicity, we denote the normalized intertwining operator $R_P(w, \pi)$ by $w \in W^G$. The main diagram becomes:



Notice that we have added a bottom "map"

$$|\cdot|_E^{0} \times |\cdot|_E^{-1} \times |\cdot|_E^{0} \times |\cdot|_E^{1} \xrightarrow{\begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix}} |\cdot|_E^{-1} \times |\cdot|_E^{0} \times |\cdot|_E^{1} \times |\cdot|_E^{0}$$

that is actually a singularity of a meromorphic operator. Hence this "map" is not well-defined so that we cannot consider it.

3.5. **Remark on the untwisted case.** Note that the argument for Theorem 1.9.1 works when we replace θ with the identity map on G. We state the analogue of Theorem 1.9.1 as in [Mok, Proposition 3.5.1 (a)].

Theorem 3.5.1. *Let $P = MN$ be a standard parabolic subgroup of $G = \mathrm{GL}_N(E)$, and let ψ be an A-parameter for M. Then for any $w \in W(M)$ with $w\pi_\psi \cong \pi_\psi$, we have*

$$I_P(\widetilde{\pi}_\psi(w)) \circ R_P(w, \pi_\psi, \psi) = \mathrm{id},$$

where $\widetilde{\pi}_\psi(w)\colon w\pi_\psi \to \pi_\psi$ is the isomorphism normalized by using a Whittaker functional on the standard module $\mathcal{I}_\psi^M$ of π_ψ.

4. Characters of component groups vs. Aubert duality

In this and the next two sections, we assume that F is non-archimedean. Fix a non-trivial additive character $\psi_F \colon F \to \mathbb{C}^\times$. Let $\mathfrak{o}_E$ be the ring of integers of E.

In this section, assuming Arthur's theory (Hypothesis 4.4.2), we provide a formula for the action of Aubert duality on the characters of component groups in certain cases (Corollaries 4.4.5 and 4.5.3). These results will be applied to tempered L-parameters for a given classical group G, and to certain A-parameters for a proper Levi subgroup M of G: in both cases, Hypothesis 4.4.2 is known in the framework of Arthur's inductive argument.

4.1. **Twisted Aubert duality for** $\mathrm{GL}_N(E)$**.** In this and next subsections, we consider twisted Aubert duality for $\mathrm{GL}_N(E)$. For a preview of the general theory of twisted Aubert duality, see Sections B.4 and B.5.

Let $\mathrm{Rep}(\widetilde{\mathrm{GL}}_N(E))$ be the category of smooth finite length representations of the non-connected group $\widetilde{\mathrm{GL}}_N(E) = \mathrm{GL}_N(E) \rtimes \langle\theta\rangle$, and let $\mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$ be its Grothendieck group. For $\widetilde{\pi} \in \mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$, one can consider its character $\Theta_{\widetilde{\pi}}$, which is a linear form on $C_c^\infty(\widetilde{\mathrm{GL}}_N(E))$. For $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$, we write

$$\widetilde{\pi}_1 \overset{\theta}{=} \widetilde{\pi}_2$$

if

$$\Theta_{\widetilde{\pi}_1}(\widetilde{f}) = \Theta_{\widetilde{\pi}_2}(\widetilde{f})$$

for any $\widetilde{f} \in C_c^\infty(\mathrm{GL}_N(E) \rtimes \theta)$. For example, if $\pi \in \mathrm{Irr}(\mathrm{GL}_N(E))$, then

$$\mathrm{Ind}_{\mathrm{GL}_N(E)}^{\widetilde{\mathrm{GL}}_N(E)}(\pi) \overset{\theta}{=} 0.$$

Let

$$\mathrm{Rep}(\widetilde{\mathrm{GL}}_N(E)) \to \mathrm{Rep}(\widetilde{\mathrm{GL}}_N(E)),\ \widetilde{\pi} \mapsto \hat{\widetilde{\pi}}$$

be the functor given by Definition B.4.6. It satisfies the following properties.

Proposition 4.1.1. (1) *The restriction* $\hat{\widetilde{\pi}}|_{\mathrm{GL}_N(E)}$ *is equal to Aubert dual of* $\widetilde{\pi}|_{\mathrm{GL}_N(E)}$.

(2) *If* $\widetilde{\pi}$ *is an irreducible representation of* $\widetilde{\mathrm{GL}}_N(E)$, *then so is* $\hat{\widetilde{\pi}}$.

(3) *For* $\widetilde{\pi} \in \mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$, *set*

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\widetilde{\pi}) = \sum_{P=\theta(P)} (-1)^{\dim(A_M^\theta)} \mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{GL}}_N(E)}(\mathrm{Jac}_{\widetilde{P}}(\widetilde{\pi})),$$

where P *runs over the set of standard parabolic subgroups of* $\mathrm{GL}_N(E)$ *which are stable under* θ *and set* $\widetilde{P} = P \rtimes \langle\theta\rangle$. *Then*

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\widetilde{\pi}) \overset{\theta}{=} (-1)^{\dim(A_{M_0}/A_{\mathrm{GL}_N(E)})}\hat{\widetilde{\pi}}$$

for $\widetilde{\pi} \in \mathrm{Irr}(\widetilde{\mathrm{GL}}_N(E))$, *where* $P_0 = M_0 N_{P_0}$ *is a minimal standard parabolic subgroup of* $\mathrm{GL}_N(E)$ *such that* $\mathrm{Jac}_{P_0}(\widetilde{\pi}|_{\mathrm{GL}_N(E)}) \neq 0$. *Note that such a* P_0 *is not unique but the sign* $(-1)^{\dim(A_{M_0}/A_{\mathrm{GL}_N(E)})}$ *is well-defined.*

Proof. These properties are proven in Propositions B.4.7 and B.5.1. For the notations of parabolic inductions and Jacquet modules, see Section B.5. Here, note that $\dim((A_M/A_{\mathrm{GL}_N(E)})^\theta) = \dim(A_M^\theta)$. □

We call $\widehat{\widetilde{\pi}}$ the *Aubert dual* of $\widetilde{\pi}$. We remark that $\widetilde{\pi} \mapsto \widehat{\widetilde{\pi}}$ is expected to be an involution, but we do not prove it and we will not use this property.

Let $\psi\colon W_E \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ be an A-parameter for $\mathrm{GL}_N(E)$ and let π_ψ be the corresponding irreducible representation of $\mathrm{GL}_N(E)$. Suppose that ψ (or equivalently, π_ψ) is conjugate-self-dual. Then, as in Section 1.4, we have Arthur's extension $\widetilde{\pi}_\psi = \pi_\psi \boxtimes \theta_A$ of π_ψ to $\widetilde{\mathrm{GL}}_N(E)$. The Aubert dual of π_ψ is equal to $\pi_{\widehat{\psi}}$, where $\widehat{\psi}$ is defined by $\widehat{\psi}(w, g_1, g_2) = \psi(w, g_2, g_1)$. However, $\widehat{\widetilde{\pi}}_\psi$ is not necessarily equal to $\widetilde{\pi}_{\widehat{\psi}}$. Namely, if we write $\widehat{\widetilde{\pi}}_\psi = \pi_{\widehat{\psi}} \boxtimes \widehat{\theta}_A$, then $\widehat{\theta}_A$ is not necessarily equal to θ_A as linear operators on $\pi_{\widehat{\psi}}$. Since $\theta^2 = 1$, we have $\widehat{\theta}_A = \pm\theta_A$. Hence $D_{\widetilde{\mathrm{GL}}_N(E)}(\widetilde{\pi}_\psi) \overset{\theta}{=} \pm\widetilde{\pi}_{\widehat{\psi}}$. We define $\beta(\psi) \in \{\pm 1\}$ such that

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\widetilde{\pi}_\psi) \overset{\theta}{=} \beta(\psi)\widetilde{\pi}_{\widehat{\psi}}.$$

Lemma 4.1.2. *Let ψ be as above, and let η be a conjugate-self-dual character of $E^\times$, which is regarded as a character of W_E by local class field theory. Then $\beta(\psi\otimes\eta) = \beta(\psi)$.*

Proof. Since η is conjugate-self-dual, we have $\eta(\det(\theta(g))) = \eta(\det(g))$ for $g \in \mathrm{GL}_N(E)$. This implies that $\widetilde{\pi}_{\psi\otimes\eta} = \widetilde{\pi}_\psi \otimes \eta$, i.e., Arthur's action of θ on $\pi_{\psi\otimes\eta}$ is the same as the one of π_ψ as a linear operator on the same vector space. By Lemma B.4.9, $\widehat{\widetilde{\pi}}_{\psi\otimes\eta} = \widehat{\widetilde{\pi}}_\psi \otimes \eta \overset{\theta}{=} \beta(\psi)\pi_{\widehat{\psi}} \otimes \eta$. Hence we have $\beta(\psi\otimes\eta) = \beta(\psi)$. □

On the other hand, since ψ is conjugate-self-dual, we can decompose the representation

$$W_E \ni w \mapsto \psi\left(w, \begin{pmatrix} |w|_E^{\frac{1}{2}} & 0 \\ 0 & |w|_E^{-\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} |w|_E^{\frac{1}{2}} & 0 \\ 0 & |w|_E^{-\frac{1}{2}} \end{pmatrix}\right) \in \mathrm{GL}_N(\mathbb{C})$$

of W_E into irreducible representations as

$$\rho_{-r} \oplus \cdots \oplus \rho_{-1} \oplus \rho'_1 \oplus \cdots \oplus \rho'_t \oplus \rho_1 \oplus \cdots \oplus \rho_r,$$

where

- ρ_i is an irreducible representation of W_E;
- $\rho_{-i} \cong {}^c\rho_i^\vee$ is the conjugate-dual of ρ_i;
- ρ'_i is an irreducible conjugate-self-dual representation of W_E such that $\rho'_i \not\cong \rho'_j$ for $1 \le i < j \le t$.

Note that such a labeling of irreducible components is not unique, but the non-negative integer r is uniquely determined from ψ. We write $r(\psi) = r$.

Proposition 4.1.3. *Suppose that $\psi = \phi$ is tempered and conjugate-self-dual. Then $\beta(\phi) = \beta(\widehat{\phi}) = (-1)^{r(\phi)}$.*

Proof. A more general assertion was stated by Mœglin–Waldspurger ([MW3, Lemma 3.2.2]). However, they gave a proof only when every irreducible component of $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is even dimensional. Here, we will give a proof for the general case.

Let $\mathcal{I}_{\widehat{\phi}}$ be the standard module whose Langlands quotient is $\pi_{\widehat{\phi}}$. If we denote by θ_W the action of θ on $\mathcal{I}_{\widehat{\phi}}$ fixing a nonzero Whittaker functional, then the action θ_A on $\pi_{\widehat{\phi}}$ is induced from θ_W by definition. On the other hand, by Lemma 3.1.2, we see that $\mathcal{I}_{\widehat{\phi}}$ contains the tempered representation π_ϕ as a subrepresentation. Moreover, the restriction of θ_W on π_ϕ gives Arthur's action θ_A on π_ϕ since the restriction of a nonzero Whittaker functional on $\mathcal{I}_{\widehat{\phi}}$ to π_ϕ is also nonzero. Thus, as representations of $\widetilde{\mathrm{GL}}_N(E)$, we have

$$\pi_\phi \boxtimes \theta_A \hookrightarrow \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W \twoheadrightarrow \pi_{\widehat{\phi}} \boxtimes \theta_A.$$

Applying the duality functor $\widetilde{\pi} \mapsto \widehat{\widetilde{\pi}}$, we see that $(\pi_\phi \boxtimes \theta_A)\widehat{\ } = \pi_{\widehat{\phi}} \boxtimes \widehat{\theta}_A$ and $(\pi_{\widehat{\phi}} \boxtimes \theta_A)\widehat{\ } = \pi_\phi \boxtimes \widehat{\theta}_A$ are contained in $(\mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W)\widehat{\ } = \widehat{\mathcal{I}}_{\widehat{\phi}} \boxtimes \widehat{\theta}_W$ as irreducible subquotients. By Proposition 4.1.1 (3), they satisfy

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W) \overset{\theta}{=} \varepsilon \cdot \widehat{\mathcal{I}}_{\widehat{\phi}} \boxtimes \widehat{\theta}_W,$$
$$D_{\widetilde{\mathrm{GL}}_N(E)}(\pi_\phi \boxtimes \theta_A) \overset{\theta}{=} \varepsilon \cdot \pi_{\widehat{\phi}} \boxtimes \widehat{\theta}_A,$$
$$D_{\widetilde{\mathrm{GL}}_N(E)}(\pi_{\widehat{\phi}} \boxtimes \theta_A) \overset{\theta}{=} \varepsilon \cdot \pi_\phi \boxtimes \widehat{\theta}_A$$

for some common sign $\varepsilon \in \{\pm 1\}$ since all irreducible subquotients of $\mathcal{I}_{\widehat{\phi}}$ share the same cuspidal support. Since $\mathcal{I}_{\widehat{\phi}}$ is an induction from a cuspidal representation, its Aubert dual $\widehat{\mathcal{I}}_{\widehat{\phi}}$ is equal to $\mathcal{I}_{\widehat{\phi}}$ in $\mathcal{R}(\mathrm{GL}_N(E))$. Moreover, as $\dim_{\mathbb{C}}(\mathrm{End}_{\mathrm{GL}_N(E)}(\mathcal{I}_{\widehat{\phi}})) = 1$ by Lemma 3.1.1, we can find $\delta \in \{\pm 1\}$ such that $\widehat{\theta}_W = \delta\theta_W$ so that $\widehat{\mathcal{I}}_{\widehat{\phi}} \boxtimes \widehat{\theta}_W = \delta \cdot \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W$ in $\mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$. Hence

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W) \overset{\theta}{=} \varepsilon\delta \cdot \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W.$$

Since π_ϕ and $\pi_{\widehat{\phi}}$ appear in $\mathcal{I}_{\widehat{\phi}}$ as subquotients with multiplicity one, by Proposition 4.1.1 (1), we see that $\widehat{\mathcal{I}}_{\widehat{\phi}} \boxtimes \widehat{\theta}_W$ contains only one irreducible representation of the form $\pi_{\widehat{\phi}} \boxtimes \theta$ (resp. $\pi_\phi \boxtimes \theta$) as subquotients. In particular, we get

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\pi_\phi \boxtimes \theta_A) \overset{\theta}{=} \varepsilon\delta \cdot \pi_{\widehat{\phi}} \boxtimes \theta_A, \quad D_{\widetilde{\mathrm{GL}}_N(E)}(\pi_{\widehat{\phi}} \boxtimes \theta_A) \overset{\theta}{=} \varepsilon\delta \cdot \pi_\phi \boxtimes \theta_A.$$

This means that $\varepsilon\delta = \beta(\phi) = \beta(\widehat{\phi})$. Therefore, what we have to show is $\varepsilon\delta = (-1)^{r(\phi)}$, which we will prove in the next lemma. □

Lemma 4.1.4. *With the above notation, we have*

$$D_{\widetilde{\mathrm{GL}}_N(E)}(\mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W) \overset{\theta}{=} (-1)^{r(\phi)} \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W.$$

4.2. **An example.** The proof of Lemma 4.1.4 is complicated. Before giving the proof generally, let us discuss a simple but non-trivial case. This example showcases the strategy of the general proof.

Suppose that $E = F$. Let χ_1, χ_2 be two quadratic characters of $F^\times$, and consider $\phi = \chi_1 \oplus \chi_2$. Then

$$\pi_\phi = \pi_{\widehat{\phi}} = \mathcal{I}_{\widehat{\phi}} = \chi_1 \times \chi_2 \in \mathrm{Irr}(\mathrm{GL}_2(F)).$$

We denote the upper triangular Borel subgroup of $\mathrm{GL}_2(F)$ by $B = TU$, where T is the diagonal torus. If we set $\widetilde{B} = B \rtimes \langle\theta\rangle$ and $\widetilde{T} = T \rtimes \langle\theta\rangle$, then by definition

$$D_{\widetilde{\mathrm{GL}}_2(F)}(\pi_\phi \boxtimes \theta_A) = \pi_\phi \boxtimes \theta_A - \mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{GL}}_2(F)}(\mathrm{Jac}_{\widetilde{B}}(\pi_\phi \boxtimes \theta_A))$$

in the Grothendieck group $\mathcal{R}(\widetilde{\mathrm{GL}}_2(F))$. Since

$$(-1)^{r(\phi)} = \begin{cases} -1 & \text{if } \chi_1 = \chi_2, \\ 1 & \text{if } \chi_1 \neq \chi_2, \end{cases}$$

the desired equation $D_{\widetilde{\mathrm{GL}}_2(F)}(\pi_\phi \boxtimes \theta_A) \overset{\theta}{=} (-1)^{r(\phi)}\pi_\phi \boxtimes \theta_A$ is equivalent to

$$\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{GL}}_2(F)}(\mathrm{Jac}_{\widetilde{B}}(\pi_\phi \boxtimes \theta_A)) \overset{\theta}{=} \begin{cases} 2 \cdot \pi_\phi \boxtimes \theta_A & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2. \end{cases}$$

Note that $\mathrm{Jac}_{\widetilde{B}}(\pi_\phi \boxtimes \theta_A)|_T = \mathrm{Jac}_B(\pi_\phi)$. By the Geometric Lemma [BZ, Theorem 5.2], we have an exact sequence

$$0 \longrightarrow \chi_2 \boxtimes \chi_1 \longrightarrow \mathrm{Jac}_B(\pi_\phi) \longrightarrow \chi_1 \boxtimes \chi_2 \longrightarrow 0$$

and hence $\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\mathrm{Jac}_B(\pi_\phi)) = 2 \cdot \pi_\phi$ in $\mathcal{R}(\mathrm{GL}_2(F))$. To understand the induced action of θ on $\mathrm{Jac}_B(\pi_\phi)$, we recall the details for this exact sequence. The surjection $\mathrm{Jac}_B(\pi_\phi) \twoheadrightarrow \chi_1 \boxtimes \chi_2$ is induced by the evaluation map

$$\pi_\phi = \chi_1 \times \chi_2 \ni f \mapsto f(\mathbf{1}) \in \chi_1 \boxtimes \chi_2.$$

The kernel of this map is

$$\mathcal{F} = \{f \in \chi_1 \times \chi_2 \,|\, \mathrm{Supp}(f) \subset Bw_0^{-1}B\},$$

where $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$. We identify w_0 with its image in the Weyl group $W^{\mathrm{GL}_2(F)}$. Then for $f \in \mathcal{F}$, the evaluation of the integral

$$J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1}) = \int_U f(w_0^{-1}u)du$$

converges absolutely. Moreover, the map $\mathcal{F} \ni f \mapsto J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1})$ induces an isomorphism $\mathrm{Jac}_B(\pi_\phi) \supset \mathrm{Jac}_B(\mathcal{F}) \xrightarrow{\sim} \chi_2 \boxtimes \chi_1$.

However, this description of $\mathrm{Jac}_B(\pi_\phi)$ does not seem convenient for us. We give another description. Let $\mathcal{F}'$ be the subspace of $\pi_\phi = \chi_1 \times \chi_2$ consisting of functions $f \in$

$\chi_1 \times \chi_2$ such that $J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1})$ is well-defined. This means that the meromorphic continuation of

$$\mathbb{C}^2 \ni \lambda = (\lambda_1, \lambda_2) \mapsto J_B(w_0, \chi_1|\cdot|_F^{\lambda_1} \boxtimes \chi_2|\cdot|_F^{\lambda_2})f_\lambda(\mathbf{1})$$

is holomorphic at $\lambda = (0,0)$, where $f_\lambda \in \chi_1|\cdot|_F^{\lambda_1} \boxtimes \chi_2|\cdot|_F^{\lambda_2}$ is such that $f_\lambda|_K = f|_K$ with $K = \mathrm{GL}_2(\mathfrak{o}_F)$. Then $\mathcal{F} \subset \mathcal{F}'$, and the map $\mathcal{F}' \ni f \mapsto J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1})$ also induces a surjection

$$\mathrm{Jac}_B(\pi_\phi) \supset \mathrm{Jac}_B(\mathcal{F}') \twoheadrightarrow \chi_2 \boxtimes \chi_1.$$

We set $\mathcal{F}'' = \{f \in \mathcal{F}' \mid J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1}) = 0\}$ so that $\mathrm{Jac}_B(\mathcal{F}')/\mathrm{Jac}_B(\mathcal{F}'') \cong \chi_2 \boxtimes \chi_1$.

Now we consider two cases separately.

Case 1: Suppose that $\chi_1 = \chi_2$. Then the action θ_A on $\pi_\phi = \chi_1 \times \chi_2$ is given by

$$f \mapsto f \circ \theta.$$

Since $(f \circ \theta)(\mathbf{1}) = f(\mathbf{1})$, the induced action on $\mathrm{Jac}_B(\pi_\phi)$ preserves the quotient $\chi_1 \boxtimes \chi_2$ and acts on it trivially. Similarly, since $\theta(w_0) = w_0$ and $\theta(U) = U$, the same holds for the subrepresentation $\chi_2 \boxtimes \chi_1$ of $\mathrm{Jac}_B(\pi_\phi)$. Hence by inducing these two pieces, we obtain that $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{GL}}_2(F)}(\mathrm{Jac}_{\widetilde{B}}(\pi_\phi \boxtimes \theta_A)) = 2 \cdot \pi_\phi \boxtimes \theta_A$.

Case 2: Suppose that $\chi_1 \neq \chi_2$. Then the map $f \mapsto f \circ \theta$ is no longer an action of θ on π_ϕ. Instead of this map, we use Theorem 1.9.1. Namely, the action θ_A on $\pi_\phi = \chi_1 \times \chi_2$ can be realized by the normalized intertwining operator

$$R_B(w_0, \chi_2 \boxtimes \chi_1) \circ \theta^* = \theta^* \circ R_B(w_0, \chi_1 \boxtimes \chi_2).$$

Since $\chi_1 \neq \chi_2$, we see that the normalizing factor

$$\gamma_A(s, \chi_1 \otimes \chi_2, \psi_F)$$

is holomorphic and nonzero at $s = 0$. Hence this action can be written as

$$\theta_A = \gamma_A(0, \chi_1 \otimes \chi_2, \psi_F) \cdot \theta^* \circ J_B(w_0, \chi_1 \boxtimes \chi_2).$$

In particular, the map

$$\mathcal{F}' \ni f \mapsto \theta_A(f)(\mathbf{1}) = \gamma_A(0, \chi_1 \otimes \chi_2, \psi_F) \cdot J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1})$$

factors through $f \mapsto J_B(w_0, \chi_1 \boxtimes \chi_2)f(\mathbf{1})$, and hence, this map is zero on $\mathcal{F}''$. Conversely, for $f \in \chi_1 \times \chi_2$, since $R_B(w_0, \chi_1 \boxtimes \chi_2) \circ R_B(w_0, \chi_2 \boxtimes \chi_1) = \mathrm{id}$ and $\theta(\mathbf{1}) = \mathbf{1}$, we have

$$J_B(w_0, \chi_1 \boxtimes \chi_2)\theta_A(f)(\mathbf{1}) = \gamma_A(0, \chi_1 \otimes \chi_2, \psi_F)^{-1} \cdot f(\mathbf{1}).$$

Hence $\theta_A(f) \in \mathcal{F}'$ and the map $f \mapsto J_B(w_0, \chi_1 \boxtimes \chi_2)\theta_A(f)(\mathbf{1})$ factors through $f \mapsto f(\mathbf{1})$. In particular, if $f(\mathbf{1}) = 0$, then $J_B(w_0, \chi_1 \boxtimes \chi_2)\theta_A(f)(\mathbf{1}) = 0$. (Here, it is not true in general that $\theta_A(f) \in \mathcal{F}$.) Therefore, the induced action on $\mathrm{Jac}_B(\pi_\phi)$ swaps $\chi_2 \boxtimes \chi_1$ and $\chi_1 \boxtimes \chi_2$. By inducing these two pieces, we obtain that $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{GL}}_2(F)}(\mathrm{Jac}_{\widetilde{B}}(\pi_\phi \boxtimes \theta_A)) \overset{\theta}{=} 0$.

Note that Theorem 1.9.1 can be used even when $\chi_1 = \chi_2$. However, in this case, $\gamma_A(0, \chi_1 \otimes \chi_2, \psi_F) = 0$ and so that $\theta_A(f)(\mathbf{1}) = 0$ for $f \in \mathcal{F}'$. Hence the argument in Case 2 does not work for Case 1. In Case 2, it is trivial that the induced action swaps $\chi_2 \boxtimes \chi_1$ and $\chi_1 \boxtimes \chi_2$ since $(\chi_1 \boxtimes \chi_2) \circ \theta \neq \chi_1 \boxtimes \chi_2$. However, this idea would not work in general, e.g., for $\phi = \chi_1^{\oplus 3} \oplus \chi_2$ with $\chi_1 \neq \chi_2$. On the other hand, the analysis using intertwining operators can be generalized.

4.3. **Proof of Lemma 4.1.4.** Now we prove Lemma 4.1.4 generally.

Proof of Lemma 4.1.4. One can write $\mathcal{I}_{\widehat{\phi}} = \mathrm{Ind}_{P_0}^{\mathrm{GL}_N(E)}(\pi_{M_0})$ with

$$\pi_{M_0} = \rho_{-r} \boxtimes \cdots \boxtimes \rho_{-1} \boxtimes \rho'_1 \boxtimes \cdots \boxtimes \rho'_t \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_r,$$

where

- P_0 corresponds to a partition $(d_r, \dots, d_1, d'_1, \dots, d'_t, d_1, \dots, d_r)$ of N;
- ρ_i is an irreducible cuspidal representation of $\mathrm{GL}_{d_i}(E)$;
- $\rho_{-i} \cong {}^c\rho_i^\vee$ is the conjugate-dual of ρ_i;
- ρ'_i is an irreducible conjugate-self-dual cuspidal representation of $\mathrm{GL}_{d'_j}(E)$ such that $\rho'_i \not\cong \rho'_j$ for $1 \leq i < j \leq t$.

Then $r(\phi) = r$.

Set $W = W^{\mathrm{GL}_N(E)}$. For a θ-stable parabolic subgroup $P = MN_P$, by the geometric lemma ([BZ, Theorem 5.2]), up to a semisimplification, we can write

$$\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}}) = \bigoplus_{w \in W^M \backslash W / W^{M_0}} J^w_{\widehat{\phi}}$$

for some representation $J^w_{\widehat{\phi}}$ (possibly zero). We recall the relation between $\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}})$ and $J^w_{\widehat{\phi}}$ more precisely. Fix a total order $\geq$ on $W^M \backslash W / W^{M_0}$ such that $w' \geq w \implies \dim(P_0 w'^{-1} P) \geq \dim(P_0 w^{-1} P)$. For $w \in W^M \backslash W / W^{M_0}$, we define $\mathcal{F}_w$ as the subspace of $\mathcal{I}_{\widehat{\phi}}$ consisting of functions f such that

$$\mathrm{Supp}(f) \subset \bigcup_{\substack{w' \in W^M \backslash W / W^{M_0} \\ w' \geq w}} P_0 w'^{-1} P.$$

If $f \in \mathcal{F}_w$, then the evaluation of the unnormalized intertwining operator

$$(J(w, \pi_{M_0})f)(\mathbf{1})$$

converges absolutely. Here, we write $J(w, \pi_{M_0}) = J_{P_0}(w, \pi_{M_0})$ for simplicity. The map $\mathcal{F}_w \ni f \mapsto (J(w, \pi_{M_0})f)(\mathbf{1})$ gives a surjection

$$\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}}) \supset \mathrm{Jac}_P(\mathcal{F}_w) \twoheadrightarrow J^w_{\widehat{\phi}}.$$

By varying w, we obtain a filtration of $\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}})$. For details, see [BZ, Section 5].

We modify this description for $\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}})$. For $\lambda \in \mathfrak{a}^*_{M_0,\mathbb{C}}$, let $\pi_{M_0,\lambda}$ be the unramified twist of π_{M_0} as in Section 1.7. Set $K = \mathrm{GL}_N(\mathfrak{o}_E)$ to be the standard maximal compact open subgroup of $\mathrm{GL}_N(E)$. Then for $f \in \mathcal{I}_{\widehat{\phi}} = \mathrm{Ind}_{P_0}^{\mathrm{GL}_N(E)}(\pi_{M_0})$, one can define

$f_\lambda \in \mathrm{Ind}_{P_0}^{\mathrm{GL}_N(E)}(\pi_{M_0,\lambda})$ by requiring $f_\lambda|_K = f|_K$. For $w \in W^M\backslash W/W^{M_0}$, we have the unnormalized intertwining operator $J(w, \pi_{M_0,\lambda})$ as a meromorphic family of operators. Let $\mathcal{F}'_w$ be the subspace of $\mathcal{I}_{\widehat{\phi}} = \mathrm{Ind}_{P_0}^{\mathrm{GL}_N(E)}(\pi_{M_0})$ consisting of functions f such that the meromorphic function

$$\lambda \mapsto J(w, \pi_{M_0,\lambda})f_\lambda(\mathbf{1})$$

is holomorphic at $\lambda = 0$. Then $\mathcal{F}_w \subset \mathcal{F}'_w$ so that we obtain a well-defined surjection

$$\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}}) \supset \mathrm{Jac}_P(\mathcal{F}'_w) \twoheadrightarrow J^w_{\widehat{\phi}}.$$

Its kernel is of the form $\mathrm{Jac}_P(\mathcal{F}''_w)$, where $\mathcal{F}''_w = \{f \in \mathcal{F}'_w \mid J(w, \pi_{M_0})f(\mathbf{1}) = 0\}$.

Using Theorem 1.9.1, we realize an action θ_W on $\mathcal{I}_{\widehat{\phi}}$ by a twisted intertwining operator. Set $N' = d'_1 + \cdots + d'_t$. Consider $\mathrm{GL}_{N'}(E)$ and its standard parabolic subgroup $P'_0 = M'_0 N_{P'_0}$ corresponding to the partition $(d'_1, \dots, d'_t)$. Define

$$\pi_{M'_0} = \rho'_1 \boxtimes \cdots \boxtimes \rho'_t.$$

Then $\mathrm{Ind}_{P'_0}^{\mathrm{GL}_{N'}(E)}(\pi_{M'_0})$ is an irreducible conjugate-self-dual representation of $\mathrm{GL}_{N'}(E)$. Let $w_0 \in W(\theta(M'_0), M'_0)$ be the unique element such that $w_0(\pi_{M'_0} \circ \theta) \cong \pi_{M'_0}$. We regard w_0 as an element in $W(\theta(M_0), M_0)$. By Theorem 1.9.1 for $\pi_{M'_0}$, we have

$$\theta_W = I_{P_0}(\widetilde{\pi}_{M_0}(w_0 \rtimes \theta)) \circ R(w_0, \pi_{M_0} \circ \theta) \circ \theta^*.$$

Here, we write $R(w_0, \pi_{M_0} \circ \theta) = R_{\theta(P_0)}(w_0, \pi_{M_0} \circ \theta)$ for simplicity.

Now the map $w \mapsto \theta(ww_0)$ gives a well-defined involutive action of θ on $W^M\backslash W/W^{M_0}$ since $w_0^{-1}W^{M_0}w_0 = W^{\theta(M_0)} = \theta(W^{M_0})$ and $w_0\theta(w_0) \in W^{M_0}$. If we let P_w be the standard parabolic subgroup of $\mathrm{GL}_N(E)$ such that wM_0w^{-1} is its Levi subgroup, then $\theta(P_w)$ contains $\theta(ww_0)M_0\theta(ww_0)^{-1}$ as a Levi subgroup. One can check that

$$\left.\frac{\gamma_A(0, \pi_{M_0,\lambda}, \rho^\vee_{w^{-1}P_w|P_0}, \psi_F)}{\gamma_A(0, \pi_{M_0,\lambda}, \rho^\vee_{\theta(ww_0)^{-1}\theta(P_w)|P_0}, \psi_F)}\right|_{\lambda=0} = \prod_{1 \le i < j \le t} \gamma_A(0, \rho'_i \otimes {\rho'_j}^\vee, \psi_E)^{m_{i,j}}$$

for some $m_{i,j} \in \mathbb{Z}$. Since $\rho'_i \not\cong \rho'_j$ for $1 \le i < j \le t$, the right-hand side is in $\mathbb{C}^\times$.

We denote the evaluation map $f \mapsto f(\mathbf{1})$ by $\mathrm{ev}_{\mathbf{1}}$. For $w \in W^M\backslash W/W^{M_0}$, we claim that

$$(\mathrm{ev}_{\mathbf{1}} \circ J(w, \pi_{M_0})) \circ (I_{P_0}(\widetilde{\pi}_{M_0}(w_0 \rtimes \theta)) \circ R(w_0, \pi_{M_0} \circ \theta) \circ \theta^*)$$

factors through $\mathrm{ev}_{\mathbf{1}} \circ J(\theta(ww_0), \pi_{M_0})$. For simplicity, let $\mathcal{V}$ be the space of π_0, and we regard $\widetilde{\pi}_{M_0}(w_0 \rtimes \theta)$ as a linear isomorphism $\Phi\colon \mathcal{V} \to \mathcal{V}$. Since $R(\theta(w), \theta(w_0)\pi_{M_0}) \circ R(\theta(w_0), \pi_{M_0}) = R(\theta(ww_0), \pi_{M_0})$ by Proposition 1.7.2, as linear maps, we have

$$\begin{aligned}
&(\mathrm{ev}_{\mathbf{1}} \circ J(w, \pi_{M_0})) \circ (I_{P_0}(\widetilde{\pi}_{M_0}(w_0 \rtimes \theta)) \circ R(w_0, \pi_{M_0} \circ \theta) \circ \theta^*) \\
&= \Phi \circ (\mathrm{ev}_{\mathbf{1}} \circ J(w, w_0(\pi_{M_0} \circ \theta))) \circ (\theta^* \circ R(\theta(w_0), \pi_{M_0})) \\
&= \Phi \circ (\mathrm{ev}_{\mathbf{1}} \circ J(\theta(w), \theta(w_0)\pi_{M_0})) \circ R(\theta(w_0), \pi_{M_0}) \\
&= \prod_{1 \le i < j \le t} \gamma_A(0, \rho'_i \otimes {\rho'_j}^\vee, \psi_E)^{-m_{i,j}} \cdot \Phi \circ (\mathrm{ev}_{\mathbf{1}} \circ J(\theta(ww_0), \pi_{M_0})).
\end{aligned}$$

This implies that $\theta_W(\mathcal{F}'_{\theta(ww_0)}) \subset \mathcal{F}'_w$ and $\theta_W(\mathcal{F}''_{\theta(ww_0)}) \subset \mathcal{F}''_w$, which is the advantage of $\mathcal{F}'_w$ over $\mathcal{F}_w$. Hence the induced action θ_W on $\mathrm{Jac}_P(\mathcal{I}_{\widehat{\phi}})$ sends $J_{\widehat{\phi}}^{\theta(ww_0)}$ to $J_{\widehat{\phi}}^w$. Since $w \mapsto \theta(ww_0)$ is an involution on $W^M \backslash W / W^{M_0}$, we see that θ_W swaps $J_{\widehat{\phi}}^w$ and $J_{\widehat{\phi}}^{\theta(ww_0)}$. Hence if $\theta(ww_0) \neq w$, then we have

$$\left(\mathrm{Ind}_P^{\mathrm{GL}_N(E)}(J_{\widehat{\phi}}^w) + \mathrm{Ind}_P^{\mathrm{GL}_N(E)}(J_{\widehat{\phi}}^{\theta(ww_0)})\right) \boxtimes \theta_W \overset{\theta}{=} 0.$$

On the other hand, if $\theta(ww_0) = w$, then θ_W preserves $J_{\widehat{\phi}}^w$. Moreover, the same argument as above shows that $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{GL}}_N(E)}(J_{\widehat{\phi}}^w \boxtimes \theta_W) = \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W$ in $\mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$. Therefore

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{GL}}_N(E)}(\mathrm{Jac}_{\widetilde{P}}(\mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W)) \overset{\theta}{=} \bigoplus_{\substack{w \in (W^M \backslash W / W^{M_0})^\theta \\ J_{\widehat{\phi}}^w \neq 0}} \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W,$$

where $(W^M \backslash W / W^{M_0})^\theta$ is the subset of the double coset space fixed by the action $w \mapsto \theta(ww_0)$.

Suppose that $P = MN_P$ corresponds to a partition $(n_m, \dots, n_1, n_0, n_1, \dots, n_m)$ of N. Here, we assume that $n_i > 0$ for $i > 0$, but n_0 is possibly zero. Note that $\dim(A_M^\theta) = m$. As in [Z, Section 1.6], the double coset space $W^M \backslash W / W^{M_0}$ is canonically identified with the set of matrices of the form

$$A = \begin{pmatrix} a_{-m,-r} & \dots & a_{-m,-1} & a'_{-m,1} & \dots & a'_{-m,t} & a_{-m,1} & \dots & a_{-m,r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m,-r} & \dots & a_{m,-1} & a'_{m,1} & \dots & a'_{m,t} & a_{m,1} & \dots & a_{m,r} \end{pmatrix} \in \mathrm{M}_{2m+1,2r+t}(\mathbb{Z})$$

such that

(1) all entries are non-negative;
(2) $\sum_{i=-m}^m a_{i,\pm j} = d_j$ for $1 \leq j \leq r$;
(3) $\sum_{i=-m}^m a'_{i,j} = d'_j$ for $1 \leq j \leq t$;
(4) $\sum_{j=1}^r (a_{i,-j} + a_{i,j}) + \sum_{j=1}^t a'_{i,j} = n_{|i|}$ for $-m \leq i \leq m$.

Since $\rho_{\pm i}$ and ρ'_j are cuspidal, we see that $J_{\widehat{\phi}}^w \neq 0$ if and only if

(5) $a_{i,\pm j} \in \{0, d_j\}$ for $-m \leq i \leq m$ and $1 \leq j \leq r$; and
(6) $a'_{i,j} \in \{0, d'_j\}$ for $-m \leq i \leq m$ and $1 \leq j \leq t$.

In particular, for each $\pm j$ (resp. j), there exists unique i such that $a_{i,\pm j} = d_j$ (resp. $a'_{i,j} = d_j$). Let X_P be the set of matrices A as above satisfying the conditions (1)–(6). For $A \in X_P$, the corresponding $J_{\widehat{\phi}}^w$ is given by

$$J_{\widehat{\phi}}^w = \bigotimes_{i=-m}^m \left(\left(\bigtimes_{\substack{1 \leq j \leq r \\ a_{i,j} \neq 0}} \rho_j \right) \times \left(\bigtimes_{\substack{1 \leq j \leq t \\ a'_{i,j} \neq 0}} \rho'_j \right) \times \left(\bigtimes_{\substack{1 \leq j \leq r \\ a_{i,-j} \neq 0}} \rho_{-j} \right) \right) \in \bigotimes_{i=-m}^m \mathcal{R}(\mathrm{GL}_{n_i}(E)).$$

The action $w \mapsto \theta(ww_0)$ on $W^M\backslash W/W^{M_0}$ gives an action of θ on X_P. This is given by

$$\begin{cases} a_{i,j} \mapsto a_{-i,-j} & (-m \le i \le m, 1 \le j \le r), \\ a'_{i,j} \mapsto a'_{-i,j} & (-m \le i \le m, 1 \le j \le t). \end{cases}$$

Therefore, A is fixed by this action if and only if

(7) $a_{-i,-j} = a_{i,j}$ for $-m \le i \le m$ and $1 \le j \le r$; and
(8) $a'_{-i,j} = a'_{i,j}$ for $-m \le i \le m$ and $1 \le j \le t$.

Since for each $1 \le j \le t$, there is only one i such that $a'_{i,j} \ne 0$, we must have $a'_{i,j} = 0$ for $i \ne 0$ and $a'_{0,j} = d'_j$.

Therefore, for a fixed $0 \le m \le r$, there is a bijection between

$$\left\{(P, w) \,\middle|\, P = \theta(P) = MN_P,\, \dim(A_M^\theta) = m,\, w \in (W^M\backslash W/W^{M_0})^\theta,\, J^w_{\widehat{\phi}} \ne 0\right\}$$

and

$$\{(\{I_i\}_{i=1}^m, \epsilon) \,|\, \emptyset \ne I_i \subset \{1, \dots, r\},\, I_i \cap I_{i'} = \emptyset\, (i \ne i'),\, \epsilon\colon I_1 \sqcup \dots \sqcup I_m \to \{\pm 1\}\}\,.$$

If (P, w) corresponds to $(\{I_i\}_{i=1}^m, \epsilon)$, then the corresponding matrix $A \in X_P$ is given such that

- if $1 \le j \le r$ and $j \in I_i$ for some $1 \le i \le m$, then $a_{\epsilon(j)i,j} = d_j$ and $a_{i',j} = 0$ for $i' \ne \epsilon(j)i$;
- if $1 \le j \le r$ and $j \notin I_i$ for all $1 \le i \le m$, then $a_{0,j} = d_j$ and $a_{i',j} = 0$ for $i' \ne 0$;
- if $1 \le j \le t$, then $a'_{0,j} = d'_j$ and $a'_{i',j} = 0$ for $i' \ne 0$.

Moreover, $J^w_{\widehat{\phi}}$ is equal to

$$\left(\bigtimes_{j\in I_1} \rho_{\epsilon(j)j}\right) \boxtimes \cdots \boxtimes \left(\bigtimes_{j\in I_m} \rho_{\epsilon(j)j}\right) \boxtimes \sigma_0 \boxtimes \left(\bigtimes_{j\in I_m} \rho_{-\epsilon(j)j}\right) \boxtimes \cdots \boxtimes \left(\bigtimes_{j\in I_1} \rho_{-\epsilon(j)j}\right)$$

for some $\sigma_0 \in \mathrm{Irr}(\mathrm{GL}_{n_0}(E))$. In particular, P corresponds to the partition

$$\left(\sum_{j\in I_1} d_j, \dots, \sum_{j \in I_m} d_j, n_0, \sum_{j\in I_m} d_j, \dots, \sum_{j\in I_1} d_j\right).$$

By setting $k_i = |I_i|$, we see that

$$\begin{aligned} &|\{(\{I_i\}_{i=1}^m, \epsilon) \,|\, \emptyset \ne I_i \subset \{1, \dots, r\},\, I_i \cap I_{i'} = \emptyset\, (i \ne i'),\, \epsilon\colon I_1 \sqcup \dots \sqcup I_m \to \{\pm 1\}\}| \\ &= \sum_{\substack{k_1,\dots,k_m \ge 1 \\ k_1+\dots+k_m \le r}} 2^{k_1+\dots+k_m} \frac{r(r-1)\cdots(r-k_1-\dots-k_m+1)}{k_1!\cdots k_m!} \\ &= \left(\frac{d}{dx}\right)^r e^x (e^{2x}-1)^m \bigg|_{x=0}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} D_{\widetilde{\mathrm{GL}}_N(E)}(\mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W) &\overset{\theta}{=} \left(\sum_{m=0}^{r} (-1)^m \left(\frac{d}{dx}\right)^r e^x (e^{2x}-1)^m \bigg|_{x=0} \right) \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W \\ &= \left(\left(\frac{d}{dx}\right)^r e^{-x}(1-(1-e^{2x})^{r+1}) \bigg|_{x=0} \right) \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W \\ &= \left(\left(\frac{d}{dx}\right)^r e^{-x} \bigg|_{x=0} \right) \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W = (-1)^r \mathcal{I}_{\widehat{\phi}} \boxtimes \theta_W. \end{aligned}$$

Here, we use the fact that $e^{-x}(1-e^{2x})^{r+1}$ has a zero at $x=0$ with order $r+1$. Since $r(\phi)=r$, we obtain the assertion. □

4.4. **ECR vs. Aubert duality.** Next, we consider Aubert duality for classical groups. We will use the notations in Section 1.2.

Let G be one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{O}_{2n}(F), \quad \mathrm{U}_n.$$

For $\pi \in \mathrm{Rep}(G^\circ)$, its Aubert dual is defined by

$$D_{G^\circ}(\pi) = \sum_{P^\circ} (-1)^{\dim(A_{M^\circ})} \mathrm{Ind}_{P^\circ}^{G^\circ}(\mathrm{Jac}_{P^\circ}(\pi))$$

in the Grothendieck group $\mathcal{R}(G^\circ)$, where $P^\circ = M^\circ N_P$ runs over the set of standard parabolic subgroups of G°. Note that $A_{G^\circ} = \{1\}$ unless $G^\circ = \mathrm{SO}_2(F)$. If $\pi \in \mathrm{Irr}(G^\circ)$, then $D_{G^\circ}(\pi) = \beta(\pi)\hat{\pi}$ for an irreducible representation $\hat{\pi}$ with a sign $\beta(\pi) \in \{\pm 1\}$ ([Au, Théorème 1.7 (4)]). As in Theorem B.2.3 (2), this sign is given by $\beta(\pi) = (-1)^{\dim(A_{M^\circ})}$, where $P^\circ = M^\circ N_P$ is a minimal standard parabolic subgroup of G° such that $\mathrm{Jac}_{P^\circ}(\pi) \neq 0$. Such a P° may not be unique, but the sign $(-1)^{\dim(A_{M^\circ})}$ is well-defined.

When $G = \mathrm{O}_{2n}(F)$, we also consider the twisted Aubert duality defined in Definition B.4.6. Fix a Borel subgroup $B^\circ = T^\circ U$ of $G^\circ = \mathrm{SO}_{2n}(F)$. If we denote the normalizer of (T°, B°) in G by T, then $T \cap G^\circ = T^\circ$ and $T/T^\circ \cong G/G^\circ$. Fix a representative $\epsilon \in T$ of the non-trivial coset in T/T° as in Section 1.2. For $\pi \in \mathrm{Rep}(G)$, we define

$$D_G(\pi) = \sum_{P^\circ} (-1)^{\dim(A_{M^\circ}^\epsilon)} \mathrm{Ind}_P^G(\mathrm{Jac}_P(\pi)),$$

where

- $P^\circ = M^\circ N_P$ now runs over the set of standard parabolic subgroups of $G^\circ = \mathrm{SO}_{2n}(F)$ which are stable under the conjugate action of ϵ;
- $P = P^\circ \cdot T \subset G$;
- A_M^ϵ is the subgroup of A_M fixed by the action of ϵ.

The Jacquet module $\mathrm{Jac}_P(\pi)$ is defined as usual (see Section 1.3). It is a representation of $M = M^\circ \cdot T$ and satisfies

$$\mathrm{Jac}_P(\pi)|_{M^\circ} = \mathrm{Jac}_{P^\circ}(\pi|_{G^\circ}).$$

By Propositions B.4.7 (3) and B.5.1, for $\pi \in \mathrm{Irr}(G)$, one can find $\hat{\pi} \in \mathrm{Irr}(G)$ such that the trace of $D_G(\pi) - \beta(\pi)\hat{\pi}$ is zero on $f_G \in C_c^\infty(G \setminus G^\circ)$. Here

$$\beta(\pi) = \begin{cases} \beta(\pi_1) = \beta(\pi_2) & \text{if } \pi|_{G^\circ} = \pi_1 \oplus \pi_2, \\ \beta(\pi|_{G^\circ}) & \text{if } \pi|_{G^\circ} \text{ is irreducible.} \end{cases}$$

Note that when $\pi|_{G^\circ} = \pi_1 \oplus \pi_2$, if $\mathrm{Jac}_P(\pi_1) \neq 0$, then $\mathrm{Jac}_{P'}(\pi_2) \neq 0$ with P' the conjugate of P by ϵ.

Example 4.4.1. Let us consider $G = \mathrm{O}_2(F)$. Then $G^\circ = \mathrm{SO}_2(F)$ is a torus so that it has only one parabolic subgroup $P^\circ = G^\circ$. For $\pi \in \mathrm{Irr}(G)$, we have

$$\beta(\pi) = (-1)^{\dim(A_{G^\circ})} = \begin{cases} -1 & \text{if } G^\circ = \mathrm{SO}_2(F) \text{ is split,} \\ 1 & \text{otherwise,} \end{cases}$$
$$(-1)^{\dim(A^\epsilon_{G^\circ})} = 1.$$

Hence $D_G(\pi) = \pi$. Since $\hat{\pi}$ is equal to $\beta(\pi)D_G(\pi)$ on $\mathrm{O}_2(F) \setminus \mathrm{SO}_2(F)$, we see that $\hat{\pi} \neq \pi$ if and only if $G^\circ = \mathrm{SO}_2(F)$ is split.

Now we compare Aubert duality for G with twisted Aubert duality for $\widetilde{\mathrm{GL}}_N(E)$. To do this, we consider the following hypothesis.

Hypothesis 4.4.2. Fix an A-parameter ψ for G. Then there exist multi-sets Π_ψ and $\Pi_{\widehat{\psi}}$ over $\mathrm{Irr}(G)$ equipped with $\langle \cdot, \pi \rangle_\psi$ and $\langle \cdot, \pi' \rangle_{\widehat{\psi}}$ satisfying (**ECR1**) and (**ECR2**) in Section 1.6. Moreover, we assume that for any proper Levi subgroup M of G and any A-parameter ψ_M for M, there exists a multi-set Π_{ψ_M} over $\mathrm{Irr}(M)$ equipped with $\langle \cdot, \pi_M \rangle_{\psi_M}$ satisfying (**ECR1**) and (**ECR2**) in Section 1.6.

Remark 4.4.3. Notice that Hypothesis 4.4.2 does not require members in A-packets to be unitary. For a proper Levi subgroup M, Hypothesis 4.4.2 assumes Arthur's results for all A-parameters ψ_M, whereas, for G, it assumes only for a fixed A-parameter ψ and its dual $\widehat{\psi}$. In particular, if $\psi = \phi$ is a tempered L-parameters ϕ for G, after establishing (**ECR1**) and (**ECR2**) for $\widehat{\phi}$ in the next section, one can use results in this section for ϕ and $\widehat{\phi}$.

Lemma 4.4.4. *Fix $\psi \in \Psi(G)$. Assume Hypothesis 4.4.2.*

(1) *The A-packet $\Pi_{\widehat{\psi}}$ is given by*

$$\Pi_{\widehat{\psi}} = \{\hat{\pi} \mid \pi \in \Pi_\psi\}.$$

Moreover, for $\pi \in \Pi_\psi$, we have $\beta(\psi)\langle s_{\widehat{\psi}}, \hat{\pi} \rangle_{\widehat{\psi}} = \langle s_\psi, \pi \rangle_\psi \beta(\pi)$.

(2) *For $\pi \in \Pi_\psi$ and $s \in A_\psi$, we have*

$$\frac{\langle \hat{s}, \hat{\pi} \rangle_{\widehat{\psi}}}{\langle s, \pi \rangle_\psi} = \frac{\beta(\psi)}{\beta(\psi_+)\beta(\psi_-)},$$

where $\hat{s} \in A_{\widehat{\psi}}$ is the element corresponding to s via the canonical identification

$$A_\psi \xrightarrow{\sim} A_{\widehat{\psi}},\ e(\rho, a, b) \mapsto e(\rho, b, a),$$

and $\psi_\pm$ is given by s as in (**ECR2**) *in Section 1.6.*

Proof. We show (1). Since (twisted) Aubert duality commutes with the twisted endoscopic character identity (see [X2, (A.1)]), when $\widetilde{f} \in C_c^\infty(\mathrm{GL}_N(E) \rtimes \theta)$ and $f_G \in C_c^\infty(G^\circ)$ have matching orbital integrals, we have

$$\begin{aligned}\beta(\psi) \sum_{\hat{\pi} \in \Pi_{\widehat{\psi}}} \langle s_{\widehat{\psi}}, \hat{\pi} \rangle_{\widehat{\psi}} \Theta_{\hat{\pi}}(f_G) &= \beta(\psi)(G : G^\circ) \Theta_{\widetilde{\pi}_{\widehat{\psi}}}(\widetilde{f}) \\ &= (G : G^\circ) \Theta_{D_{\widetilde{\mathrm{GL}}_N(E)}(\widetilde{\pi}_\psi)}(\widetilde{f}) \\ &= \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle_\psi \Theta_{D_{G^\circ}(\pi)}(f_G) \\ &= \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle_\psi \beta(\pi) \Theta_{\hat{\pi}}(f_G).\end{aligned}$$

By the linear independence of the characters Θ_π together with the surjectivity of $\widetilde{f} \mapsto f_G$, we see that $\Pi_{\widehat{\psi}} = \{\hat{\pi} \,|\, \pi \in \Pi_\psi\}$. Moreover, comparing the coefficients, we have $\beta(\psi)\langle s_{\widehat{\psi}}, \hat{\pi} \rangle_{\widehat{\psi}} = \langle s_\psi, \pi \rangle_\psi \beta(\pi)$.

Next, we show (2). Similar to (1), by [Hi, Theorem 1.5] or [X2, (A.1)], when $f_G \in C_c^\infty(G)$ and $f_{G_+} \otimes f_{G_-} \in C_c^\infty(G_+^\circ \times G_-^\circ)$ have matching orbital integrals, we have

$$\begin{aligned}&\frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s \cdot s_\psi, \pi \rangle_\psi \beta(\pi) \Theta_{\hat{\pi}}(f_G) \\ &= \frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s \cdot s_\psi, \pi \rangle_\psi \Theta_{D_{G^\bullet}(\pi)}(f_G) \\ &= \prod_{\kappa \in \{\pm\}} \frac{1}{(G_\kappa : G_\kappa^\circ)} \sum_{\pi_\kappa \in \Pi_{\psi_\kappa \otimes \eta_\kappa}} \langle s_{\psi_\kappa \otimes \eta_\kappa}, \pi_\kappa \rangle_{\psi_\kappa \otimes \eta_\kappa} \Theta_{D_{G_\kappa^\circ}(\pi_\kappa)}(f_{G_\kappa}) \\ &= \prod_{\kappa \in \{\pm\}} \frac{1}{(G_\kappa : G_\kappa^\circ)} \sum_{\pi_\kappa \in \Pi_{\psi_\kappa \otimes \eta_\kappa}} \langle s_{\psi_\kappa \otimes \eta_\kappa}, \pi_\kappa \rangle_{\psi_\kappa} \beta(\pi_\kappa) \Theta_{\hat{\pi}_\kappa}(f_{G_\kappa}).\end{aligned}$$

Here, we set

$$D_{G^\bullet} = \begin{cases} D_G & \text{if } G = \mathrm{O}_{2n}(F) \text{ and } s \notin A_\psi^+, \\ D_{G^\circ} & \text{otherwise,} \end{cases}$$

and we assume that $f_G \in C_c^\infty(G^\circ)$ if $D_{G^\bullet} = D_{G^\circ}$, whereas $f_G|_{G^\circ} = 0$ if $D_{G^\bullet} = D_G$. Using (1) and Lemma 4.1.2, we have

$$\begin{aligned}&\frac{\beta(\psi)}{(G : G^\circ)} \sum_{\pi \in \Pi_\psi} \langle s, \pi \rangle_\psi \langle s_{\widehat{\psi}}, \hat{\pi} \rangle_{\widehat{\psi}} \Theta_{\hat{\pi}}(f_G) \\ &= \prod_{\kappa \in \{\pm\}} \frac{\beta(\psi_\kappa \otimes \eta_\kappa)}{(G_\kappa : G_\kappa^\circ)} \sum_{\pi_\kappa \in \Pi_{\psi_\kappa \otimes \eta_\kappa}} \langle s_{\widehat{\psi}_\kappa \otimes \eta_\kappa}, \hat{\pi}_\kappa \rangle_{\widehat{\psi}_\kappa \otimes \eta_\kappa} \Theta_{\hat{\pi}_\kappa}(f_{G_\kappa})\end{aligned}$$

$$= \frac{\beta(\psi_+)\beta(\psi_-)}{(G:G^\circ)} \sum_{\hat{\pi}\in\Pi_{\widehat{\psi}}} \langle \hat{s}\cdot s_{\widehat{\psi}}, \hat{\pi}\rangle_{\widehat{\psi}} \Theta_{\hat{\pi}}(f_G).$$

Comparing the coefficients, we have $\beta(\psi)\langle s,\pi\rangle_\psi = \beta(\psi_+)\beta(\psi_-)\langle \hat{s},\hat{\pi}\rangle_{\widehat{\psi}}$, as desired. □

By Proposition 4.1.3, we know $\beta(\psi)$ for tempered parameters $\psi = \phi$. It is useful to state the following result.

Corollary 4.4.5. *Let ϕ be a tempered L-parameter for G. Assume the existence of A-packets Π_ϕ and $\Pi_{\widehat{\phi}}$ associated to ϕ and $\widehat{\phi}$ which satisfy* (**ECR1**) *and* (**ECR2**). *Then we have*

$$\frac{\langle \hat{s},\hat{\pi}\rangle_{\widehat{\phi}}}{\langle s,\pi\rangle_\phi} = (-1)^{r(\phi)-r(\phi_+)-r(\phi_-)}.$$

Remark 4.4.6. Notice that the same proof shows the converse of Lemma 4.4.4 in the following sense. Namely, if we assume (**ECR1**) and Lemma 4.4.4 (1) (resp. (**ECR2**) and Lemma 4.4.4 (2)) for an A-parameter ψ, then we obtain (**ECR1**) (resp. (**ECR2**)) for the dual A-parameter $\widehat{\psi}$. In fact, (**ECR1**) and (**ECR2**) for A-parameters of the form $\psi = \widehat{\phi}$, where ϕ is a tempered L-parameter, will be proven in this way. See Theorem 5.4.1 below.

Note that the same statement as Corollary 4.4.5 was recently given by Liu–Lo–Shahidi [LLS, Theorem 5.9].

We shall give some example of this corollary for $G = \mathrm{O}_4(F)$. In this example, for $A, B \in \mathcal{R}(G_0)$, we write $A \le B$ if $B - A$ is a non-negative combination of irreducible representations.

Example 4.4.7. Write $G = \mathrm{O}_4(F)$. We denote by $B^\circ = T^\circ U$ the standard Borel subgroup of G° so that its Levi is isomorphic to $T^\circ = \mathrm{GL}_1(F)\times \mathrm{SO}_2(F)$. This is the unique proper standard parabolic subgroup which is stable under the conjugate action of $T = \mathrm{GL}_1(F)\times\mathrm{O}_2(F)$. Set $B = TU$. For simplicity, write $e_\chi = e(\chi,1,1) \in A_\phi$ for some ϕ containing χ.

(1) Let χ and χ' be quadratic characters of $F^\times$ with $\chi \ne \chi'$. Consider

$$\phi = \chi\oplus\chi\oplus\chi'\oplus\chi' \in \Phi(G)$$

so that $G^\circ = \mathrm{SO}_4(F)$ is split. Then $|\mathcal{A}_\phi| = 4$ and $|\mathcal{S}_\phi| = 2$. We can write $\Pi_\phi = \{\pi, \pi\otimes\det, \pi', \pi'\otimes\det\}$ such that

$$\begin{aligned} \langle e_\chi, \pi\rangle_\phi &= 1, & \langle e_{\chi'}, \pi\rangle_\phi &= 1,\\ \langle e_\chi, \pi\otimes\det\rangle_\phi &= -1, & \langle e_{\chi'}, \pi\otimes\det\rangle_\phi &= -1,\\ \langle e_\chi, \pi'\rangle_\phi &= -1, & \langle e_{\chi'}, \pi'\rangle_\phi &= 1,\\ \langle e_\chi, \pi'\otimes\det\rangle_\phi &= 1, & \langle e_{\chi'}, \pi'\otimes\det\rangle_\phi &= -1. \end{aligned}$$

Similarly, write $\Pi_{\chi\oplus\chi} = \{\pi_\chi, \pi_\chi \otimes \det\}$ and $\Pi_{\chi'\oplus\chi'} = \{\pi_{\chi'}, \pi_{\chi'} \otimes \det\}$ such that $\langle\cdot, \pi_\chi\rangle_{\chi\oplus\chi} = \mathbf{1}$ and $\langle\cdot, \pi_{\chi'}\rangle_{\chi'\oplus\chi'} = \mathbf{1}$. Then

$$\mathrm{Ind}_B^G(\chi \boxtimes \pi_{\chi'}) = \pi \oplus \pi', \quad \mathrm{Ind}_B^G(\chi' \boxtimes \pi_\chi) = \pi \oplus (\pi' \otimes \det).$$

Hence, in the Grothendieck group $\mathcal{R}(T)$, we have

$$\mathrm{Jac}_B(\pi) \leq 2\chi \otimes \pi_{\chi'} + \chi' \otimes \pi_\chi + \chi' \otimes (\pi_\chi \otimes \det),$$
$$\mathrm{Jac}_B(\pi) \leq 2\chi' \otimes \pi_\chi + \chi \otimes \pi_{\chi'} + \chi \otimes (\pi_{\chi'} \otimes \det)$$

so that

$$\mathrm{Jac}_B(\pi) = \chi \otimes \pi_{\chi'} + \chi' \otimes \pi_\chi.$$

Since $\beta(\pi) = 1$, we have

$$\hat{\pi} \overset{\theta}{=} D_G(\pi) = \pi - (\pi + \pi') - (\pi + (\pi' \otimes \det)) \overset{\theta}{=} \pi \otimes \det.$$

One can check that

$$\frac{\langle e_\chi, \pi \otimes \det\rangle_\phi}{\langle e_\chi, \pi\rangle_\phi} = \frac{\langle e_{\chi'}, \pi \otimes \det\rangle_\phi}{\langle e_{\chi'}, \pi\rangle_\phi} = -1,$$

which is the statement of Corollary 4.4.5. Similarly, we have $\hat{\pi}' = \pi' \otimes \det$, and one can also check Corollary 4.4.5 for this case.

(2) Let χ, χ' and χ'' be distinct quadratic characters of $F^\times$. Consider

$$\phi = \chi \oplus \chi \oplus \chi' \oplus \chi'' \in \Phi(G)$$

so that $G^\circ = \mathrm{SO}_4(F)$ is not split. Then $|\mathcal{A}_\phi| = 4$ and $|\mathcal{S}_\phi| = 2$. We can write $\Pi_\phi = \{\pi, \pi \otimes \det, \pi', \pi' \otimes \det\}$ such that

$$\langle e_\chi, \pi\rangle_\phi = 1, \quad \langle e_{\chi'}, \pi\rangle_\phi = 1, \quad \langle e_{\chi''}, \pi\rangle_\phi = 1$$
$$\langle e_\chi, \pi'\rangle_\phi = -1, \quad \langle e_{\chi'}, \pi'\rangle_\phi = 1, \quad \langle e_{\chi''}, \pi'\rangle_\phi = 1.$$

Similarly, write $\Pi_{\chi'\oplus\chi''} = \{\pi_0, \pi_0 \otimes \det\}$ with $\langle\cdot, \pi_0\rangle_{\chi'\oplus\chi''} = \mathbf{1}$. Then $\mathrm{Ind}_B^G(\chi \boxtimes \pi_0) = \pi \oplus \pi'$ so that $\mathrm{Jac}_B(\pi) = \chi \otimes \pi_0$. Since $\beta(\pi) = -1$, we have

$$\hat{\pi} \overset{\theta}{=} -D_G(\pi) = -\pi + (\pi + \pi') = \pi'.$$

On the other hand, since

$$(-1)^{r(\phi)-r(\phi_+)-r(\phi_-)} = \begin{cases} -1 & \text{if } s = e_\chi, \\ 1 & \text{if } s = e_{\chi'}, e_{\chi''}, \end{cases}$$

we get Corollary 4.4.5 for π.

(3) Let χ and χ' be quadratic characters of $F^\times$ with $\chi \neq \chi'$. Consider

$$\phi = \chi \oplus \chi \oplus \chi \oplus \chi' \in \Phi(G)$$

so that $G^\circ = \mathrm{SO}_4(F)$ is not split. Then $|\mathcal{A}_\phi| = 2$ and $|\mathcal{S}_\phi| = 1$. We can write $\Pi_\phi = \{\pi, \pi \otimes \det\}$ such that $\langle\cdot, \pi\rangle_\phi = \mathbf{1}$. Then $\pi = \mathrm{Ind}_B^G(\chi \boxtimes \pi_0)$ with $\pi_0 \in \Pi_{\chi\oplus\chi'}$ such that $\langle\cdot, \pi_0\rangle_{\chi\oplus\chi'} = \mathbf{1}$. Hence $\mathrm{Jac}_B(\pi) = 2\chi \otimes \pi_0$. Since $\beta(\pi) = -1$, we have

$$\hat{\pi} \overset{\theta}{=} -D_G(\pi) = -\pi + 2\pi = \pi.$$

On the other hand, since

$$(-1)^{r(\phi)-r(\phi_+)-r(\phi_-)} = 1$$

for $s = e_\chi$ and e'_χ, we obtain Corollary 4.4.5 for π.

(4) Let χ be a quadratic character of $F^\times$. Consider

$$\phi = \chi \oplus \chi \oplus \chi \oplus \chi \in \Phi(G)$$

so that $G^\circ = \mathrm{SO}_4(F)$ is split. Then $|\mathcal{A}_\phi| = 2$ and $|\mathcal{S}_\phi| = 1$. We can write $\Pi_\phi = \{\pi, \pi \otimes \det\}$ such that $\langle \cdot, \pi\rangle_\phi = \mathbf{1}$. Then $\pi = \mathrm{Ind}_B^G(\chi \boxtimes \pi_\chi)$ with $\pi_\chi \in \Pi_{\chi\oplus\chi}$ such that $\langle \cdot, \pi_\chi\rangle_{\chi\oplus\chi} = \mathbf{1}$. Note that

$$\mathrm{Jac}_B(\pi) = 3\chi \otimes \pi_\chi + \chi \otimes (\pi_\chi \otimes \det).$$

Since $\beta(\pi) = 1$, we have

$$\hat{\pi} \overset{\theta}{=} D_G(\pi) = \pi - (3\pi + (\pi \otimes \det)) \overset{\theta}{=} -\pi \overset{\theta}{=} \pi \otimes \det.$$

On the other hand, since

$$(-1)^{r(\phi)-r(\phi_+)-r(\phi_-)} = -1$$

for $s = e_\chi$, we get Corollary 4.4.5 for π.

4.5. **Computation of $\beta(\psi)$.** To show Theorem 1.10.5 (2), we need to compare $\langle \hat{s}, \hat{\pi}\rangle_{\widehat{\psi}}$ with $\langle s, \pi\rangle_\psi$ in a little more general situation than the tempered case. The results in this subsection will only be used in the proof of Lemma 7.6.2.

In the next proposition, we compute $\beta(\psi)$ when ψ is irreducible as a representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ assuming Hypothesis 4.4.2.

Proposition 4.5.1. *Assume Hypothesis 4.4.2. If ψ is irreducible and conjugate-self-dual, then $\beta(\psi) = (-1)^{r(\psi)}$.*

Proof. Using Lemma 4.1.2, up to replacing ψ with $\psi \otimes \eta$ for some η if necessary, we may assume that $\psi \in \Psi(G)$ for some classical group G.

Since ψ is irreducible, we have $\mathcal{A}_\psi \cong \mathcal{A}_{\widehat{\psi}} = \{1\}$. By Lemma 4.4.4 (1), we have $\beta(\psi) = \beta(\pi)$ for every $\pi \in \Pi_\psi$. By [Ar2, Proposition 7.4.1] and [Mok, Proposition 8.4.1] which we can use because we are assuming Hypothesis 4.4.2, we can take π to be an element in the associated L-packet Π_{ϕ_ψ}. If π is the unique element in this L-packet corresponding to the trivial character of $\mathcal{A}_{\phi_\psi}$, and if we write $\psi = \rho \boxtimes S_a \boxtimes S_b$, then by Theorem C.3.3, we have

$$\pi \hookrightarrow \rho|\cdot|^{x_1} \times \cdots \times \rho|\cdot|^{x_n} \rtimes \sigma$$

for some supercuspidal representation σ and real numbers $x_1, \ldots, x_n \in \mathbb{R}$, where $n = [\frac{ab}{2}]$. For the notion of the parabolically induced representations, see Section C.1. Since $r(\psi) = [\frac{ab}{2}]$, we have $\beta(\psi) = \beta(\pi) = (-1)^{r(\psi)}$. $\square$

By Propositions 4.1.3 and 4.5.1, we know $\beta(\psi) = (-1)^{r(\psi)}$ when ψ is irreducible or (co-)tempered. We will need a few more cases. In the following proposition, for $\psi\colon W_E \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$, we define ψ^D and ψ^A by

$$\psi^D(w, g_1, g_2) = \psi(w, g_1, g_1), \quad \psi^A(w, g_1, g_2) = \psi(w, g_2, g_2).$$

Recall that S_a is the unique a-dimensional irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$.

Proposition 4.5.2. *Fix an A-parameter ψ of G of good parity, and assume Hypothesis 4.4.2. Suppose that $\psi = \phi_1 \boxtimes S_1 \oplus \phi_2 \boxtimes S_2$ for some representations ϕ_1, ϕ_2 of $W_E \times \mathrm{SL}_2(\mathbb{C})$ (which can be zero). If we decompose $\psi = \oplus_{i=1}^t \psi_i$ into irreducible representations, then we have*

$$\beta(\psi) = (-1)^{r(\psi)} \prod_{1 \le i < j \le t} \left. \frac{\gamma_A(s, \psi_i^A \otimes {}^c\psi_j^A, \psi_E)}{\gamma_A(s, \widehat{\psi}_i \otimes {}^c\widehat{\psi}_j, \psi_E)} \right|_{s=0}.$$

Proof. Note that the product of gamma factors is independent of the order of the irreducible components of ψ by Proposition A.1.2.

We prove the assertion by induction on $\dim(\phi_2)$. If $\phi_2 = 0$, then $\psi = \phi_1$ is tempered and the assertion is Proposition 4.1.3 since $\psi_i^A = \widehat{\psi}_i$ for any $1 \le i \le t$ in this case.

Suppose that $\phi_2 \neq 0$. Choose an irreducible subrepresentation $\phi_0 \subset \phi_2$. Set $\psi_0 = \phi_0 \boxtimes S_2$ and $\psi' = \psi - \phi_0 \boxtimes S_2$. Hence $\pi_\psi = \pi_{\psi'} \times \pi_{\psi_0}$. We denote the standard modules of π_{ψ_0} and $\pi_{\psi'}$ by $\mathcal{I}_{\psi_0}$ and $\mathcal{I}_{\psi'}$, respectively. By Lemma 3.1.2, we have the following diagram

$$\begin{array}{ccccccccc} & & \mathcal{I}_{\psi'} \times \pi_{\psi_0^D} & \hookrightarrow & \mathcal{I}_{\psi'} \times \mathcal{I}_{\psi_0} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \pi_{\psi'} \times \pi_{\psi_0^D} & \longrightarrow & \pi_{\psi'} \times \mathcal{I}_{\psi_0} & \longrightarrow & \pi_{\psi'} \times \pi_{\psi_0} & \longrightarrow & 0. \end{array}$$

Here, the bottom sequence is exact since $\mathcal{I}_{\psi_0} = \pi_{\phi_0}|\cdot|_E^{\frac{1}{2}} \times \pi_{\phi_0}|\cdot|_E^{-\frac{1}{2}}$ is of length two. By the same argument as Lemma 3.1.1, we have

$$\dim_{\mathbb{C}}(\mathrm{End}_{\mathrm{GL}_N(E)}(\pi_{\psi'} \times \mathcal{I}_{\psi_0})) = 1.$$

Indeed, the canonical map

$$\mathrm{End}_{\mathrm{GL}_N(E)}(\pi_{\psi'} \times \mathcal{I}_{\psi_0}) \to \mathrm{End}_{\mathrm{GL}_N(E)}(\pi_{\psi'} \times \pi_{\psi_0}) \cong \mathbb{C}$$

is injective since $\pi_{\psi'} \times \pi_{\psi_0}$ is the unique irreducible quotient of $\pi_{\psi'} \times \mathcal{I}_{\psi_0}$ and appears in $\pi_{\psi'} \times I_{\psi_0}$ as a subquotient with multiplicity one. By the functoriality of the Aubert involution (Theorem B.2.3 (1)), we see that $\mathrm{End}_{\mathrm{GL}_N(E)}(\hat{\pi}_{\psi'} \times \widehat{\mathcal{I}}_{\psi_0})$ is also one dimensional.

The action θ_W of θ on the standard module $\mathcal{I}_{\psi'} \times \mathcal{I}_{\psi_0}$ which fixes a Whittaker functional gives an action θ_W on $\pi_{\psi'} \times \mathcal{I}_{\psi_0}$. This is the unique action inducing Arthur's actions θ_A both on $\pi_{\psi'} \times \pi_{\psi_0^D}$ and on $\pi_{\psi'} \times \pi_{\psi_0}$. If we denote by $\widehat{\theta}_W$ and $\widehat{\theta}_A$ the actions induced by $\widetilde{\pi} \mapsto \widehat{\widetilde{\pi}}$, then by Proposition B.4.7 (2) and Theorem B.2.3 (4), we see that

$$(\pi_{\widehat{\psi}'} \times \widehat{\mathcal{I}}_{\psi_0}) \boxtimes \widehat{\theta}_W = (\pi_{\widehat{\psi}'} \times \pi_{\widehat{\psi}_0}) \boxtimes \widehat{\theta}_A + (\pi_{\widehat{\psi}'} \times \pi_{\psi_0^A}) \boxtimes \widehat{\theta}_A$$

in the Grothendieck group $\mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$. Since $D_{\widetilde{\mathrm{GL}}_N(E)}(\pi_\psi \boxtimes \theta_A) \overset{\theta}{=} \beta(\psi)\pi_{\widehat{\psi}} \boxtimes \theta_A$, if we take an action θ'_W of θ on $\hat{\pi}_{\psi'} \times \widehat{\mathcal{I}}_{\psi_0}$ such that

$$(\pi_{\widehat{\psi}'} \times \widehat{\mathcal{I}}_{\psi_0}) \boxtimes \theta'_W = (\pi_{\widehat{\psi}'} \times \pi_{\widehat{\psi}_0}) \boxtimes \theta_A + (\pi_{\widehat{\psi}'} \times \pi_{\psi_0^A}) \boxtimes \theta'_A$$

in $\mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$ for some θ'_A, then we have

$$\theta'_A = \frac{\beta(\psi' \oplus \psi_0^D)}{\beta(\psi)}\theta_A.$$

We realize this action θ'_W on $\hat{\pi}_{\psi'} \times \widehat{\mathcal{I}}_{\psi_0}$ using Theorem 1.9.1. Namely, if we let $P = MN_P$ be the standard maximal parabolic subgroup of $\mathrm{GL}_N(E)$ such that $\pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0}$ is an irreducible representation of M, and if we let $w \in W(\theta(M), M)$ be the unique non-trivial element, then we have $w(\pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0} \circ \theta) \cong \pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0}$, and the normalized intertwining operator $\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\widehat{\psi}_0})$ realizes Arthur's action θ_A on $\pi_{\widehat{\psi}} = \pi_{\widehat{\psi}'} \times \pi_{\widehat{\psi}_0}$. Recall that

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\widehat{\psi}_0}) = I_P(\widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\widehat{\psi}_0}(w \rtimes \theta)) \circ \theta^* \circ R_P(\theta(w), \pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0}, \psi).$$

The operator $R_P(\theta(w), \pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0}, \psi)$ can be extended to a meromorphic family of normalized intertwining operators on $\pi_{\widehat{\psi}'}|\cdot|_E^{s'} \times \widehat{\mathcal{I}}_{\psi_0}|\cdot|_E^{s}$ for $(s, s') \in \mathbb{C}$. It can be decomposed into the composition of the normalized intertwining operators

$$\begin{aligned}\hat{\pi}_{\psi'}|\cdot|_E^{s'} \times \hat{\pi}_{\phi_0}|\cdot|_E^{s-\frac{1}{2}} \times \hat{\pi}_{\phi_0}|\cdot|_E^{s+\frac{1}{2}} &\to \hat{\pi}_{\phi_0}|\cdot|_E^{s-\frac{1}{2}} \times \hat{\pi}_{\psi'}|\cdot|_E^{s'} \times \hat{\pi}_{\phi_0}|\cdot|_E^{s+\frac{1}{2}} \\ &\to \hat{\pi}_{\phi_0}|\cdot|_E^{s-\frac{1}{2}} \times \hat{\pi}_{\phi_0}|\cdot|_E^{s+\frac{1}{2}} \times \hat{\pi}_{\psi'}|\cdot|_E^{s'}\end{aligned}$$

up to a scalar valued meromorphic function which is holomorphic at $(s, s') = (0, 0)$ (see Lemma 1.7.3). Since these two intertwining operators are holomorphic at $(s, s') = (0, 0)$ by [MW2, I.6.3, Lemma (ii)], we see that $R_P(\theta(w), \pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0}, \psi)$ can be considered as a well-defined operator on $\pi_{\widehat{\psi}'} \times \widehat{\mathcal{I}}_{\psi_0}$. On the other hand, the linear isomorphism

$$\widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\widehat{\psi}_0}(w \rtimes \theta) \colon w(\pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0} \circ \theta) \xrightarrow{\sim} \pi_{\widehat{\psi}'} \boxtimes \pi_{\widehat{\psi}_0}$$

can be extended to a linear isomorphism

$$w(\pi_{\widehat{\psi}'} \boxtimes \widehat{\mathcal{I}}_{\psi_0} \circ \theta) \xrightarrow{\sim} \pi_{\widehat{\psi}'} \boxtimes \widehat{\mathcal{I}}_{\psi_0}.$$

Therefore, the action θ'_W on $\pi_{\widehat{\psi}'} \times \widehat{\mathcal{I}}_{\psi_0}$ can be realized by $\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\widehat{\psi}_0})$.

By Propositions 4.1.3 and 4.5.1 together with $r(\psi_0^D) = r(\psi_0)$, if we apply the functor $\widetilde{\pi} \mapsto \widehat{\widetilde{\pi}}$ to the equation

$$\mathcal{I}_{\psi_0} \boxtimes \theta_W = \pi_{\psi_0^D} \boxtimes \theta_A + \pi_{\psi_0} \boxtimes \theta_A$$

in $\mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$, then we obtain

$$\widehat{\mathcal{I}}_{\psi_0} \boxtimes \widehat{\theta}_W = \pi_{\psi_0^A} \boxtimes \theta_A + \pi_{\widehat{\psi}_0} \boxtimes \theta_A$$

for some $\widehat{\theta}_W$. Hence the above isomorphism $w(\pi_{\widehat{\psi}'} \boxtimes \widehat{\mathcal{I}}_{\psi_0} \circ \theta) \xrightarrow{\sim} \pi_{\widehat{\psi}'} \boxtimes \widehat{\mathcal{I}}_{\psi_0}$ induces

$$\widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\psi_0^A}(w \rtimes \theta) \colon w(\pi_{\widehat{\psi}'} \boxtimes \pi_{\psi_0^A} \circ \theta) \xrightarrow{\sim} \pi_{\widehat{\psi}'} \boxtimes \pi_{\psi_0^A}.$$

Therefore the normalized intertwining operator $\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\widehat{\psi}_0})$ on $\pi_{\widehat{\psi}'} \times \widehat{\mathcal{I}}_{\psi_0}$ induces

$$\widetilde{R}_P(\theta \circ w, \widetilde{\pi}_{\widehat{\psi}'} \boxtimes \widetilde{\pi}_{\psi_0^A}) \times \left. \frac{\gamma_A(s, \widehat{\psi}' \otimes {}^c\widehat{\psi}_0, \psi_E)}{\gamma_A(s, \widehat{\psi}' \otimes {}^c\psi_0^A, \psi_E)} \right|_{s=0}$$

on $\pi_{\widehat{\psi}'} \times \pi_{\psi_0^A}$. This means that

$$\frac{\beta(\psi' \oplus \psi_0^D)}{\beta(\psi)} = \left. \frac{\gamma_A(s, \widehat{\psi}' \otimes {}^c\widehat{\psi}_0, \psi_E)}{\gamma_A(s, \widehat{\psi}' \otimes {}^c\psi_0^A, \psi_E)} \right|_{s=0}.$$

Now, we write $\psi' = \oplus_{i=1}^t \psi_i$ for the irreducible decomposition. Then by induction, we see that $\beta(\psi' \oplus \psi_0^D)$ is equal to

$$(-1)^{r(\psi' \oplus \psi_0^D)} \left(\prod_{1 \le i < j \le t} \left. \frac{\gamma_A(s, \psi_i^A \otimes {}^c\psi_j^A, \psi_E)}{\gamma_A(s, \widehat{\psi}_i \otimes {}^c\widehat{\psi}_j, \psi_E)} \right|_{s=0} \right) \left(\left. \frac{\gamma_A(s, \psi'^A \otimes {}^c\psi_0^A, \psi_E)}{\gamma_A(s, \widehat{\psi}' \otimes {}^c\psi_0^A, \psi_E)} \right|_{s=0} \right).$$

Since $r(\psi' \oplus \psi_0^D) = r(\psi)$, we have

$$\beta(\psi) = (-1)^{r(\psi)} \left(\prod_{0 \le i < j \le t} \left. \frac{\gamma_A(s, \psi_i^A \otimes {}^c\psi_j^A, \psi_E)}{\gamma_A(s, \widehat{\psi}_i \otimes {}^c\widehat{\psi}_j, \psi_E)} \right|_{s=0} \right) \left(\left. \frac{\gamma_A(s, \psi'^A \otimes {}^c\psi_0^A, \psi_E)}{\gamma_A(s, \widehat{\psi}' \otimes {}^c\widehat{\psi}_0, \psi_E)} \right|_{s=0} \right).$$

This completes the proof. □

Corollary 4.5.3. *Fix an A-parameter ψ of G of good parity, and assume Hypothesis 4.4.2. Suppose that $\psi = \phi_1 \boxtimes S_1 \oplus \phi_2 \boxtimes S_2$ for some representations ϕ_1, ϕ_2 of $W_E \times \mathrm{SL}_2(\mathbb{C})$. For $s = \sum_{i \in I_-} e(\rho_i, a_i, b_i) \in A_\psi$, define $\psi_\pm$ by*

$$\psi_- = \bigoplus_{i \in I_-} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}, \quad \psi_+ = \psi - \psi_-$$

as in Section 1.6. Then

$$\frac{\langle \hat{s}, \hat{\pi} \rangle_{\widehat{\psi}}}{\langle s, \pi \rangle_\psi} = (-1)^{r(\psi) - r(\psi_+) - r(\psi_-)} \left. \frac{\gamma_A(s, \psi_+^A \otimes {}^c\psi_-^A, \psi_E)}{\gamma_A(s, \widehat{\psi}_+ \otimes {}^c\widehat{\psi}_-, \psi_E)} \right|_{s=0}.$$

Proof. This follows from Lemma 4.4.4 (2) and Proposition 4.5.2. □

5. Endoscopic character relations for co-tempered parameters

The purpose of this section is to prove Theorem 1.10.5 (1). In this section, we do not impose Hypothesis 4.4.2.

5.1. **Equation** $(*)$**.** Let $\psi = \widehat{\phi}$ be a co-tempered A-parameter for G. We will construct the A-packet Π_ψ together with the pairing $\langle \cdot, \pi \rangle_\psi$ for $\pi \in \Pi_\psi$. According to Lemma 4.4.4 and Corollary 4.4.5, the correct definitions should be as follows:

- $\Pi_{\widehat{\phi}} = \{\hat{\pi} \,|\, \pi \in \Pi_\phi\}$;
- $\langle \hat{s}, \hat{\pi} \rangle_{\widehat{\phi}} = (-1)^{r(\phi)-r(\phi_+)-r(\phi_-)} \langle s, \pi \rangle_\phi$ for $s \in \mathcal{A}_\phi$ corresponding to $\hat{s} \in \mathcal{A}_{\widehat{\phi}}$.

As explained in Remark 4.4.6, to show (**ECR1**) for this packet $\Pi_{\widehat{\phi}}$, we need Lemma 4.4.4 (1), i.e., the equation $\beta(\phi)\langle s_{\widehat{\phi}}, \hat{\pi} \rangle_{\widehat{\phi}} = \langle s_\phi, \pi \rangle_\phi \beta(\pi)$. It is not trivial that this equation is compatible with the definition of $\langle \hat{s}, \hat{\pi} \rangle_{\widehat{\phi}}$ above.

If $s = \widehat{s_{\widehat{\phi}}} \in \mathcal{A}_\phi$ corresponds to $\hat{s} = s_{\widehat{\phi}} \in \mathcal{A}_{\widehat{\phi}}$, we see that $r(\phi) = r(\phi_+) + r(\phi_-)$. Hence our definition shows that $\langle s_{\widehat{\phi}}, \hat{\pi} \rangle_{\widehat{\phi}} = \langle \widehat{s_{\widehat{\phi}}}, \pi \rangle_\phi$. Using this together with $s_\phi = 1$, the equation $\beta(\phi)\langle s_{\widehat{\phi}}, \hat{\pi} \rangle_{\widehat{\phi}} = \langle s_\phi, \pi \rangle_\phi \beta(\pi)$ can be rewritten as

$$\beta(\phi)\beta(\pi) = \langle \widehat{s_{\widehat{\phi}}}, \pi \rangle_\phi.$$

In this section, we only assume Hypothesis 4.4.2 for tempered L-parameters. To clarify our situation, we state this hypothesis explicitly.

Hypothesis 5.1.1. For any quasi-split classical group G' with $\mathrm{rank}(G') \leq \mathrm{rank}(G)$, and for any tempered L-parameter ϕ' for G', there exists a subset $\Pi_{\phi'}$ of $\mathrm{Irr}_{\mathrm{temp}}(G')$ equipped with $\langle \cdot, \pi' \rangle_{\phi'}$ satisfying (**ECR1**) and (**ECR2**) of Section 1.6.

This hypothesis is the same as Hypothesis C.0.1 and hence, we can use the results in Appendix C. We also notice that the proof of Proposition 4.1.3 did not use Hypothesis 4.4.2. Therefore we know that $\beta(\phi) = (-1)^{r(\phi)}$ even now.

Now we can state what we have to show in this section.

Proposition 5.1.2. *Assume Hypothesis 5.1.1 (but not 4.4.2). Let ϕ be a tempered L-parameter for G. Then for $\pi \in \Pi_\phi$, we have*

$$(*) \qquad \beta(\phi)\beta(\pi) = \langle \widehat{s_{\widehat{\phi}}}, \pi \rangle_\phi.$$

Remark 5.1.3. This statement was recently proven by Liu–Lo–Shahidi [LLS]. However, it is not obvious to us whether they use any results that are unavailable in the setting we need, i.e. at the point of Arthur's argument in [Ar2, Section 7.1]. For safety, we will give a proof of Proposition 5.1.2.

Let us explain our strategy for the proof of Proposition 5.1.2. One can check that both $\beta(\phi)$ and $\beta(\pi)$ depend only on the cuspidal support of π. However, it is complicated to list the cuspidal support of π for all $\pi \in \Pi_\phi$. Instead of this, we give several reductions by using Theorem C.3.3. The final case is where π is supercuspidal, which we can treat by a direct computation.

Let $P = MN_P$ be a standard parabolic subgroup of G with Levi subgroup $M \cong \mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_t}(E) \times G_0$. For $\tau_i \in \mathrm{Rep}(\mathrm{GL}_{k_i}(E))$ and $\pi_0 \in \mathrm{Rep}(G_0)$, we denote the normalized parabolic induction by

$$\tau_1 \times \cdots \times \tau_t \rtimes \pi_0 = \mathrm{Ind}_P^G(\tau_1 \boxtimes \cdots \boxtimes \tau_t \boxtimes \pi_0).$$

5.2. **Reductions.** As the first reduction, we assume that we can decompose ϕ as

$$\phi = \phi_1 \oplus \phi_0 \oplus {}^c\phi_1^\vee.$$

Then we have a canonical inclusion $\mathcal{A}_{\phi_0} \hookrightarrow \mathcal{A}_\phi$, which satisfies $\widehat{s_{\widehat{\phi}_0}} \mapsto \widehat{s_{\widehat{\phi}}}$. If we denote by τ the irreducible tempered representation of $\mathrm{GL}_m(E)$ corresponding to ϕ_1 with $m = \dim(\phi_1)$, then we have

- $\tau \rtimes \pi_0$ is semisimple and multiplicity-free for $\pi_0 \in \Pi_{\phi_0}$;
- $\pi \in \Pi_\phi$ if and only if $\pi \hookrightarrow \tau \rtimes \pi_0$ for some $\pi_0 \in \Pi_{\phi_0}$;
- if $\pi \hookrightarrow \tau \rtimes \pi_0$, then

$$\langle \cdot, \pi \rangle_\phi |_{\mathcal{A}_{\phi_0}} = \langle \cdot, \pi_0 \rangle_{\phi_0}.$$

For these statements, see the proofs of [Ar2, Proposition 2.4.3] and [Mok, Proposition 3.4.4]. For such $\pi \hookrightarrow \tau \rtimes \pi_0$, by definition of $\beta(\phi)$ and $\beta(\pi)$, we have

$$\frac{\beta(\phi)}{\beta(\phi_0)} = (-1)^r = \frac{\beta(\pi)}{\beta(\pi_0)},$$

where r is the length of

$$W_E \ni w \mapsto \phi_1 \left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \right)$$

as a representation of W_E. Hence if we knew $(*)$ for π_0, we would have

$$\beta(\phi)\beta(\pi) = \beta(\phi_0)\beta(\pi_0) = \langle \widehat{s_{\widehat{\phi}_0}}, \pi_0 \rangle_{\phi_0} = \langle \widehat{s_{\widehat{\phi}}}, \pi \rangle_\phi,$$

which is $(*)$ for π. Therefore we may assume that:

(1) ϕ is a discrete parameter, i.e., ϕ is of good parity and multiplicity-free.

When ϕ is a discrete parameter for G, we write

$$\phi = \bigoplus_\rho \bigoplus_{i=1}^t \rho \boxtimes S_{2a_{\rho,i}+1},$$

where $a_{\rho,i} \in (1/2)\mathbb{Z}$ and $0 \le a_{\rho,1} < \cdots < a_{\rho,t}$ with $t = t_\rho \ge 0$. Then ρ is conjugate-self-dual, and the parity of $2a_{\rho,i} + 1$ depends only on the sign of ρ. Recall that $\mathcal{A}_\phi$ is a quotient of

$$A_\phi = \bigoplus_\rho \bigoplus_{i=1}^t \mathbb{Z}/2\mathbb{Z} e(\rho, 2a_{\rho,i} + 1).$$

Here, as we abbreviated $\rho \boxtimes S_d \boxtimes S_1$ to $\rho \boxtimes S_d$, we write $e(\rho, d) = e(\rho, d, 1)$ for simplicity. Moreover, $\widehat{s_{\widehat{\phi}}} \in \mathcal{A}_\phi$ is the image of

$$\sum_{\substack{\rho \\ a_{\rho,1} \notin \mathbb{Z}}} \sum_{i=1}^t e(\rho, 2a_{\rho,i} + 1).$$

Let $\pi \in \Pi_\phi$. As the second reduction, we assume that one can find ρ and $1 < i \leq t$ such that

$$\langle e(\rho, 2a_{\rho,i}+1), \pi\rangle_\phi = \langle e(\rho, 2a_{\rho,i-1}+1), \pi\rangle_\phi.$$

In this case, by applying Theorem C.3.3 repeatedly together with [Ar2, Proposition 2.4.3] and [Mok, Proposition 3.4.4], we have

$$\pi \hookrightarrow \rho|\cdot|^{a_{\rho,i}} \times \rho|\cdot|^{a_{\rho,i}-1} \times \cdots \times \rho|\cdot|^{-a_{\rho,i-1}} \rtimes \pi_0,$$

where $\pi_0 \in \Pi_{\phi_0}$ with

$$\phi_0 = \phi - \rho \boxtimes (S_{2a_{\rho,i}+1} \oplus S_{2a_{\rho,i-1}+1}),$$

and

$$\langle \cdot, \pi\rangle_\phi|_{\mathcal{A}_{\phi_0}} = \langle \cdot, \pi_0\rangle_{\phi_0}.$$

Hence

$$\frac{\beta(\phi)}{\beta(\phi_0)} = (-1)^{a_{\rho,i}+a_{\rho,i-1}+1} = \frac{\beta(\pi)}{\beta(\pi_0)}.$$

Note that via the canonical inclusion $\mathcal{A}_{\phi_0} \hookrightarrow \mathcal{A}_\phi$,

$$\widehat{s_{\widehat{\phi}_0}} \mapsto \begin{cases} \widehat{s_{\widehat{\phi}}} & \text{if } a_{\rho,i} \in \mathbb{Z}, \\ \widehat{s_{\widehat{\phi}}} - (e(\rho, 2a_{\rho,i}+1) + e(\rho, 2a_{\rho,i-1}+1)) & \text{if } a_{\rho,i} \notin \mathbb{Z}. \end{cases}$$

Hence if we knew $(*)$ for π_0, using $\langle e(\rho, 2a_{\rho,i}+1), \pi\rangle_\phi = \langle e(\rho, 2a_{\rho,i-1}+1), \pi\rangle_\phi$, we would have

$$\beta(\phi)\beta(\pi) = \beta(\phi_0)\beta(\pi_0) = \langle \widehat{s_{\widehat{\phi}_0}}, \pi_0\rangle_{\phi_0} = \langle \widehat{s_{\widehat{\phi}}}, \pi\rangle_\phi,$$

which is $(*)$ for π. Therefore, we may assume that:

(2) $\langle e(\rho, 2a_{\rho,i}+1), \pi\rangle_\phi \neq \langle e(\rho, 2a_{\rho,i-1}+1), \pi\rangle_\phi$ for any ρ and $1 < i \leq t$.

As the third reduction, we assume that one can find ρ with $t \geq 1$ and $a_{\rho,1} \notin \mathbb{Z}$ such that

$$\langle e(\rho, 2a_{\rho,1}+1), \pi\rangle_\phi = 1.$$

In this case, by applying Theorem C.3.3 repeatedly, we have

$$\pi \hookrightarrow \rho|\cdot|^{a_{\rho,1}} \times \rho|\cdot|^{a_{\rho,1}-1} \times \cdots \times \rho|\cdot|^{\frac{1}{2}} \rtimes \pi_0,$$

where $\pi_0 \in \Pi_{\phi_0}$ with

$$\phi_0 = \phi - \rho \boxtimes S_{2a_{\rho,1}+1},$$

and

$$\langle \cdot, \pi\rangle_\phi|_{\mathcal{A}_{\phi_0}} = \langle \cdot, \pi_0\rangle_{\phi_0}.$$

Hence

$$\frac{\beta(\phi)}{\beta(\phi_0)} = (-1)^{a_{\rho,1}+\frac{1}{2}} = \frac{\beta(\pi)}{\beta(\pi_0)}.$$

Note that via the canonical inclusion $\mathcal{A}_{\phi_0} \hookrightarrow \mathcal{A}_\phi$,

$$\widehat{s_{\widehat{\phi}_0}} \mapsto \widehat{s_{\widehat{\phi}}} - e(\rho, 2a_{\rho,1}+1).$$

Hence the equation $(*)$ for π_0 implies $(*)$ for π. Therefore we may assume that:

(3) $\langle e(\rho, 2a_{\rho,1}+1), \pi\rangle_\phi = -1$ if $a_{\rho,1} \notin \mathbb{Z}$.

Note that an irreducible tempered representation π satisfying (1), (2) and (3) is called *strongly positive discrete series* in Mœglin–Tadić's terminology.

Let $\pi \in \Pi_\phi$ be a strongly positive discrete series representation, i.e., it satisfies the conditions (1)–(3). Write again

$$\phi = \bigoplus_\rho \bigoplus_{i=1}^{t} \rho \boxtimes S_{2a_{\rho,i}+1},$$

where $a_{\rho,i} \in (1/2)\mathbb{Z}$ and $0 \leq a_{\rho,1} < \cdots < a_{\rho,t}$ with $t = t_\rho \geq 0$.

As the fourth reduction, we assume that one can find ρ with $t \geq 1$ such that $a_{\rho,i} - a_{\rho,i-1} > 1$ for some $1 \leq i \leq t$. Here, formally we set $a_{\rho,0} = -\frac{1}{2}$. In this case, by applying Theorem C.3.3, we have

$$\pi \hookrightarrow \rho|\cdot|^{a_{\rho,i}} \rtimes \pi_0,$$

where $\pi_0 \in \Pi_{\phi_0}$ with

$$\phi_0 = \phi - \rho \boxtimes S_{2a_{\rho,i}+1} \oplus \rho \boxtimes S_{2a_{\rho,i}-1},$$

and $\langle \cdot, \pi\rangle_\phi = \langle \cdot, \pi_0\rangle_{\phi_0}$ via the canonical identification $\mathcal{A}_\phi \cong \mathcal{A}_{\phi_0}$ (see Proposition C.3.1). Hence

$$\frac{\beta(\phi)}{\beta(\phi_0)} = -1 = \frac{\beta(\pi)}{\beta(\pi_0)}.$$

Note that $s_{\widehat{\phi_0}} = s_{\widehat{\phi}}$ via $\mathcal{A}_{\phi_0} \cong \mathcal{A}_\phi$. Hence if we knew $(*)$ for π_0, we would have

$$\beta(\phi)\beta(\pi) = \beta(\phi_0)\beta(\pi_0) = \langle s_{\widehat{\phi_0}}, \pi_0\rangle_{\phi_0} = \langle s_{\widehat{\phi}}, \pi\rangle_\phi,$$

which is $(*)$ for π.

Therefore, we may finally assume that

(1) ϕ is a discrete parameter;
(2) $\langle e(\rho, 2a_{\rho,i}+1), \pi\rangle_\phi \neq \langle e(\rho, 2a_{\rho,i-1}+1), \pi\rangle_\phi$ for any ρ and $1 < i \leq t$;
(3) $\langle e(\rho, 2a_{\rho,1}+1), \pi\rangle_\phi = -1$ if $a_{\rho,1} \notin \mathbb{Z}$;
(4) $a_{\rho,1} = 0$ or $a_{\rho,1} = \frac{1}{2}$;
(5) $a_{\rho,i} = a_{\rho,i-1} + 1$ for $i > 1$.

Such a representation π is *supercuspidal* by Corollary C.3.5.

5.3. **The case of supercuspidal representations.** Now we assume that $\pi \in \Pi_\phi$ is supercuspidal, i.e., it satisfies the conditions (1)–(5) in the last subsection. We check $(*)$ for π directly.

Write

$$\phi = \bigoplus_\rho \bigoplus_{i=1}^{t} \rho \boxtimes S_{2a_{\rho,i}+1},$$

where $a_{\rho,i} \in (1/2)\mathbb{Z}$ and $0 \leq a_{\rho,1} < \cdots < a_{\rho,t}$ with $t = t_\rho \geq 0$. Then

$$\phi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right) \cong \bigoplus_\rho \bigoplus_{i=1}^{t} \rho \otimes (|\cdot|^{a_{\rho,i}} \oplus |\cdot|^{a_{\rho,i}-1} \oplus \cdots \oplus |\cdot|^{-a_{\rho,i}}).$$

Note that if $a_{\rho,1} \in \mathbb{Z}$, then ρ appears with multiplicity $t = t_\rho$. Hence,

$$r(\phi) = \sum_{\substack{\rho \\ a_{\rho,1} \notin \mathbb{Z}}} \sum_{i=1}^{t} \left(a_{\rho,i} + \frac{1}{2} \right) + \sum_{\substack{\rho \\ a_{\rho,1} \in \mathbb{Z}}} \left(\left[\frac{t}{2} \right] + \sum_{i=1}^{t} a_{\rho,i} \right),$$

where $[x]$ denotes the largest integer not greater than x. By the conditions (4) and (5) in the last subsection, we note that when $a_{\rho,1} \in \mathbb{Z}$, we have $a_{\rho,i} = i - 1$ so that

$$\left[\frac{t}{2} \right] + \sum_{i=1}^{t} a_{\rho,i} = \left[\frac{t}{2} \right] + \sum_{i=1}^{t} (i-1) = \left[\frac{t}{2} \right] + \frac{t(t-1)}{2} \equiv 0 \bmod 2.$$

Similarly, when $a_{\rho,1} \notin \mathbb{Z}$, we have $a_{\rho,i} + \frac{1}{2} = i$ so that

$$\sum_{i=1}^{t} \left(a_{\rho,i} + \frac{1}{2} \right) = \sum_{i=1}^{t} i = \frac{t(t+1)}{2}.$$

Hence

$$\beta(\phi) = (-1)^{r(\phi)} = \prod_{\substack{\rho \\ a_{\rho,1} \notin \mathbb{Z}}} (-1)^{\frac{t(t+1)}{2}}.$$

On the other hand, since π is supercuspidal, we have $\beta(\pi) = 1$. Finally, by the conditions (2) and (3), we have

$$\langle \widehat{s}_{\widehat{\phi}}, \pi \rangle_\phi = \prod_{\substack{\rho \\ a_{\rho,1} \notin \mathbb{Z}}} \prod_{i=1}^{t} \langle e(\rho, 2a_{\rho,i} + 1), \pi \rangle_\phi = \prod_{\substack{\rho \\ a_{\rho,1} \notin \mathbb{Z}}} \prod_{i=1}^{t} (-1)^i = \prod_{\substack{\rho \\ a_{\rho,1} \notin \mathbb{Z}}} (-1)^{\frac{t(t+1)}{2}}.$$

Therefore, we conclude that

$$\beta(\phi)\beta(\pi) = \langle \widehat{s}_{\widehat{\phi}}, \pi \rangle_\phi,$$

as desired. We obtain $(*)$ for π, and hence we complete the proof of Proposition 5.1.2.

5.4. **ECR for co-tempered A-parameters.** Now we can prove Theorem 1.10.5 (1). More precisely, we have the following:

Theorem 5.4.1. *Assume Hypothesis 5.1.1 (but not 4.4.2). Let ϕ be a tempered L-parameter for G. Define $\Pi_{\widehat{\phi}}$ and $\langle \hat{s}, \hat{\pi} \rangle_{\widehat{\phi}}$ for $\hat{s} \in A_{\widehat{\phi}}$ by*

$$\Pi_{\widehat{\phi}} = \{ \hat{\pi} \mid \pi \in \Pi_\phi \}$$

and

$$\langle \hat{s}, \hat{\pi} \rangle_{\widehat{\phi}} = (-1)^{r(\phi) - r(\phi_+) - r(\phi_-)} \langle s, \pi \rangle_\phi.$$

Then $\langle \cdot, \hat{\pi} \rangle_{\widehat{\phi}}$ factors through $A_{\widehat{\phi}} \twoheadrightarrow \mathcal{A}_{\widehat{\phi}}$. Moreover, (**ECR1**) *and* (**ECR2**) *hold for $\widehat{\phi}$.*

Proof. Since $r(\phi) = r(\phi_+) + r(\phi_-)$ if $s = \widehat{s}_{\widehat{\phi}}$, we have $\langle s_{\widehat{\phi}}, \hat{\pi}\rangle_{\widehat{\phi}} = \langle \widehat{s}_{\widehat{\phi}}, \pi\rangle_\phi = \beta(\phi)\beta(\pi)$ by Proposition 5.1.2. Hence, when $\widetilde{f} \in C_c^\infty(\mathrm{GL}_N(E) \rtimes \theta)$ and $f_G \in C_c^\infty(G^\circ)$ have matching orbital integrals, by [X2, (A.1)], we have

$$\begin{aligned}
\sum_{\hat{\pi} \in \Pi_{\widehat{\phi}}} \langle s_{\widehat{\phi}}, \hat{\pi}\rangle_{\widehat{\phi}} \Theta_{\hat{\pi}}(f_G) &= \beta(\phi) \sum_{\hat{\pi} \in \Pi_{\widehat{\phi}}} \beta(\pi)\Theta_{\hat{\pi}}(f_G) \\
&= \beta(\phi) \sum_{\pi \in \Pi_\phi} \Theta_{D_{G^\circ}(\pi)}(f_G) \\
&= (G : G^\circ)\beta(\phi)\Theta_{D_{\widetilde{\mathrm{GL}}_N(E)}(\widetilde{\pi}_\phi)}(\widetilde{f}) \\
&= (G : G^\circ)\Theta_{\widetilde{\pi}_{\widehat{\phi}}}(\widetilde{f}).
\end{aligned}$$

This shows (**ECR1**).

Similarly, since $\beta(\phi) = (-1)^{r(\phi)}$ and $\beta(\phi_\pm) = \beta(\phi_\pm \otimes \eta_\pm)$ by Proposition 4.1.3 and Lemma 4.1.2, when $f_G \in C_c^\infty(G)$ and $f_{G_+} \otimes f_{G_-} \in C_c^\infty(G_+^\circ \times G_-^\circ)$ have matching orbital integrals, by [Hi, Theorem 1.5] or [X2, (A.1)], we have

$$\begin{aligned}
&\frac{1}{(G : G^\circ)} \sum_{\hat{\pi} \in \Pi_{\widehat{\phi}}} \langle \hat{s} \cdot s_{\widehat{\phi}}, \hat{\pi}\rangle_{\widehat{\phi}} \Theta_{\hat{\pi}}(f_G) \\
&= \frac{1}{(G : G^\circ)} \sum_{\hat{\pi} \in \Pi_{\widehat{\phi}}} (-1)^{r(\phi) - r(\phi_+) - r(\phi_-)} \langle s, \pi\rangle_\phi \cdot \beta(\phi)\beta(\pi)\Theta_{\hat{\pi}}(f_G) \\
&= \frac{1}{(G : G^\circ)} \beta(\phi_+)\beta(\phi_-) \sum_{\hat{\pi} \in \Pi_{\widehat{\phi}}} \langle s, \pi\rangle_\phi \Theta_{D_{G^\bullet}(\pi)}(f_G) \\
&= \prod_{\kappa \in \{\pm\}} \frac{1}{(G_\kappa : G_\kappa^\circ)} \beta(\phi_\kappa \otimes \eta_\kappa) \sum_{\pi_\kappa \in \Pi_{\phi_\kappa \otimes \eta_\kappa}} \Theta_{D_{G_\kappa^\circ}(\pi_\kappa)}(f_{G_\kappa}) \\
&= \prod_{\kappa \in \{\pm\}} \frac{1}{(G_\kappa : G_\kappa^\circ)} \sum_{\pi_\kappa \in \Pi_{\phi_\kappa \otimes \eta_\kappa}} \beta(\phi_\kappa \otimes \eta_\kappa)\beta(\pi_\kappa)\Theta_{\hat{\pi}_\kappa}(f_{G_\kappa}) \\
&= \prod_{\kappa \in \{\pm\}} \frac{1}{(G_\kappa : G_\kappa^\circ)} \sum_{\hat{\pi}_\kappa \in \Pi_{\widehat{\phi}_\kappa \otimes \eta_\kappa}} \langle s_{\widehat{\phi}_\kappa \otimes \eta_\kappa}, \hat{\pi}_\kappa\rangle_{s_{\widehat{\phi}_\kappa} \otimes \eta_\kappa} \Theta_{\hat{\pi}_\kappa}(f_{G_\kappa}).
\end{aligned}$$

Here, the definition of $D_{G^\bullet}$ and the assumption on f_G are the same as in the proof of Lemma 4.4.4 (2). This shows (**ECR2**). □

6. Local intertwining relations for classical groups

The purpose of this and next sections is to prove Theorem 1.10.5 (2). This is a key result that is essential in the global method for establishing Arthur's theory of endoscopic classification. Arthur's initial approach was to prove some special cases of Theorem 1.10.5 (2), which would suffice for the global method, by an argument based on

Hecke algebras. However, our attempts to realize this approach led to very complicated computations.

To show Theorem 1.10.5 (2), we will instead adapt the method for Theorem 1.9.1 to the case of classical groups. However, unlike the $\mathrm{GL}_N(E)$ case, the unitary parabolic inductions of classical groups are not necessarily irreducible. Because of this fact, we can apply our method only to "half" of the cases. The final key ingredient is Corollary B.3.3, which was obtained in an arXiv version of [KMSW]. This result tells us that "half" of the cases is enough.

6.1. **Hypothesis.** Let F be a non-archimedean local field of characteristic zero. Fix a non-trivial character $\psi_F\colon F \to \mathbb{C}^\times$. Recall that G is one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{O}_{2n}(F), \quad \mathrm{U}_n.$$

In this section, we assume Hypothesis 1.10.4, which we restate here for the reader's convenience.

Hypothesis 6.1.1. We assume (**ECR1**), (**ECR2**) and (**A-LIR**) for

- all tempered L-parameters ϕ for G;
- all A-parameters ψ for G' with G' any classical group such that $\mathrm{rank}(G') < \mathrm{rank}(G)$.

In particular, we have the A-packet Π_{ψ_M} for $\psi_M \in \Psi(M)$ for any proper Levi subgroup M of G, since M is a product of such a G' and general linear groups.

Remark 6.1.2. (1) By Theorem 5.4.1, one has an A-packet $\Pi_{\widehat{\phi}}$ associated to co-tempered A-parameters $\widehat{\phi}$ for G, satisfying (**ECR1**) and (**ECR2**).
(2) For any proper Levi subgroup M of G, Hypothesis 6.1.1 is stronger than Hypotheses 4.4.2 and B.3.1. Hence we can use Corollaries 4.5.3 and B.3.3 for $\psi_M \in \Psi(M)$.
(3) As explained in Remark 1.10.3, we may identify Arthur's LIR (**A-LIR**) with our (**LIR**) stated in Section 1.10, except for Section 7.6.
(4) Note that we assume (**ECR1**) and (**ECR2**) not only for proper Levi subgroups M of G, but also for any classical group G' with $\mathrm{rank}(G') < \mathrm{rank}(G)$. Hence Hypothesis 6.1.1 contains Hypothesis C.0.1 so that we can use Theorems C.3.3, C.3.4 and Corollary C.3.5.
(5) On the other hand, Mœglin's explicit construction of A-packets uses endoscopic character relations for higher rank cases. Hence one should not use several results in [X2] even for proper Levi subgroups of G. In particular, we cannot use the fact that Π_{ψ_M} is multiplicity-free for $\psi_M \in \Psi(M)$ in general.

6.2. **Reduction to the maximal parabolic case.** Let $P = MN_P$ be a standard parabolic subgroup of G. Write $M = \mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_t}(E) \times G_0$. Let $\psi_M = \widehat{\phi}_M = \psi_t \oplus \cdots \oplus \psi_1 \oplus \psi_0$ be a co-tempered A-parameter for M such that ψ_i is an A-parameter for $\mathrm{GL}_{k_i}(E)$ for $1 \leq i \leq t$, and $\psi_0 \in \Psi(G_0)$. Suppose that ψ_i is irreducible and conjugate-self-dual for any $1 \leq i \leq t$.

In this subsection, we reduce Theorem 1.10.5 (2) to the case where $t = 1$. Namely, we prove the following.

Lemma 6.2.1. *Assume Hypothesis 6.1.1. We further assume that* (**LIR**) *holds for any irreducible components $\pi \subset I_{P'}(\pi_{M'})$, where*

- *$P' = M'N_{P'}$ is a maximal parabolic subgroup of G so that $M' \cong \mathrm{GL}_k(E) \times G'_0$;*
- *$\psi_{M'} = \widehat{\phi}_{M'} = \psi_{\mathrm{GL}} \oplus \psi'_0$ is a co-tempered A-parameter for M' with ψ_{GL} irreducible and conjugate-self-dual;*
- *$\pi_{M'} \in \Pi_{\psi_{M'}}$.*

Then (**LIR**) *holds for any irreducible components $\pi \subset I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$.*

Proof. By Proposition 1.7.2 and [KMSW, Lemma 2.5.3], the map

$$\mathfrak{N}_\psi \ni u \mapsto \langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)$$

is multiplicative. (See also the paragraph in [Ar2] containing (2.4.2).) Hence we may assume that $u = u_i$ for $1 \leq i \leq t$ or $u = \sigma \in \mathfrak{S}_t$ such that $\psi_{\sigma(i)} \cong \psi_i$ for $1 \leq i \leq t$ since

$$\mathfrak{N}_\psi = \{u_1^{\epsilon_1} \dots u_t^{\epsilon_t} \,|\, \epsilon_i \in \mathbb{Z}/2\mathbb{Z}\} \rtimes \{\sigma \in \mathfrak{S}_t \,|\, \psi_{\sigma(i)} \cong \psi_i \, (1 \leq i \leq t)\}.$$

See Section 1.10.

First, we assume that $u = \sigma \in \mathfrak{S}_t$ such that $\psi_{\sigma(i)} \cong \psi_i$ for $1 \leq i \leq t$. Let $P' = M'N_{P'}$ be a maximal parabolic subgroup of G such that $M' \cong \mathrm{GL}_{k_1+\dots+k_t}(E) \times G_0$, and let $\psi_{M'} = (\psi_t \oplus \dots \oplus \psi_1) \oplus \psi_0$ be the A-parameter for M' given by ψ_M. Then

$$\Pi_{\psi_{M'}} = \left\{ \mathrm{Ind}_{P \cap M'}^{M'}(\pi_M) \,\middle|\, \pi_M \in \Pi_{\psi_M} \right\}.$$

By [KMSW, Lemma 2.7.2], $\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)$ descends to the normalized intertwining operator for $\mathrm{GL}_{k_1+\dots+k_t}(E)$. (See also the proof of Lemma 2.4.2 in [Ar2].) Hence by Theorem 3.5.1, this operator must be the identity map. Since $s_u = 1$ in this case, we obtain (**LIR**).

Next, we assume that $u = u_i$ for $1 \leq i \leq t$. We will prove the assertion by induction on t. We can assume that $t > 1$. Let $P' = M'N_{P'}$ be a maximal parabolic subgroup of G such that $M' \cong \mathrm{GL}_{k_t}(E) \times G'_0$, and let

$$\psi_{M'} = \psi_t \oplus (\psi_{t-1} \oplus \dots \oplus \psi_1 \oplus \psi_0 \oplus {}^c\psi_1^\vee \oplus \dots \oplus {}^c\psi_{t-1}^\vee)$$

be the A-parameter for M' given by ψ_M. Then for any $\pi_M \in \Pi_{\psi_M}$ and any irreducible component $\pi \subset I_P(\pi_M)$, there is a unique $\pi_{M'} \in \Pi_{\psi_{M'}}$ such that $\pi_{M'} \subset \mathrm{Ind}_{P \cap M'}^{M'}(\pi_M)$ and $\pi \subset I_{P'}(\pi_{M'})$. Moreover, there is a canonical injection $A_{\psi_{M'}} \hookrightarrow A_\psi$ and we have

$$\langle \cdot, \pi \rangle_\psi |_{A_{\psi_{M'}}} = \langle \cdot, \pi_{M'} \rangle_{\psi_{M'}}.$$

These facts follow from the tempered case by taking Aubert duality and using Corollary 4.4.5.

Suppose that $u = u_i$ with $i \neq t$ so that $s_{u_i} \in A_{\psi_{M'}}$. Then by [KMSW, Lemma 2.7.2], $\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)$ descends to the normalized intertwining operator for G'_0, i.e.,

$$\langle \widetilde{u}_i, \widetilde{\pi}_M \rangle R_P(w_{u_i}, \widetilde{\pi}_M, \psi_M) = \mathrm{Ind}_{P'}^G \left(\mathrm{id}_{\pi_{\psi_t}} \otimes \langle \widetilde{u}_i, \widetilde{\pi}_{M'_0} \rangle R_{P \cap G'_0}(w_{u_i}, \widetilde{\pi}_{M'_0}, \psi_{M'_0}) \right),$$

where $M'_0 = M \cap G'_0$, $\psi_{M'_0} = \psi_M - \psi_t \in \Psi(M'_0)$ and we write $\pi_M = \pi_{\psi_t} \boxtimes \pi_{M'_0}$ with $\pi_{M'_0} \in \Pi_{\psi_{M'_0}}$. By the induction hypothesis, (**LIR**) is known for G'_0 in place of G, so this operator acts on $I_{P'}(\pi_{M'})$ by the scalar $\langle s_{u_i}, \pi_{M'} \rangle_{\psi_{M'}} = \langle s_{u_i}, \pi \rangle_\psi$. Hence we obtain (**LIR**) for this case.

Suppose finally that $u = u_t$. Then by [KMSW, Lemma 2.7.2],

$$\langle \widetilde{u}_t, \widetilde{\pi}_M \rangle R_P(w_{u_t}, \widetilde{\pi}_M, \psi_M)|_{I_{P'}(\pi_{M'})} = \langle \widetilde{u}_t, \widetilde{\pi}_{M'} \rangle R_{P'}(w_{u_t}, \widetilde{\pi}_{M'}, \psi_{M'}).$$

Since we are assuming (**LIR**) for the maximal parabolic case, we know that

$$\langle \widetilde{u}_t, \widetilde{\pi}_{M'} \rangle R_{P'}(w_{u_t}, \widetilde{\pi}_{M'}, \psi_{M'})|_\pi = \langle s_{u_t}, \pi \rangle_\psi.$$

Hence we obtain (**LIR**) for this case. This completes the proof. □

6.3. **Halving the problem.** In the rest of this section, we will focus on the maximal parabolic case.

Fix a standard maximal parabolic subgroup $P = MN_P$ of G with $M \cong \mathrm{GL}_k(E) \times G_0$. Let $\psi_M = \psi_{\mathrm{GL}} \oplus \psi_0$ be an A-parameter for M, and $\psi = \psi_{\mathrm{GL}} \oplus \psi_0 \oplus {}^c\psi_{\mathrm{GL}}^\vee$ be the A-parameter for G given by ψ_M. Suppose that ψ_{GL} is irreducible and conjugate-self-dual. The A-packet Π_ψ is given by the (multi-)set of irreducible components of $I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$. Let $u = u_1 \in \mathfrak{N}_\psi$. Recall that the normalized self-intertwining operator

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) \colon I_P(\pi_M) \to I_P(\pi_M)$$

is defined by

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) f(g) = \langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u) \left(R_P(w_u, \pi_M, \psi_M) f(g) \right)$$

with a linear isomorphism

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u) \colon \pi_M \xrightarrow{\sim} \pi_M$$

satisfying that the diagram

$$\begin{CD} \pi_M @>{\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)}>> \pi_M \\ @V{\pi_M(\widetilde{w_u}^{-1} m \widetilde{w_u})}VV @VV{\pi_M(m)}V \\ \pi_M @>{\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)}>> \pi_M \end{CD}$$

is commutative for any $m \in M$. The definition of this isomorphism will be recalled in the proof of the next lemma.

We can write $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0$, where π_{GL} is an irreducible conjugate-self-dual representation of $\mathrm{GL}_k(E)$. Similar to the definition of θ_A in Section 1.4, by using a Whittaker functional on the standard module of π_{GL}, one can normalize a linear isomorphism $\mathcal{A}_{w_u} \colon \pi_{\mathrm{GL}} \xrightarrow{\sim} \pi_{\mathrm{GL}}$ such that the diagram

$$\begin{CD} \pi_M @>{\mathcal{A}_{w_u} \otimes \mathrm{id}_{\pi_0}}>> \pi_M \\ @V{\pi_M(\widetilde{w_u}^{-1} m \widetilde{w_u})}VV @VV{\pi_M(m)}V \\ \pi_M @>{\mathcal{A}_{w_u} \otimes \mathrm{id}_{\pi_0}}>> \pi_M \end{CD}$$

is commutative for any $m \in M$.

Lemma 6.3.1. *We have* $\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u) = \mathcal{A}_{w_u} \otimes \mathrm{id}_{\pi_0}$.

Proof. Note that $u = u_1 \in \mathfrak{N}_\psi$ normalizes $\widehat{M^\circ}$. If we regard u as an element of $W(\widehat{M^\circ})$, we have its Tits lifting $\widetilde{u}$. Conjugation by $\widetilde{u}$ normalizes $\widehat{M^\circ}$ and preserves the pinning inherited from $\widehat{G^\circ}$. Write $\widehat{\theta}$ for the resulting automorphism on $\widehat{M^\circ}$, and θ for its dual, an automorphism of M°. Write $u = s\widetilde{u}$. Then s lies in the θ-twisted centralizer of ψ_M. The pair (s, ψ_M) determines a twisted endoscopic datum (M'°, s, ξ) and a parameter $\psi_{M'}$ such that $\psi_M = \xi \circ \psi_{M'}$. The isomorphism $\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)$ is normalized by requiring the identity

$$\sum_{\pi_{M'} \in \Pi_{\psi_{M'}}} \langle s_{\psi_{M'}}, \pi_{M'} \rangle_{\psi_{M'}} \Theta_{\pi_{M'}}(f_{M'}) = \sum_{\pi_M \in \Pi_{\psi_M}} \langle s_{\psi_M}, \pi_M \rangle_{\psi_M} \mathrm{tr}(\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u) \circ \pi_M(f_M))$$

holds whenever $f_M \in C_c^\infty(M)$ and $f_{M'} \in C_c^\infty(M')$ have matching orbital integrals. We will check that this equation also holds after replacing $\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u)$ with $\mathcal{A}_{w_u} \otimes \mathrm{id}_{\pi_0}$.

Since the conjugation action of $\widetilde{u}$ on $\widehat{M^\circ}$ preserves the standard pinning, and it acts on $\mathrm{GL}_k(\mathbb{C})$ by the pinned outer automorphism and on $\widehat{G_0^\circ}$ by the identity, we see that $s \in Z(\widehat{M^\circ})$ and $M'^\circ = G_1 \times G_0^\circ$ with G_1 a twisted endoscopic datum for $\mathrm{GL}_k(E)$.

We may assume that $f_M = f_{\mathrm{GL}} \otimes f_0$ and $f_{M'} = f_1 \otimes f_0$. Then

$$\begin{aligned} &\sum_{\pi_M \in \Pi_{\psi_M}} \langle s_{\psi_M}, \pi_M \rangle_{\psi_M} \mathrm{tr}(\mathcal{A}_{w_u} \otimes \mathrm{id}_{\pi_0} \circ \pi_M(f_M)) \\ &= \mathrm{tr}(\mathcal{A}_{w_u} \circ \pi_{\mathrm{GL}}(f_{\mathrm{GL}})) \sum_{\pi_0 \in \Pi_{\psi_0}} \langle s_{\psi_0}, \pi_0 \rangle_{\psi_0} \Theta_{\pi_0}(f_0), \end{aligned}$$

whereas

$$\begin{aligned} &\sum_{\pi_{M'} \in \Pi_{\psi_{M'}}} \langle s_{\psi_{M'}}, \pi_{M'} \rangle_{\psi_{M'}} \Theta_{\pi_{M'}}(f_{M'}) \\ &= \left(\sum_{\pi_1 \in \Pi_{\psi_1}} \langle s_{\psi_1}, \pi_1 \rangle_{\psi_1} \Theta_{\pi_1}(f_1) \right) \left(\sum_{\pi_0 \in \Pi_{\psi_0}} \langle s_{\psi_0}, \pi_0 \rangle_{\psi_0} \Theta_{\pi_0}(f_0) \right), \end{aligned}$$

where we write $\psi_{M'} = \psi_1 \oplus \psi_0 \in \Psi(G_1 \times G_0)$. Since $\psi_{\mathrm{GL}} = \xi \circ \psi_1$, by (**ECR1**) for ψ_1, we have

$$\mathrm{tr}(\mathcal{A}_{w_u} \circ \pi_{\mathrm{GL}}(f_{\mathrm{GL}})) = \sum_{\pi_1 \in \Pi_{\psi_1}} \langle s_{\psi_1}, \pi_1 \rangle_{\psi_1} \Theta_{\pi_1}(f_1).$$

Therefore, we obtain the desired identity. □

Now by Lemma 6.2.1, we may assume that $\psi_M = \widehat{\phi}_M$ where $\phi_M = \phi_{\mathrm{GL}} \oplus \phi_0$ is a tempered L-parameter for M such that $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ is irreducible and conjugate-self-dual. Then Π_{ψ_M} is given by Aubert duality from Π_{ϕ_M}. In particular, for $\pi_M \in \Pi_{\psi_M}$, the parabolically induced representation $I_P(\pi_M) = \mathrm{Ind}_P^G(\pi_M)$ is a direct sum of at most two irreducible unitary representations. Indeed, by taking Aubert duality, this fact is reduced to the tempered case, which follows from [Ar2, Theorem 1.5.1], [Mok, Theorem 2.5.1] and $[\mathcal{A}_\phi : \mathcal{A}_{\phi_0}] \leq 2$.

Lemma 6.3.2. *Assume Hypothesis 6.1.1. Let $\psi_M = \widehat{\phi}_M$ be a co-tempered A-parameter, and $\pi_M \in \Pi_{\psi_M}$. Assume that $I_P(\pi_M)$ is reducible, hence $I_P(\pi_M) = \pi_1 \oplus \pi_2$. Write*

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)|_{\pi_i} = \varepsilon_i \cdot \mathrm{id}_{\pi_i}$$

for $\varepsilon_i \in \mathbb{C}^\times$. Then we have $\varepsilon_2 = -\varepsilon_1$.

Proof. Let $\overline{P}$ be the parabolic subgroup of G opposite to P. If we set $\sigma_M = \widehat{\pi}_M$ and $\sigma_i = \widehat{\pi}_i$, then $I_{\overline{P}}(\sigma_M) = \sigma_1 \oplus \sigma_2$ is a direct sum of irreducible tempered representations. Moreover, we have a normalized intertwining operator

$$\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M) \colon I_{\overline{P}}(\sigma_M) \to I_{\overline{P}}(\sigma_M).$$

If we write $\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M)|_{\sigma_i} = \varepsilon'_i \cdot \mathrm{id}_{\sigma_i}$, then by (**LIR**) for the tempered case, we know that $\varepsilon'_2 = -\varepsilon'_1$.

Recall that Aubert duality is a functor. We claim that the two intertwining operators

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) \quad \text{and} \quad \langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M)$$

are dual to each other up to a nonzero constant $c \in \mathbb{C}^\times$. This means that $\varepsilon_i = c\varepsilon'_i$ for $i = 1, 2$. Therefore, we have $\varepsilon_2 = c\varepsilon'_2 = -c\varepsilon'_1 = -\varepsilon_1$.

This claim is Corollary B.3.3 (1) in almost all cases. The exceptional case is where $G = \mathrm{O}_{2n}(F)$ and $I_P(\pi_M) = \mathrm{Ind}_{P^\circ}^{G}(\pi_M^\circ)$ for some irreducible representation π_M° of M°. Then we can write $I_P(\pi_M) = I^+_{P^\circ}(\pi_M^\circ) \oplus I^-_{P^\circ}(\pi_M^\circ)$, where $I^+_{P^\circ}(\pi_M^\circ)$ (resp. $I^-_{P^\circ}(\pi_M^\circ)$) is the subspace of $\mathrm{Ind}_{P^\circ}^{G}(\pi_M^\circ)$ consisting of functions f on G whose supports are contained in G° (resp. $G \setminus G^\circ$). Corollary B.3.3 (2) says that

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)|_{I^\pm_{P^\circ}(\pi_{M^\circ})} \quad \text{and} \quad \langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M)|_{(I^\pm_{P^\circ}(\pi_{M^\circ}))\widehat{}}$$

are dual to each other up to a nonzero constant $c_\pm \in \mathbb{C}^\times$.

We notice that the image of π_i under the canonical projection $I_P(\pi_M) \twoheadrightarrow I^\pm_{P^\circ}(\pi_M^\circ)$ is nonzero. For this, if $\pi_i \subset I^\mp_{P^\circ}(\pi_M^\circ)$, then for any $f \in \pi_i$, we have

$$I_P(\pi_M)(\epsilon)f \in \pi_i \cap I^\pm_{P^\circ}(\pi_M^\circ) = \{0\}$$

so that we would have $\pi_i = \{0\}$. This is a contradiction.

Now fix $f \in \pi_i$ with $f \neq 0$, and write $f = f_+ + f_-$ with $f_\pm \in I^\pm_{P^\circ}(\pi_M^\circ)$. Then

$$\begin{aligned}
&\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) f \\
&= c_+ (\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M))\widehat{}\, f_+ + c_- (\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M))\widehat{}\, f_- \\
&= c_+ (\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M))\widehat{}\, f + (c_- - c_+)(\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M))\widehat{}\, f_-.
\end{aligned}$$

Since Aubert duality is a functor, we have

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M) f, \quad c_+ (\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M))\widehat{}\, f \in \pi_i$$

so that

$$(c_- - c_+)(\langle \widetilde{u}, \widetilde{\sigma}_M \rangle R_{\overline{P}}(w_u, \widetilde{\sigma}_M, \phi_M))\widehat{\ } f_- \in \pi_i \cap I_{P^\circ}^{\delta}(\pi_{M^\circ}) = \{0\},$$

where $\delta = -\det(\widetilde{w}_u)$. Since $f_- \neq 0$, we obtain that $c_+ = c_-$, which shows the claim. □

Lemma 6.3.2 is a key step to prove Theorem 1.10.5 (2). It "halves" the problem, i.e., by this lemma, it is enough to prove (**LIR**) for only one direct summand $\pi \subset I_P(\pi_M)$ for each $\pi_M \in \Pi_{\psi_M}$.

6.4. **Highly non-tempered summands.** In the previous subsection, we showed that it is enough to prove (**LIR**) for one irreducible summand of $I_P(\pi_M)$. Here, we introduce a notion that will isolate a suitable summand.

Let $P = MN_P$ be a standard maximal parabolic subgroup of G with $M \cong \mathrm{GL}_k(F) \times G_0$, and let $\psi_M = \psi_{\mathrm{GL}} \oplus \psi_0$ be an A-parameter for M. In this and next subsections, we only assume that ψ_{GL} is irreducible and conjugate-self-dual. Namely, ψ_M is not necessarily co-tempered here. Note that then $I_P(\pi_M)$ could have more than two irreducible summands for $\pi_M \in \Pi_{\psi_M}$.

Recall from Section 3.1 that for a multi-segment $\mathfrak{m}$, we denote by $\mathcal{I}(\mathfrak{m})$ the standard module associated to $\mathfrak{m}$. For a segment $\mathfrak{s} = [x, y]_\rho$ with ρ unitary supercuspidal, we call the value $\frac{x+y}{2}$ the *midpoint* of $\mathfrak{s}$. The Langlands classification for G_0 says that for $\pi_0 \in \mathrm{Irr}(G_0)$, one has a multi-segment $\mathfrak{m}_0$ and an irreducible tempered representation τ_0 such that every segment $\mathfrak{s} \in \mathfrak{m}_0$ has a positive midpoint, and $\mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0$ is the standard module of π_0. Thus, π_0 is the unique irreducible quotient of $\mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0$. When $G_0 = \mathrm{O}_{2n_0}(F)$, it follows from the Langlands classification for $\mathrm{SO}_{2n_0}(F)$. Note that if $\pi_0 \in \mathrm{Irr}(\mathrm{O}_{2n_0}(F))$ is the Langlands quotient of $\mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0$ with $\tau_0 \in \mathrm{Irr}(\mathrm{O}_{2n_0'}(F))$, then $\tau_0|_{\mathrm{SO}_{2n_0'}(F)}$ is reducible if and only if $n_0' > 0$ and $\pi_0|_{\mathrm{SO}_{2n_0}(F)}$ is reducible.

Recall that ψ_{GL} is assumed to be irreducible. We write $\psi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1} \boxtimes S_{2\beta+1}$, and set

$$\mathfrak{m}_{\mathrm{GL}} = [-\alpha+\beta, \alpha+\beta]_{\rho_{\mathrm{GL}}} + [-\alpha+\beta-1, \alpha+\beta-1]_{\rho_{\mathrm{GL}}} + \cdots + [-\alpha-\beta, \alpha-\beta]_{\rho_{\mathrm{GL}}}.$$

Then the Langlands quotient π_{GL} of $\mathcal{I}(\mathfrak{m}_{\mathrm{GL}})$ is the representation corresponding to ψ_{GL}.

Definition 6.4.1. *Define $\kappa \in \{1, \frac{1}{2}\}$ such that $\beta - \kappa \in \mathbb{Z}$. For $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0 \in \Pi_{\psi_M}$, write $\mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0$ for the standard module of π_0. We say that an irreducible summand π of $I_P(\pi_M)$ is* highly non-tempered *if there is a tempered representation τ with*

$$\begin{cases} \tau = \tau_0 & \text{if } \kappa = \dfrac{1}{2}, \\ \tau \hookrightarrow \Delta([-\alpha, \alpha]_{\rho_{\mathrm{GL}}}) \rtimes \tau_0 & \text{if } \kappa = 1 \end{cases}$$

such that $\mathcal{I}(\mathfrak{m}) \rtimes \tau$ is the standard module of π, where

$$\mathfrak{m} = \mathfrak{m}_0 + 2[-\alpha+\beta, \alpha+\beta]_{\rho_{\mathrm{GL}}} + 2[-\alpha+\beta-1, \alpha+\beta-1]_{\rho_{\mathrm{GL}}} + \cdots + 2[-\alpha+\kappa, \alpha+\kappa]_{\rho_{\mathrm{GL}}}.$$

Lemma 6.4.2. *For any $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0 \in \Pi_{\psi_M}$, there exists a highly non-tempered summand π of $I_P(\pi_M)$. Moreover, it is unique if $\kappa = \frac{1}{2}$, and there are at most two such summands if $\kappa = 1$.*

Proof. The lemma is a special case of [Tad2, Proposition 1.3]. But for completeness, we give a proof.

We use the notations as above. Write $\mathcal{I}(\mathfrak{m}) = \tau_1|\cdot|^{s_1} \times \cdots \times \tau_r|\cdot|^{s_r}$ with τ_i being tempered and $s_1 > \cdots > s_r > 0$. By Tadić's formula ([Tad1, Theorems 5.4, 6.5], [Ban, Theorem 7.3]), with a suitable parabolic subgroup $P' = M'N_{P'}$ and an irreducible tempered representation τ, we have

$$\mathrm{Jac}_{P'}(I_P(\pi_M)) \geq {}^c\tau_1^\vee|\cdot|^{-s_1} \otimes \cdots \otimes {}^c\tau_r^\vee|\cdot|^{-s_r} \otimes \tau$$

in the Grothendieck group $\mathcal{R}(M')$. Here, for $A, B \in \mathcal{R}(M')$, we write $A \leq B$ if $B - A$ is a non-negative combination of irreducible representations. Conversely, if $\mathrm{Jac}_{P'}(I_P(\pi_M)) \geq \tau'_1|\cdot|^{-s_1}\otimes\cdots\otimes\tau'_r|\cdot|^{-s_r}\otimes\tau$ for some irreducible tempered representations $\tau'_1, \dots, \tau'_r$ and τ, then we must have $\tau'_i \cong {}^c\tau_i^\vee$ for $i = 1, \dots, r$, and $\tau = \tau_0$ if $\kappa = \frac{1}{2}$, whereas $\tau \hookrightarrow \Delta([-\alpha, \alpha]_{\rho_{\mathrm{GL}}}) \rtimes \tau_0$ if $\kappa = 1$. Moreover, such an irreducible representation appears in $\mathrm{Jac}_{P'}(I_P(\pi_M))$ with multiplicity one.

Suppose that an irreducible subquotient π of $I_P(\pi_M)$ satisfies that $\mathrm{Jac}_{P'}(\pi) \geq {}^c\tau_1^\vee|\cdot|^{-s_1} \otimes \cdots \otimes {}^c\tau_r^\vee|\cdot|^{-s_r} \otimes \tau$. Then by looking at the central character, (after replacing τ if necessary) we see that the right hand-side is a quotient of $\mathrm{Jac}_{P'}(\pi)$ in the category $\mathrm{Rep}(M')$, which is equivalent by the Frobenius reciprocity to saying that

$$\pi \hookrightarrow {}^c\tau_1^\vee|\cdot|^{-s_1} \times \cdots \times {}^c\tau_r^\vee|\cdot|^{-s_r} \rtimes \tau.$$

By [AG2, Lemma 2.2], one can see that the standard module of π is $\mathcal{I}(\mathfrak{m}) \rtimes \tau$. In particular, since $I_P(\pi_M)$ is semisimple, we conclude that π is a highly non-tempered summand.

On the other hand, $\mathrm{Jac}_{P'}(I_P(\pi_M))$ contains ${}^c\tau_1^\vee|\cdot|^{-s_1} \otimes \cdots \otimes {}^c\tau_r^\vee|\cdot|^{-s_r} \otimes \tau$ with multiplicity at most one for each τ. The number of highly non-tempered summands are at most the number of the choices of τ. By definition of τ, this number is 1 or 2 according to $\kappa = \frac{1}{2}$ or $\kappa = 1$. □

For the rest of this subsection, we fix a highly non-tempered summand $\pi \subset I_P(\pi_M)$. Let $\mathcal{I}(\mathfrak{m}_{\mathrm{GL}})$, $\mathcal{I}(\mathfrak{m}) \rtimes \tau$ and $\mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0$ be as above. Write

$$\mathcal{I}(\mathfrak{m}_{\mathrm{GL}}) = \Delta([-\alpha, \alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{\beta} \times \Delta([-\alpha, \alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{\beta-1} \times \cdots \times \Delta([-\alpha, \alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{-\beta},$$
$$\mathcal{I}(\mathfrak{m}_0) = \tau_r|\cdot|^{e_r} \times \cdots \times \tau_1|\cdot|^{e_1},$$

where τ_i is an irreducible tempered representation of $\mathrm{GL}_{k_i}(E)$ and $e_r > \cdots > e_1 > 0$.

To define several objects, we realize G as an isometry group $G(W)$ (or its identity component) of a vector space W over E equipped with a non-degenerate sesquilinear form. We write

$$W = V_+ \oplus W_0 \oplus V_-,$$

where $V_\pm$ is a totally isotropic subspace with $V_+ \oplus V_-$ non-degenerate, and where W_0 is the orthogonal complement of $V_+ \oplus V_-$. Suppose that the standard parabolic subgroup $P = MN_P$ is the stabilizer of V_+, and the Levi subgroup M of P is the stabilizer of V_+ and V_-. Hence $M = \mathrm{GL}(V_+) \times G(W_0)$. We decompose

$$V_\pm = V_\pm^{(\beta)} \oplus V_\pm^{(\beta-1)} \oplus \cdots \oplus V_\pm^{(-\beta)},$$

$$W_0 = \left(\bigoplus_{i=1}^{r} W_0^{(e_i)}\right) \oplus W_0^{(0)} \oplus \left(\bigoplus_{i=1}^{r} W_0^{(-e_i)}\right),$$

such that

- $\dim(V_\pm^{(b)}) = d_0$ with $d_0 = \dim(\rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1})$ and $b \in \{\beta, \beta-1, \dots, -\beta\}$;
- $V_+^{(b)} \oplus V_-^{(-b)}$ is non-degenerate for $b \in \{\beta, \beta-1, \dots, -\beta\}$;
- $W_0^{(\pm e_i)}$ is a totally isotropic subspace of dimension k_i for $1 \le i \le r$;
- $W_0^{(e_i)} \oplus W_0^{(-e_i)}$ is non-degenerate for $1 \le i \le r$.

For $j = 0, 1, 2$, we define a total order $\prec_j$ on the set

$$\mathcal{V} = \left\{V_+^{(b)}, V_-^{(b)}\right\}_{-\beta \le b \le \beta} \cup \left\{W_0^{(e_i)}, W_0^{(-e_i)}\right\}_{1 \le i \le r} \cup \{W_0^{(0)}\}$$

as follows.

(0) When $j = 0$,
  - $V_+^{(b)} \prec_0 W_0^{(e)} \prec_0 V_-^{(b)}$;
  - if $b > b'$, then $V_\pm^{(b')} \prec_0 V_\pm^{(b)}$;
  - if $e > e'$, then $W_0^{(e')} \prec_0 W_0^{(e)}$.

(1) When $j = 1$,
  - $V_+^{(b)} \prec_1 W_0^{(e)} \prec_1 V_-^{(b)}$;
  - if $b > b'$, then $V_\pm^{(b)} \prec_1 V_\pm^{(b')}$;
  - if $e > e'$, then $W_0^{(e)} \prec_1 W_0^{(e')}$.

(2) When $j = 2$,
  - For $X, Y \in \{V_\pm, W_0\}$, if $b > b'$, then $X^{(b)} \prec_2 Y^{(b')}$;
  - if $b = e$, then $V_+^{(b)} \prec_2 W_0^{(e)} \prec_2 V_-^{(b)}$.

For $j = 0, 1, 2$, if we write

$$\{V \in \mathcal{V} \mid V \prec_j W_0^{(0)}\} = \{V_1, \dots, V_t\}$$

with $V_1 \prec_j \cdots \prec_j V_t \prec_j W_0^{(0)}$, then we define a parabolic subgroup $P'_j = M_1 N_{P'_j}$ as the stabilizer of the flag

$$V_1 \subset V_1 \oplus V_2 \subset \cdots \subset V_1 \oplus \cdots \oplus V_t,$$

where M_1 is the common Levi subgroup of G stabilizing all $V \in \mathcal{V}$. We may assume that $P_1 = P'_1$ is a standard parabolic subgroup. Note that $P'_1 \subset P$. Let $w_1, w_2 \in W^G$ be such that $\widetilde{w}_1 P_1 \widetilde{w}_1^{-1} = P'_0$ and such that $\widetilde{w}_2^{-1} P'_2 \widetilde{w}_2 = P_2 = M_2 N_{P_2}$ is a standard parabolic subgroup. We may regard w_1 and w_2 as

$$w_1 \in W(M_1^\circ), \quad w_2 \in W(M_2^\circ, M_1^\circ).$$

If we consider that

- $\mathrm{GL}(V_\pm^{(b)})$ acts on $\Delta([-\alpha, \alpha]_{\rho_{\mathrm{GL}}})|\cdot|^b$;
- $\mathrm{GL}(W_0^{(e_i)})$ acts on $\tau_i|\cdot|^{e_i}$ if $e_i > 0$ (resp. ${}^c\tau_i^\vee|\cdot|^{e_i}$ if $e_i < 0$);
- $G(W_0^{(0)})$ acts on τ_0,

then we obtain irreducible representations π_{M_0}, π_{M_1} of M_1 and π_{M_2} of M_2 such that

$$\begin{aligned} I_{P_0}(\pi_{M_0}) &= {}^c\mathcal{I}(\mathfrak{m}_{\mathrm{GL}})^\vee \times {}^c\mathcal{I}(\mathfrak{m}_0)^\vee \rtimes \tau_0, \\ I_{P_1}(\pi_{M_1}) &= \mathcal{I}(\mathfrak{m}_{\mathrm{GL}}) \times \mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0, \\ I_{P_2}(\pi_{M_2}) &= \begin{cases} \mathcal{I}(\mathfrak{m}) \times \Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}}) \rtimes \tau_0 & \text{if } 2\beta+1 \equiv 1 \bmod 2, \\ \mathcal{I}(\mathfrak{m}) \rtimes \tau_0 & \text{if } 2\beta+1 \equiv 0 \bmod 2, \end{cases} \end{aligned}$$

respectively, where $P_0 = P_1$ is the standard parabolic subgroup with Levi $\widetilde{w}_1 M_1 \widetilde{w}_1^{-1}$. Here, we note that ρ_{GL} is conjugate-self-dual. Then we have

$$w_2 \pi_{M_2} = \pi_{M_1}, \quad w_1 \pi_{M_1} = \pi_{M_0}.$$

See Section 1.7 for these notations.

Recall that

$$\begin{aligned} \pi_{M_1} = &\left(\Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{\beta} \times \cdots \times \Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{-\beta}\right) \\ &\times \left(\tau_r|\cdot|^{e_r} \times \cdots \times \tau_1|\cdot|^{e_1} \rtimes \tau_0\right). \end{aligned}$$

For $\lambda = (\lambda_\beta, \lambda_{\beta-1} \dots, \lambda_{-\beta}) \in \mathbb{C}^{2\beta+1}$ and $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{C}^r$, we set

$$\begin{aligned} \pi_{M_1,(\lambda,\mu)} = &\left(\Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{\lambda_\beta} \times \cdots \times \Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}})|\cdot|^{\lambda_{-\beta}}\right) \\ &\times \left(\tau_r|\cdot|^{\mu_r} \times \cdots \times \tau_1|\cdot|^{\mu_1} \rtimes \tau_0\right). \end{aligned}$$

We define $\pi_{M_0,(\lambda,\mu)}$ and $\pi_{M_2,(\lambda,\mu)}$ similarly. Let $\phi_{M_j,(\lambda,\mu)}$ be the L-parameter of $\pi_{M_j,(\lambda,\mu)}$. Recall that the intertwining operators

$$\begin{aligned} R_{P_1}(w_1, \pi_{M_1,(\lambda,\mu)}, \phi_{M_1,(\lambda,\mu)})&\colon I_{P_1}(\pi_{M_1,(\lambda,\mu)}) \to I_{P_0}(\pi_{M_0,(\lambda,\mu)}), \\ R_{P_2}(w_2, \pi_{M_2,(\lambda,\mu)}, \phi_{M_2,(\lambda,\mu)})&\colon I_{P_2}(\pi_{M_2,(\lambda,\mu)}) \to I_{P_1}(\pi_{M_1,(\lambda,\mu)}) \end{aligned}$$

are defined by the meromorphic continuation of

$$\begin{aligned} R_{P_j}(w_j, \pi_{M_j,(\lambda,\mu)}, \phi_{M_j,(\lambda,\mu)}) f_{j,(\lambda,\mu)}(g) = \gamma_A(0, \phi_{M_j,(\lambda,\mu)}, \rho^\vee_{w_j^{-1}P_{j-1}|P_j}, \psi_F) \\ \times \lambda(w_j)^{-1} \int_{N_{P_{j-1}} \cap \widetilde{w}_j N_{P_j} \widetilde{w}_j^{-1} \backslash N_{P_{j-1}}} f_{j,(\lambda,\mu)}(\widetilde{w}_j^{-1} n g) dn \end{aligned}$$

for $j = 1, 2$. As in [BW, Chapter XI, Proposition 2.6 (1)], this integral converges when we specialize it at $\lambda = (\beta, \beta-1, \dots, -\beta)$ and $\mu = (e_1, \dots, e_r)$. Hence $R(w_1) = R_{P_1}(w_1, \pi_{M_1}, \phi_{M_1})$ and $R(w_2) = R_{P_2}(w_2, \pi_{M_2}, \phi_{M_2})$ are well-defined and nonzero. Moreover, the image of $R(w_1)$ is exactly equal to $I_P(\pi_M)$.

Set $w'_u = w_2^{-1} w_1^{-1} w_u w_1 w_2$. Using the order $\prec_2$, we see that a representative of w'_u preserves $W_0^{(e)}$ for any e, and exchanges $V_+^{(b)}$ with $V_-^{(b)}$ for any b. Since w_u and w'_u preserve the Levi subgroups M_0 and M_2, respectively, we obtain normalized intertwining operators

$$\begin{aligned} R_{P_0}(w_u, \pi_{M_0,(\lambda,\mu)}, \phi_{M_0,(\lambda,\mu)})&\colon I_{P_0}(\pi_{M_0,(\lambda,\mu)}) \to I_{P_0}(w_u \pi_{M_0,(\lambda,\mu)}), \\ R_{P_2}(w'_u, \pi_{M_2,(\lambda,\mu)}, \phi_{M_2,(\lambda,\mu)})&\colon I_{P_2}(\pi_{M_2,(\lambda,\mu)}) \to I_{P_2}(w'_u \pi_{M_2,(\lambda,\mu)}). \end{aligned}$$

We will show in Lemma 6.4.3 (2) and (3) below that $R_{P_2}(w'_u, \pi_{M_2,(\lambda,\mu)}, \phi_{M_2,(\lambda,\mu)})$ is holomorphic at $\lambda = (\beta, \dots, -\beta)$ and $\mu = (e_1, \dots, e_r)$. Moreover, since $w_u\pi_{M_0} \cong \pi_{M_0}$ and $w'_u\pi_{M_2} \cong \pi_{M_2}$, one can normalize isomorphisms

$$A_{w_u} \otimes \mathrm{id} \colon w_u\pi_{M_0} \xrightarrow{\sim} \pi_{M_0}, \quad A_{w'_u} \otimes \mathrm{id} \colon w'_u\pi_{M_2} \xrightarrow{\sim} \pi_{M_2}$$

by Whittaker functionals on standard modules of general linear groups. By composing $I_{P_2}(A_{w'_u} \otimes \mathrm{id})$, we have a self-intertwining operator

$$R_{P_2}(w'_u, \widetilde{\pi}_{M_2}, \phi_{M_2}) \colon I_{P_2}(\pi_{M_2}) \to I_{P_2}(\pi_{M_2}).$$

On the other hand, $R_{P_0}(w_u, \pi_{M_0}, \phi_{M_0})$ might be a singularity of the meromorphic family $R_{P_0}(w_u, \pi_{M_0,(\lambda,\mu)}, \phi_{M_0,(\lambda,\mu)})$ at $\lambda = (\beta, \dots, -\beta)$ and $\mu = (e_1, \dots, e_r)$. Hence we just write

$$I_{P_0}(\pi_{M_0}) \xrightarrow{R_{P_0}(w_u,\widetilde{\pi}_{M_0},\phi_{M_0})} I_{P_0}(\pi_{M_0}).$$

Lemma 6.4.3. (1) *The image of the map $R(w_1) \circ R(w_2) \colon \mathcal{I}(\mathfrak{m}) \rtimes \tau \to I_P(\pi_M)$ is exactly equal to π.*
(2) *If $2\beta + 1$ is even, then $R_{P_2}(w'_u, \widetilde{\pi}_{M_2}, \phi_{M_2})$ is the identity map.*
(3) *If $2\beta + 1$ is odd, then*

$$R_{P_2}(w'_u, \widetilde{\pi}_{M_2}, \phi_{M_2}) = \langle e(\rho_{\mathrm{GL}}, 2\alpha + 1, 1), \tau\rangle_{\phi_\tau} \cdot \mathrm{id},$$

on $\mathcal{I}(\mathfrak{m}) \rtimes \tau$, where ϕ_τ is the L-parameter of τ.

Proof. For (1), we claim that π appears in $I_{P_1}(\pi_{M_1})$ as a subquotient with multiplicity one. Since the Jordan–Hölder series of $I_{P_1}(\pi_{M_1})$ and $I_{P_2}(\pi_{M_2})$ are the same, it is enough to consider $I_{P_2}(\pi_{M_2})$. If $I_{P_2}(\pi_{M_2})$ is a standard module, then the claim follows from the famous fact that the Langlands quotient appears in its standard module with multiplicity one (See e.g., [BW, Chapter XI, Lemma 2.13]). Otherwise, $2\beta + 1$ is odd and hence $I_{P_2}(\pi_{M_2})$ is a direct sum of two standard modules. Since the general linear parts of these two standard modules are the same, by computing Jacquet modules, one sees that π appears only in one of them. Hence π must appear in $I_{P_1}(\pi_{M_1})$ with multiplicity one.

Now, since $I_P(\pi_M)$ is a unitary induction, and hence semisimple, the image of $R(w_1) \circ R(w_2)$ is isomorphic to a subrepresentation of the maximal semisimple quotient of $\mathcal{I}(\mathfrak{m}) \rtimes \tau$. Since this maximal semisimple quotient is equal to π, we see that the image is equal to π, or $R(w_1) \circ R(w_2) = 0$. Since $R(w_2)$ is nonzero, π appears in its image. Hence if $R(w_1) \circ R(w_2)$ were to be zero, then π would appear in the kernel of $R(w_1)$. On the other hand, since the image of $R(w_1)$ is equal to $I_P(\pi_M)$, which contains π, it would imply that π must appear in $I_{P_1}(\pi_{M_1})$ with multiplicity greater than one. This contradicts the claim. Therefore we obtain (1).

Next, we prove (2) and (3). We can decompose $R(w'_u) = R_{P_2}(w'_u, \widetilde{\pi}_{M_2}, \phi_{M_2})$ as $R(w'_u) = R(w_{\mathrm{GL}}) \circ R(w_{u_0})$, where $R(w_{\mathrm{GL}})$ is obtained from normalized intertwining operators for several general linear groups. Since π_{M_2} and $w'_u\pi_{M_2}$ are essentially tempered and since $I_{P_2}(\pi_{M_2})$ and $I_{P_2}(w'_u\pi_{M_2})$ are standard modules, we can apply Theorem 3.5.1

to $R(w_{\mathrm{GL}})$, and obtain that $R(w_{\mathrm{GL}}) = \mathrm{id}$. This proves (2). On the other hand, if $2\beta+1$ is odd, by (**LIR**) for the tempered representation τ, we see that

$$R(w_{u_0}) = \langle e(\rho_{\mathrm{GL}}, 2\alpha+1, 1), \tau\rangle_{\phi_\tau} \cdot \mathrm{id}$$

on $\mathcal{I}(\mathfrak{m}) \rtimes \tau$. This proves (3). □

6.5. **The main diagram.** Using the notations in the previous subsection, with a constant $c \in \mathbb{C}^\times$, we now consider the following *main diagram*. When $\kappa = 1$, it is:

$$\begin{array}{ccc}
\mathcal{I}(\mathfrak{m}) \rtimes \tau & \xrightarrow{R_{P_2}(w'_u, \widetilde{\pi}_{M_2}, \phi_{M_2})} & \mathcal{I}(\mathfrak{m}) \rtimes \tau \\
\big\downarrow & & \big\downarrow \\
\mathcal{I}(\mathfrak{m}) \times \Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}}) \rtimes \tau_0 & \xrightarrow{R_{P_2}(w'_u, \widetilde{\pi}_{M_2}, \phi_{M_2})} & \mathcal{I}(\mathfrak{m}) \times \Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}}) \rtimes \tau_0 \\
{\scriptstyle R(w_2)}\big\downarrow & & {\scriptstyle R(w_2)}\big\downarrow \\
\mathcal{I}(\mathfrak{m}_{\mathrm{GL}}) \times \mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0 & & \mathcal{I}(\mathfrak{m}_{\mathrm{GL}}) \times \mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0 \\
{\scriptstyle R(w_1)}\big\downarrow & & {\scriptstyle R(w_1)}\big\downarrow \\
I_P(\pi_M) & \xrightarrow{c^{-1}\langle \widetilde{u}, \widetilde{\pi}_M\rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)} & I_P(\pi_M) \\
\big\downarrow & & \big\downarrow \\
{}^c\mathcal{I}(\mathfrak{m}_{\mathrm{GL}})^\vee \times {}^c\mathcal{I}(\mathfrak{m}_0)^\vee \rtimes \tau_0 & \xrightarrow{R_{P_0}(w_u, \widetilde{\pi}_{M_0}, \phi_{M_0})} & {}^c\mathcal{I}(\mathfrak{m}_{\mathrm{GL}})^\vee \times {}^c\mathcal{I}(\mathfrak{m}_0)^\vee \rtimes \tau_0.
\end{array}$$

When $\kappa = \frac{1}{2}$, we replace $\Delta([-\alpha,\alpha]_{\rho_{\mathrm{GL}}}) \times \tau_0$ with τ_0 in the second line, or equivalently, we remove the second line. The following is the main result in this section.

Theorem 6.5.1. *The main diagram is commutative with*

$$c = \left.\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)}\right|_{s=0},$$

where we write $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0$, and ϕ_{π_0} is the L-parameter of π_0.

Proof. As in the proof of Theorem 3.4.2, we will prove the assertion in five steps.

Step 1: With complex parameters $\lambda \in \mathbb{C}^{2\beta+1}$ and $\mu \in \mathbb{C}^r$, we can consider the following diagram of meromorphic families of operators:

$$\begin{array}{ccc}
I_{P_2}(\pi_{M_2,(\lambda,\mu)}) & \xrightarrow{R_{P_2}(w'_u, \pi_{M_2,(\lambda,\mu)})} & I_{P_2}(w'_u \pi_{M_2,(\lambda,\mu)}) \\
{\scriptstyle R_{P_2}(w_2, \pi_{M_2,(\lambda,\mu)})}\big\downarrow & & \big\downarrow{\scriptstyle R_{P_2}(w_2, w'_u\pi_{M_2,(\lambda,\mu)})} \\
I_{P_1}(\pi_{M_1,(\lambda,\mu)}) & & I_{P_1}(w_1^{-1} w_u w_1 \pi_{M_1,(\lambda,\mu)}) \\
{\scriptstyle R_{P_1}(w_1, \pi_{M_1,(\lambda,\mu)})}\big\downarrow & & \big\downarrow{\scriptstyle R_{P_1}(w_1, w_1^{-1} w_u \pi_{M_0,(\lambda,\mu)})} \\
I_{P_0}(\pi_{M_0,(\lambda,\mu)}) & \xrightarrow{R_{P_0}(w_u, \pi_{M_0,(\lambda,\mu)})} & I_{P_0}(w_u \pi_{M_0,(\lambda,\mu)}).
\end{array}$$

Here, we omit the L-parameters in the notations of the normalized intertwining operators. If $\lambda \in (\sqrt{-1}\mathbb{R})^{2\beta+1}$ and $\mu \in (\sqrt{-1}\mathbb{R})^{r}$, then by Proposition 1.7.2, all maps in this diagram are regular and the diagram is commutative. By analytic continuation, we see that this diagram is commutative whenever all maps are regular.

Step 2: Let $s \in \mathbb{C}$ be a new complex parameter. We will specialize the diagram in Step 1 at $\lambda = (s+\beta, s+\beta-1, \dots, s-\beta)$ and $\mu = (e_1, \dots, e_r)$. We write $\pi_{M_i,s}$ for the corresponding $\pi_{M_i,(\lambda,\mu)}$. We claim that all operators in the diagram are well-defined as meromorphic families of operators in s.

In fact, the bottom operator $R_{P_0}(w_u, \pi_{M_0,s})$ is not always a pole of the family of the operators $R_{P_0}(w_u, \pi_{M_0,(\lambda,\mu)})$ (but might be a pole at $s=0$). On the other hand, as we have seen in Lemma 6.4.3, the other five operators are still regular at $s=0$.

Hence we can specialize the diagram in Step 1 at $\lambda = (s+\beta, s+\beta-1, \dots, s-\beta)$ and $\mu = (e_1, \dots, e_r)$, and obtain the following commutative diagram:

$$\begin{array}{ccc}
I_{P_2}(\pi_{M_2,s}) & \xrightarrow{R_{P_2}(w_u', \pi_{M_2,s})} & I_{P_2}(w_u'\pi_{M_2,s}) \\
{\scriptstyle R_{P_2}(w_2, \pi_{M_2,s})}\downarrow & & \downarrow{\scriptstyle R_{P_2}(w_2, w_u'\pi_{M_2,s})} \\
I_{P_1}(\pi_{M_1,s}) & & I_{P_1}(w_1^{-1}w_u w_1 \pi_{M_1,s}) \\
{\scriptstyle R_{P_1}(w_1, \pi_{M_1,s})}\downarrow & & \downarrow{\scriptstyle R_{P_1}(w_1, w_1^{-1}w_u\pi_{M_0,s})} \\
I_{P_0}(\pi_{M_0,s}) & \xrightarrow{R_{P_0}(w_u, \pi_{M_0,s})} & I_{P_0}(w_u\pi_{M_0,s}).
\end{array}$$

Step 3: Note that the image of $R_{P_1}(w_1, \pi_{M_1,s})\colon I_{P_1}(\pi_{M_1,s}) \to I_{P_0}(\pi_{M_0,s})$ is equal to $I_P(\pi_{M,s}) = I_P(\pi_{\mathrm{GL}}|\cdot|^s \boxtimes \pi_0)$. If we set

$$c(s) = \frac{\gamma_A(0, \psi_{M,s}, \rho^{\vee}_{w_u^{-1}P|P}, \psi_F)}{\gamma_A(0, \phi_{M_0,s}, \rho^{\vee}_{w_u^{-1}P|P}, \psi_F)},$$

then using a canonical homeomorphism

$$N_P \cap \widetilde{w}_u N_P \widetilde{w}_u^{-1} \backslash N_P \cong N_{P_0} \cap \widetilde{w}_u N_{P_0} \widetilde{w}_u^{-1} \backslash N_{P_0},$$

we obtain a commutative diagram of meromorphic families of operators

$$\begin{array}{ccc}
I_P(\pi_{M,s}) & \xrightarrow{c(s)^{-1} R_P(w_u, \pi_{M,s}, \psi_{M,s})} & I_P(w_u\pi_{M,s}) \\
\downarrow & & \downarrow \\
I_{P_0}(\pi_{M_0,s}) & \xrightarrow{R_{P_0}(w_u, \pi_{M_0,s})} & I_{P_0}(w_u\pi_{M_0,s})
\end{array}$$

where the vertical maps are the canonical inclusions. Combining this diagram with the one in Step 2, we obtain the following commutative diagram:

$$\begin{array}{ccc}
I_{P_2}(\pi_{M_2,s}) & \xrightarrow{R_{P_2}(w'_u,\pi_{M_2,s})} & I_{P_2}(w'_u\pi_{M_2,s}) \\
{\scriptstyle R_{P_2}(w_2,\pi_{M_2,s})}\downarrow & & \downarrow{\scriptstyle R_{P_2}(w_2,w'_u\pi_{M_2,s})} \\
I_{P_1}(\pi_{M_1,s}) & & I_{P_1}(w_1^{-1}w_uw_1\pi_{M_1,s}) \\
{\scriptstyle R_{P_1}(w_1,\pi_{M_1,s})}\downarrow & & \downarrow{\scriptstyle R_{P_1}(w_1,w_1^{-1}w_u\pi_{M_0,s})} \\
I_P(\pi_{M,s}) & \xrightarrow{c(s)^{-1}R_P(w_u,\pi_{M,s},\psi_{M,s})} & I_P(w_u\pi_{M,s}).
\end{array}$$

We note that

$$c(s)=\frac{\gamma_A(s,\psi_{\mathrm{GL}}\otimes\psi_0^\vee,\psi_E)}{\gamma_A(s,\psi_{\mathrm{GL}}\otimes\phi_{\pi_0}^\vee,\psi_E)}$$

so that $c(0)=c$ by Proposition A.1.2. We will see that $c(s)$ is regular at $s=0$.

Step 4: We would like to specialize the commutative diagram above at $s=0$. As we have noted in Step 2, the five operators appearing in the top, left and right of the last diagram are regular at $s=0$. In particular, the composition

$$R_{P_1}(w_1,w_1^{-1}w_u\pi_{M_0,s})\circ R_{P_2}(w_2,w'_u\pi_{M_2,s})\circ R_{P_2}(w'_u,\pi_{M_2,s}),$$

and hence

$$c(s)^{-1}R_P(w_u,\pi_{M,s},\psi_{M,s})\circ R_{P_1}(w_1,\pi_{M_1,s})\circ R_{P_2}(w_2,\pi_{M_2,s})$$

are regular and nonzero at $s=0$. We can specialize the last diagram at $s=0$, and obtain the commutative diagram

$$\begin{array}{ccc}
I_{P_2}(\pi_{M_2}) & \xrightarrow{R_{P_2}(w'_u,\pi_{M_2})} & I_{P_2}(w'_u\pi_{M_2}) \\
{\scriptstyle R_{P_2}(w_2,\pi_{M_2})}\downarrow & & \downarrow{\scriptstyle R_{P_2}(w_2,w'_u\pi_{M_2})} \\
I_{P_1}(\pi_{M_1}) & & I_{P_1}(w_1^{-1}w_uw_1\pi_{M_1}) \\
{\scriptstyle R_{P_1}(w_1,\pi_{M_1})}\downarrow & & \downarrow{\scriptstyle R_{P_1}(w_1,w_1^{-1}w_u\pi_{M_0})} \\
I_P(\pi_M) & \xrightarrow{c^{-1}R_P(w_u,\pi_M,\psi_M)} & I_P(w_u\pi_M).
\end{array}$$

Note that $R_P(w_u,\pi_{M,s},\psi_{M,s})$ is well-defined by [Ar2, Proposition 2.3.1] and [Mok, Proposition 3.3.1]. Since $c^{-1}R_P(w_u,\pi_M,\psi_M)$ is nonzero, we conclude that $c(s)$ is regular at $s=0$.

Step 5: If we realize $w'_u\pi_{M_2}$, π_{M_2}, $w_1^{-1}w_uw_1\pi_{M_1}$ and π_{M_1} on the same vector space, say $\mathcal{V}$, then $A_{w'_u}\otimes\mathrm{id}$ is a linear isomorphism $\Phi\colon\mathcal{V}\to\mathcal{V}$ satisfying that

$$\Phi\circ\pi_{M_2}(\widetilde{w}_u'^{-1}m_2\widetilde{w}'_u)=\pi_{M_2}(m_2)\circ\Phi,\quad m_2\in M_2.$$

If we write $w''_u = w_1^{-1} w_u w_1$ so that $w'_u = w_2^{-1} w''_u w_2$, then by Lemma 1.7.1, this condition can be rewritten as

$$\Phi \circ \pi_{M_1}(\widetilde{w}''^{-1}_u m_1 \widetilde{w}''_u) = \pi_{M_1}(m_1) \circ \Phi, \quad m_1 \in M_1.$$

Therefore, Φ is also equal to the normalized isomorphism $A_{w''_u} \otimes \mathrm{id} \colon w''_u \pi_{M_1} \xrightarrow{\sim} \pi_{M_1}$, and hence the diagram

$$\begin{array}{ccc}
I_{P_2}(w'_u \pi_{M_2}) & \xrightarrow{I_{P_2}(A_{w'_u} \otimes \mathrm{id})} & I_{P_2}(\pi_{M_2}) \\
{\scriptstyle R_{P_2}(w_2, w'_u \pi_{M_2})} \downarrow & & \downarrow {\scriptstyle R_{P_2}(w_2, \pi_{M_2})} \\
I_{P_1}(w''_u \pi_{M_1}) & \xrightarrow{I_{P_1}(A_{w''_u} \otimes \mathrm{id})} & I_{P_1}(\pi_{M_1})
\end{array}$$

is commutative. On the other hand, by the definition of $\mathcal{A}_{w_u}$, we see that the diagram

$$\begin{array}{ccc}
\mathrm{Ind}_{P_1 \cap M}^{M}(w''_u \pi_{M_1}) & \xrightarrow{I_{P_1 \cap M}(A_{w''_u} \otimes \mathrm{id})} & I_{P_1 \cap M}^{M}(\pi_{M_1}) \\
{\scriptstyle R_{P_1 \cap M}(w_1, w_1^{-1} w_u \pi_{M_0})} \downarrow & & \downarrow {\scriptstyle R_{P_1 \cap M}(w_1, \pi_{M_1})} \\
w_u \pi_M & \xrightarrow{\mathcal{A}_{w_u} \otimes \mathrm{id}} & \pi_M
\end{array}$$

is also commutative. By the functoriality of I_P, combining these two diagrams, and by Lemma 6.3.1, we obtain the commutative diagram

$$\begin{array}{ccc}
I_{P_2}(w'_u \pi_{M_2}) & \xrightarrow{I_{P_2}(A_{w'_u} \otimes \mathrm{id})} & I_{P_2}(\pi_{M_2}) \\
{\scriptstyle R_{P_2}(w_2, w'_u \pi_{M_2})} \downarrow & & \downarrow {\scriptstyle R_{P_2}(w_2, \pi_{M_2})} \\
I_{P_1}(w_1^{-1} w_u w_1 \pi_{M_1}) & & I_{P_1}(\pi_{M_1}) \\
{\scriptstyle R_{P_1}(w_1, w_1^{-1} w_u \pi_{M_0})} \downarrow & & \downarrow {\scriptstyle R_{P_1}(w_1, \pi_{M_1})} \\
I_P(w_u \pi_M) & \xrightarrow{I_P(\langle \widetilde{u}, \widetilde{\pi}_M \rangle \widetilde{\pi}_M(w_u))} & I_P(\pi_M).
\end{array}$$

Combining this with the diagram obtained in Step 4, we conclude that the main diagram is commutative.

This completes the proof of Theorem 6.5.1. □

By the commutativity of the main diagram together with Lemma 6.4.3, we have

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \psi_M)|_\pi = \begin{cases} c \cdot \mathrm{id}_\pi & \text{if } 2\beta + 1 \equiv 0 \bmod 2, \\ c \langle e(\rho_{\mathrm{GL}}, 2\alpha + 1, 1), \tau \rangle_{\phi_\tau} \cdot \mathrm{id}_\pi & \text{if } 2\beta + 1 \equiv 1 \bmod 2. \end{cases}$$

Therefore, we obtain the following summary.

Corollary 6.5.2. *Assume Hypothesis 6.1.1. Let $\psi_M = \psi_{\mathrm{GL}} \oplus \psi_0$ be an A-parameter for M such that $\psi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1} \boxtimes S_{2\beta+1}$ is irreducible and conjugate-self-dual. For*

$\pi_M \in \Pi_{\psi_M}$, let $\pi \subset I_P(\pi_M)$ be a highly non-tempered summand, with the standard module $\mathcal{I}(\mathfrak{m}) \rtimes \tau$. Then (**LIR**) *holds for $\pi \subset I_P(\pi_M)$ if and only if the equation*

$$(\star)\qquad \left.\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)}\right|_{s=0} = \begin{cases} \langle s_u, \pi\rangle_\psi & \text{if } 2\beta+1 \equiv 0 \bmod 2, \\ \dfrac{\langle s_u, \pi\rangle_\psi}{\langle e(\rho_{\mathrm{GL}}, 2\alpha+1, 1), \tau\rangle_{\phi_\tau}} & \text{if } 2\beta+1 \equiv 1 \bmod 2 \end{cases}$$

holds, where ϕ_{π_0} (resp. ϕ_τ) is the L-parameter of π_0 (resp. τ).

7. Computations of local factors

This section continues the work of Section 6. Our goal is to show (**LIR**) for each irreducible summand $\pi \subset I_P(\pi_M)$ and each $\pi_M \in \Pi_{\psi_M}$, where $P = MN_P$ is a parabolic subgroup of a classical group G, and ψ_M is an arbitrary co-tempered A-parameter for M (Theorem 1.10.5 (2)). Lemma 6.2.1 reduces the problem to the case where P is maximal so that $M \cong \mathrm{GL}_k(E) \times G_0$. By the key lemma (Lemma 6.3.2), we may then assume that $\pi \subset I_P(\pi_M)$ is a highly non-tempered summand, which exists by Lemma 6.4.2. For such a representation, (**LIR**) is equivalent to the scalar equation $(\star)$ in Corollary 6.5.2. In this section, we check by hand the validity of the equation $(\star)$ for our case.

7.1. **Preliminaries.** Let $P = MN$ be a maximal parabolic subgroup of G, and let $\psi_M = \widehat{\phi}_M = \psi_{\mathrm{GL}} \oplus \psi_0$ be a co-tempered A-parameter for M. Write $\phi_M = \phi_{\mathrm{GL}} \oplus \phi_0$ so that $\widehat{\phi}_{\mathrm{GL}} = \psi_{\mathrm{GL}}$ and $\widehat{\phi}_0 = \psi_0$. By Lemma 6.2.1, we may assume that ϕ_{GL} is irreducible and conjugate-self-dual. Let ϕ (resp. ψ) be the L-parameter (resp. A-parameter) for G given by ϕ_M (resp. ψ_M).

First, we reduce the problem to the case of good parity.

Lemma 7.1.1. *Write*

$$\phi_0 = \phi_{0,\mathrm{bad}} \oplus \phi_{0,\mathrm{good}} \oplus {}^c\phi_{0,\mathrm{bad}}^\vee,$$

where $\phi_{0,\mathrm{good}}$ is the sum of irreducible conjugate-self-dual representations of the same type as ϕ_0, and $\phi_{0,\mathrm{bad}}$ is a sum of irreducible representations of other types. Set $\psi_0 = \widehat{\phi}_0$ and $\psi_{0,\mathrm{good}} = \widehat{\phi}_{0,\mathrm{good}}$. For $\pi_0 \in \Pi_{\psi_0}$, let $\pi_{0,\mathrm{good}} \in \Pi_{\psi_{0,\mathrm{good}}}$ be the representation determined by $\langle \cdot, \pi_0\rangle_{\psi_0} = \langle \cdot, \pi_{0,\mathrm{good}}\rangle_{\psi_{0,\mathrm{good}}}$ via the canonical identification $A_{\psi_0} = A_{\psi_{0,\mathrm{good}}}$. Then we have

$$\phi_{\psi_0} - \phi_{\psi_{0,\mathrm{good}}} = \phi_{\pi_0} - \phi_{\pi_{0,\mathrm{good}}}.$$

On the other hand, if we take ψ_{good} and π_{good} similarly, then $\langle s_u, \pi\rangle_\psi = \langle s_u, \pi_{\mathrm{good}}\rangle_{\psi_{\mathrm{good}}}$. In particular, the equation $(\star)$ holds for $\pi_{\mathrm{GL}} \boxtimes \pi_0$ if and only if $(\star)$ holds for $\pi_{\mathrm{GL}} \boxtimes \pi_{0,\mathrm{good}}$.

Proof. By applying Aubert duality to the tempered case, we see that

$$\pi_0 = \tau_{0,\mathrm{bad}} \rtimes \pi_{0,\mathrm{good}},$$

where $\tau_{0,\mathrm{bad}}$ is the representation of a general linear group corresponding to $\psi_{0,\mathrm{bad}} = \widehat{\phi}_{0,\mathrm{bad}}$. Since it is an irreducible parabolic induction, by [Tad2, Proposition 1.3], one can describe the Langlands data of π_0 using the ones of $\pi_{0,\mathrm{good}}$ similar to Definition 6.4.1. It shows that $(\phi_{\pi_0} - \phi_{\pi_{0,\mathrm{good}}})(w,\alpha)$ is equal to

$$(\phi_{\psi_0} - \phi_{\psi_{0,\mathrm{good}}})(w,\alpha) = (\phi_{0,\mathrm{bad}} \oplus \phi_{0,\mathrm{bad}}^\vee)\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right)$$

for $(w,\alpha) \in W_E \times \mathrm{SL}_2(\mathbb{C})$.

On the other hand, note that $\langle \cdot, \pi\rangle_\psi = \langle \cdot, \pi_{\mathrm{good}}\rangle_{\psi_{\mathrm{good}}}$ via the canonical identification $A_\psi = A_{\psi_{\mathrm{good}}}$. Since $s_u \in A_\psi$ is the same as $s_u \in A_{\psi_{\mathrm{good}}}$ via this identification, we conclude that $\langle s_u, \pi\rangle_\psi = \langle s_u, \pi_{\mathrm{good}}\rangle_{\psi_{\mathrm{good}}}$. This implies the last assertion. □

Hereafter, we always assume that ϕ_0 is of good parity, i.e., $\phi_0 = \phi_{0,\mathrm{good}}$.

To show the equation $(\star)$, we need to know the pairing $\langle s, \pi\rangle_\psi$ for $\pi \in \Pi_\psi$. Since $\psi = \widehat{\phi}$ is co-tempered, this pairing is defined such that the statement of Corollary 4.4.5 holds. We write down this corollary explicitly.

Lemma 7.1.2. *Let $\phi = \oplus_{j=1}^t \rho_j \boxtimes S_{d_j}$ be a tempered L-parameter for G of good parity, where ρ_j is an irreducible representation of W_F. Set $\psi = \widehat{\phi}$. Then for $\pi \in \Pi_\psi$ and $\sigma = \hat{\pi} \in \Pi_\phi$, we have*

$$\frac{\langle e(\rho,1,d), \pi\rangle_\psi}{\langle e(\rho,d,1), \sigma\rangle_\phi} = \begin{cases} 1 & \text{if } d \equiv 0 \bmod 2, \\ -(-1)^{|\{j \,|\, \rho_j \cong \rho\}|} & \text{if } d \equiv 1 \bmod 2. \end{cases}$$

In particular, if $d \equiv d' \bmod 2$, then we have

$$\frac{\langle e(\rho,1,d), \pi\rangle_\psi}{\langle e(\rho,1,d'), \pi\rangle_\psi} = \frac{\langle e(\rho,d,1), \sigma\rangle_\phi}{\langle e(\rho,d',1), \sigma\rangle_\phi}.$$

Proof. If $s = e(\rho,d,1) \in A_\phi$, then $\phi_- = \rho \boxtimes S_d$ and $\phi_+ = \phi - \phi_-$. By considering the cases where

- d is even;
- d is odd and $|\{j \,|\, \rho_j \cong \rho\}|$ is odd;
- d is odd and $|\{j \,|\, \rho_j \cong \rho\}|$ is even

separately, we can see that

$$(-1)^{r(\phi)-r(\phi_+)-r(\phi_-)} = \begin{cases} 1 & \text{if } d \equiv 0 \bmod 2, \\ -(-1)^{|\{j \,|\, \rho_j \cong \rho\}|} & \text{if } d \equiv 1 \bmod 2. \end{cases}$$

See Section 5.3 for the computation of $r(\phi)$. Hence the claim follows from Corollary 4.4.5. □

We write $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$. Then $s_u = e(\rho_{\mathrm{GL}}, 1, 2\alpha+1) \in A_\psi$ if ϕ_{GL} is of the same type as ϕ_0. We compute $\langle s_u, \pi\rangle_\psi$ for a highly non-tempered summand $\pi \subset I_P(\pi_M)$.

Lemma 7.1.3. *Suppose that $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ is of the same type as ϕ_0. Set $\psi_M = \widehat{\phi}_M$ with $\phi_M = \phi_{\mathrm{GL}} \oplus \phi_0$. For $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0 \in \Pi_{\psi_M}$, let $\pi \subset I_P(\pi_M)$ be a highly non-tempered summand, and let $\mathcal{I}(\mathfrak{m}) \rtimes \tau$ be its standard module.*

(1) *If ϕ_0 contains $\rho_{\mathrm{GL}} \boxtimes S_d$ with $d \geq 2\alpha+1$, then*
$$\langle s_u, \pi\rangle_\psi = \langle e(\rho_{\mathrm{GL}}, 1, d_+), \pi_0\rangle_{\psi_0},$$
where $d_+ = \min\{d \geq 2\alpha+1 \,|\, \rho_{\mathrm{GL}} \boxtimes S_d \subset \phi_0\}$.

(2) *Otherwise, if ϕ_0 contains $\rho_{\mathrm{GL}} \boxtimes S_d$ with $d < 2\alpha+1$, then*
$$\langle s_u, \pi\rangle_\psi = \langle e(\rho_{\mathrm{GL}}, 1, d_-), \pi_0\rangle_{\psi_0},$$
where $d_- = \max\{d < 2\alpha+1 \,|\, \rho_{\mathrm{GL}} \boxtimes S_d \subset \phi_0\}$. Here, when $2\alpha+1 \equiv 0 \bmod 2$, we formally understand that $\rho_{\mathrm{GL}} \boxtimes S_0 \subset \phi_0$ and $\langle e(\rho_{\mathrm{GL}}, 1, 0), \pi_0\rangle_{\psi_0} = 1$.

(3) *Otherwise, i.e., if $2\alpha+1 \equiv 1 \bmod 2$ and if ϕ_0 does not contain $\rho_{\mathrm{GL}} \boxtimes S_d$ for any $d \geq 1$, then there are precisely two choices of π, and $\langle s_u, \pi\rangle_\psi$ can take any values in $\{\pm 1\}$ depending on this choice.*

Proof. By Aubert duality, it is enough to show certain statements in the tempered case as follows.

Write $\hat{\pi}_M = \sigma_M = \sigma_{\mathrm{GL}} \otimes \sigma_0 \in \Pi_{\phi_M}$ and $\hat{\pi} = \sigma$. Recall that if $\mathcal{I}(\mathfrak{m}_0) \rtimes \tau_0$ is the standard module of π_0, then the standard module of π is $\mathcal{I}(\mathfrak{m}) \rtimes \tau$, where
$$\mathfrak{m} = \mathfrak{m}_0 + 2[\alpha, \alpha]_{\rho_{\mathrm{GL}}} + 2[\alpha-1, \alpha-1]_{\rho_{\mathrm{GL}}} + \cdots + 2[\kappa, \kappa]_{\rho_{\mathrm{GL}}}$$
with $\kappa \in \{1, \frac{1}{2}\}$ such that $\alpha - \kappa \in \mathbb{Z}$. By [AG2, Lemma 2.2], we have
$$\pi \hookrightarrow {}^cL(\mathfrak{m})^\vee \rtimes \tau, \quad \pi_0 \hookrightarrow {}^cL(\mathfrak{m}_0)^\vee \rtimes \tau_0,$$
where $L(\mathfrak{m})$ (resp. $L(\mathfrak{m}_0)$) is the Langlands quotient of $\mathcal{I}(\mathfrak{m})$ (resp. $\mathcal{I}(\mathfrak{m}_0)$). Set $d = \dim(\rho_{\mathrm{GL}})$. For $k > 0$, we denote by P_{dk} (resp. $P_{dk,0}$) the standard maximal parabolic subgroup of G (resp. G_0) with Levi subgroup of the form $\mathrm{GL}_{dk}(F) \times G'$ (resp. $\mathrm{GL}_{dk}(F) \times G'_0$).

Suppose that we are in the case (1). If $d_+ = 2\alpha+1$, then $\sigma = I_P(\sigma_M)$ is irreducible and $\langle e(\rho_{\mathrm{GL}}, 2\alpha+1, 1), \sigma\rangle_\phi = \langle e(\rho_{\mathrm{GL}}, d_+, 1), \sigma_0\rangle_{\phi_0}$. If $d_+ > 2\alpha+1$, then by Theorem C.3.3, we have
$$\mathrm{Jac}_{P_{dkl,0}}(\sigma_0) \geq (\rho_{\mathrm{GL}}|\cdot|^{\frac{d_+-1}{2}})^k \times \cdots \times (\rho_{\mathrm{GL}}|\cdot|^{\alpha+1})^k \otimes (\text{nonzero}) \geq 0$$
in the appropriate Grothendieck group with k being the multiplicity of $\rho_{\mathrm{GL}} \boxtimes S_{d_+}$ in ϕ_0, and $l = \frac{d_+-1}{2} - \alpha$. By a property of Aubert duality ([Au, Théorème 1.7 (2)]), Tadić's formula ([Tad1, Theorems 5.4, 6.5], [Ban, Theorem 7.3]) and Casselman's criterion ([Kon2, Lemma 2.4]), we see that
$$\mathrm{Jac}_{R_{dkl,0}}({}^cL(\mathfrak{m}_0)^\vee) \geq (\rho_{\mathrm{GL}}|\cdot|^{-\frac{d_+-1}{2}})^k \times \cdots \times (\rho_{\mathrm{GL}}|\cdot|^{-(\alpha+1)})^k \otimes (\text{nonzero}) \geq 0,$$
where $R_{dkl,0}$ is a suitable standard maximal parabolic subgroup of a general linear group. Then by the definition of $\mathfrak{m}$ together with [LMí, Theorem 5.11], we have
$$\mathrm{Jac}_{R_{dkl}}({}^cL(\mathfrak{m})^\vee) \geq (\rho_{\mathrm{GL}}|\cdot|^{-\frac{d_+-1}{2}})^k \times \cdots \times (\rho_{\mathrm{GL}}|\cdot|^{-(\alpha+1)})^k \otimes (\text{nonzero}) \geq 0$$
with analogous notations. This implies that
$$\mathrm{Jac}_{P_{dkl}}(\sigma) \geq (\rho_{\mathrm{GL}}|\cdot|^{\frac{d_+-1}{2}})^k \times \cdots \times (\rho_{\mathrm{GL}}|\cdot|^{\alpha+1})^k \otimes (\text{nonzero}) \geq 0$$

in the appropriate Grothendieck group. By Theorem C.3.3, this happens exactly when

$$\langle e(\rho_{\mathrm{GL}}, 2\alpha+1, 1), \sigma\rangle_\phi = \langle e(\rho_{\mathrm{GL}}, d_+, 1), \sigma_0\rangle_{\phi_0}.$$

By a similar argument, if we are in the cases (2) or (3), we have

$$\mathrm{Jac}_{P_{2dl}}(\sigma) \geq (\rho_{\mathrm{GL}}|\cdot|^{\alpha})^2 \times \cdots \times (\rho_{\mathrm{GL}}|\cdot|^{\frac{d_-+1}{2}})^2 \otimes (\text{nonzero}) \geq 0$$

in the appropriate Grothendieck group with $l = \alpha - \frac{d_- - 1}{2}$. Here, in the case (3), we set $d_- = 1$. By Theorem C.3.3, this occurs exactly when we are in the case (3), or

$$\langle e(\rho_{\mathrm{GL}}, 2\alpha+1, 1), \sigma\rangle_\phi = \langle e(\rho_{\mathrm{GL}}, d_-, 1), \sigma_0\rangle_{\phi_0}.$$

Hence we obtain (1)–(3) by Lemma 7.1.2. □

7.2. **Strategy of the proof.** We will prove ($\star$) in Corollary 6.5.2 for $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0 \in \Pi_{\psi_M}$ with $\psi_M = \widehat{\phi}_M$ a co-tempered A-parameter. Notice that the left-hand side of ($\star$) involves the L-parameter ϕ_{π_0} of π_0. In general, it is very difficult to list ϕ_{π_0} for $\pi_0 \in \Pi_{\psi_0}$. Instead of computing ϕ_{π_0} explicitly, we will give an inductive argument as follows.

The initial case is where π_0 is *almost supercuspidal*, which is defined as follows.

Definition 7.2.1. *We say that an irreducible representation π_0 of G_0 is* almost supercuspidal *if the following condition holds for every maximal parabolic subgroup P_0 of G_0. If $\mathrm{Jac}_{P_0}(\pi_0)$ contains an irreducible subquotient of the form $\rho \boxtimes \sigma$ with ρ a supercuspidal representation of $\mathrm{GL}_k(E)$, then ρ is unitary.*

The assumption that π_0 is almost supercuspidal implies the following strong properties.

Lemma 7.2.2. *Let $\psi_0 = \widehat{\phi}_0$ be a co-tempered A-parameter of good parity for a classical group G_0. Suppose that $\pi_0 \in \Pi_{\psi_0}$ is almost supercuspidal.*

(1) *The following conditions hold:*
- *If we denote the multiplicity of $\rho \boxtimes S_d$ in ϕ_0 by $m_{\phi_0}(\rho, d)$, then $m_{\phi_0}(\rho, d) \leq 1$ for any $d \geq 2$;*
- *if $\rho \boxtimes S_d \subset \phi_0$ with $d > 2$, then $\rho \boxtimes S_{d-2} \subset \phi_0$ and*
$$\langle e(\rho, 1, d), \pi_0\rangle_{\psi_0} = -\langle e(\rho, 1, d-2), \pi_0\rangle_{\psi_0};$$
- *if $\rho \boxtimes S_2 \subset \phi_0$, then*
$$\langle e(\rho, 1, 2), \pi_0\rangle_{\psi_0} = -1.$$

(2) *Let $\{\rho'_1, \ldots, \rho'_r\}$ be the set of irreducible bounded representations of W_E appearing in ϕ_0 with even multiplicity, and set*
$$2y_i + 1 = \max\{d \geq 1 \,|\, \rho'_i \boxtimes S_d \subset \phi_0\}.$$
Then the L-parameter ϕ_{π_0} of π_0 is given by
$$\phi_{\pi_0} = \phi_0 - \bigoplus_{i=1}^{r} \rho'_i \boxtimes (S_1 \oplus S_{2y_i+1}) \oplus \bigoplus_{i=1}^{r} \rho'_i(|\cdot|^{\frac{y_i}{2}} \oplus |\cdot|^{-\frac{y_i}{2}}) \boxtimes S_{y_i+1}.$$

(3) *Suppose that* $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ *is of the same type as* ϕ_0*, and that* $2\alpha + 1 \equiv 1 \bmod 2$*. Set* $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0$ *and let* $\pi \subset I_P(\pi_M)$ *be a highly non-tempered summand. We denote by* $\mathcal{I}(\mathfrak{m}) \rtimes \tau$ *the standard module of* π*. Then*

$$\langle e(\rho_{\mathrm{GL}}, 1, 1), \tau\rangle_{\phi_\tau} = \langle e(\rho_{\mathrm{GL}}, 1, d_0), \pi\rangle_{\psi},$$

where $\psi = \widehat{\phi}$ *is the A-parameter for G given by the dual of* $\phi = \phi_{\mathrm{GL}} \oplus \phi_0 \oplus {}^c\phi_{\mathrm{GL}}^\vee$*, and* $d_0 = \max\{d \geq 1 \,|\, \rho_{\mathrm{GL}} \boxtimes S_d \subset \phi\}$*.*

Proof. (1) Recall that $\hat{\pi}_0$ is tempered since it is in Π_{ϕ_0}. Since π_0 is almost supercuspidal, so is $\hat{\pi}_0$. Hence by Corollary C.3.5, we obtain several properties of $m_{\phi_0}(\rho, d)$ and $\langle \cdot, \hat{\pi}_0\rangle_{\phi_0}$. Then Lemma 7.1.2 implies the desired properties of $\langle \cdot, \pi_0\rangle_{\psi_0}$.

(2) As in [AM, Proposition 5.4], Aubert duality together with Theorem C.3.3 and Remark C.3.6 shows that if we write $y_1 \geq \cdots \geq y_t > 0 = y_{t+1} = \cdots = y_r$, then the standard module of π_0 is

$$\mathcal{I}([0, y_1]_{\rho'_1} + \cdots + [0, y_t]_{\rho'_t}) \rtimes \tau_0$$

where $\tau_0 \in \Pi_{\phi_{\tau_0}}$ with

$$\phi_{\tau_0} = \phi_0 - \bigoplus_{i=1}^{t} \rho'_i \boxtimes (S_1 \oplus S_{2y_i+1})$$

and

$$\langle e(\rho, d, 1), \tau_0\rangle_{\phi_{\tau_0}} = \begin{cases} -\langle e(\rho, d, 1), \hat{\pi}_0\rangle_{\phi_0} & \text{if } \rho \in \{\rho'_1, \dots, \rho'_r\}, \\ \langle e(\rho, d, 1), \hat{\pi}_0\rangle_{\phi_0} & \text{otherwise.} \end{cases}$$

Here, we notice that [AM, Proposition 5.4] does not use Mœglin's construction of A-packets. From this, we obtain the description for ϕ_{π_0} in (2).

(3) If $\pi \subset I_P(\pi_M)$ is a highly non-tempered summand, by Theorems C.3.3 and C.3.4, we see that

$$\hat{\pi} \hookrightarrow (\rho_{\mathrm{GL}}|\cdot|^{\alpha})^2 \times \cdots \times (\rho_{\mathrm{GL}}|\cdot|^{1})^2 \rtimes \hat{\pi}',$$

where $\hat{\pi}' \in \Pi_{\phi'}$ with $\phi' = \phi_0 \oplus \rho_{\mathrm{GL}}^{\oplus 2}$, and $\langle \cdot, \hat{\pi}'\rangle_{\phi'}$ is determined by

$$\langle \cdot, \hat{\pi}'\rangle_{\phi'}|_{\mathcal{A}_{\phi_0}} = \langle \cdot, \hat{\pi}_0\rangle_{\phi_0},$$
$$\langle e(\rho_{\mathrm{GL}}, 1, 1), \hat{\pi}'\rangle_{\phi'} = \langle e(\rho_{\mathrm{GL}}, d'_0, 1), \hat{\pi}\rangle_{\phi}$$

with $d'_0 = \min\{d \geq 1 \,|\, \rho_{\mathrm{GL}} \boxtimes S_d \subset \phi\}$. Hence π' is almost supercuspidal, and if we denote by $\mathcal{I}(\mathfrak{m}') \rtimes \tau'$ its standard module, then $\mathfrak{m} = \mathfrak{m}' + 2([\alpha, \alpha]_{\rho_{\mathrm{GL}}} + \cdots + [1, 1]_{\rho_{\mathrm{GL}}})$ and $\tau' = \tau$. In particular, by assertion (2), we have

$$\langle e(\rho_{\mathrm{GL}}, 1, 1), \tau\rangle_{\phi_\tau} = \begin{cases} -\langle e(\rho_{\mathrm{GL}}, 1, 1), \hat{\pi}'\rangle_{\phi'} & \text{if } m_{\phi'}(\rho_{\mathrm{GL}}) \equiv 0 \bmod 2, \\ \langle e(\rho_{\mathrm{GL}}, 1, 1), \hat{\pi}'\rangle_{\phi'} & \text{if } m_{\phi'}(\rho_{\mathrm{GL}}) \equiv 1 \bmod 2, \end{cases}$$

where $m_{\phi'}(\rho_{\mathrm{GL}})$ is the multiplicity of ρ_{GL} of ϕ'.

If $\rho_{\mathrm{GL}} \boxtimes S_d \not\subset \phi_0$ for any $d \geq 1$, then by Lemma 7.1.2 together with $d_0' = d_0$, we have

$$\langle e(\rho_{\mathrm{GL}}, 1, 1), \tau \rangle_{\phi_\tau} = -\langle e(\rho_{\mathrm{GL}}, 1, 1), \hat{\pi}' \rangle_{\phi'} = -\langle e(\rho_{\mathrm{GL}}, d_0', 1), \hat{\pi} \rangle_{\phi} = \langle e(\rho_{\mathrm{GL}}, 1, d_0), \pi \rangle_{\psi}.$$

From now we assume that $\rho_{\mathrm{GL}} \boxtimes S_d \subset \phi_0$ for some $d \geq 1$. Then by assertion (1), we have $\rho_{\mathrm{GL}} \boxtimes (S_1 + S_3 + \cdots + S_{d_0}) \subset \phi_0$, and $m_{\phi_0}(\rho_{\mathrm{GL}} \boxtimes S_d) = 1$ for $3 \leq d \leq d_0$ with d odd. If we write $m = m_{\phi_0}(\rho_{\mathrm{GL}})$, then $m_{\phi'}(\rho_{\mathrm{GL}}) = m + 2$, and by Lemma 7.1.2 together with $d_0' = 1$,

$$\begin{aligned}
\langle e(\rho_{\mathrm{GL}}, 1, 1), \tau \rangle_{\phi_\tau} &= (-1)^{m-1} \langle e(\rho_{\mathrm{GL}}, 1, 1), \hat{\pi}' \rangle_{\phi'} \\
&= (-1)^{m-1} \langle e(\rho_{\mathrm{GL}}, 1, 1), \hat{\pi} \rangle_{\phi} \\
&= (-1)^{m-1} \cdot (-1)^{\frac{d_0-1}{2}+m-1} \langle e(\rho_{\mathrm{GL}}, 1, 1), \pi \rangle_{\psi} \\
&= \langle e(\rho_{\mathrm{GL}}, 1, d_0), \pi \rangle_{\psi}.
\end{aligned}$$

This completes the proof of Lemma 7.2.2. □

By this lemma, we can compute all terms of $(\star)$ when π_0 is almost supercuspidal. The details are given in Section 7.3.

For the general case, suppose that π_0 is not almost supercuspidal. Then we will find a classical group G_0' with $\mathrm{rank}(G_0') < \mathrm{rank}(G_0)$, an A-parameter ψ_0' for G_0', and $\pi_0' \in \Pi_{\psi_0'}$ such that the difference

$$\phi_{\pi_0} - \phi_{\pi_0'}$$

is explicitly known (although we might not know ϕ_{π_0} nor $\phi_{\pi_0'}$ themselves). By the induction hypothesis, we can assume (**LIR**), equivalently equation $(\star)$, for $\pi'_{M'} = \pi_{\mathrm{GL}} \boxtimes \pi_0'$. Therefore, what we have to check is the equation

$$(\star\star) \qquad \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)} \right) \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0', \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0'}, \psi_E)} \right)^{-1} \Bigg|_{s=0} = \frac{\langle s_u, \pi \rangle_{\psi}}{\langle s_u, \pi' \rangle_{\psi'}},$$

where $\pi \subset I_P(\pi_M)$ and $\pi' \subset I_{P'}(\pi'_{M'})$ are highly non-tempered summands.

To give π_0', we consider Jacquet modules of the tempered representation $\hat{\pi}_0$.

Lemma 7.2.3. *Let $\psi_0 = \widehat{\phi}_0$ be a co-tempered A-parameter of good parity for a classical group G_0. Suppose that $\pi_0 \in \Pi_{\psi_0}$ is not almost supercuspidal. Then one can find an irreducible conjugate-self-dual representation ρ_1 of W_E and positive half-integers $x \leq y$ with $x \equiv y \bmod \mathbb{Z}$ such that*

1. $\rho_1 \boxtimes (S_{2x+1} \oplus S_{2x+3} \oplus \cdots \oplus S_{2y+1}) \subset \phi_0$;
2. *if we denote the multiplicity of $\rho \boxtimes S_d$ in ϕ_0 by $m_{\phi_0}(\rho, d)$, then*
$$m_{\phi_0}(\rho_1, 2i+1) = \begin{cases} 1 & \text{if } x < i \leq y, \\ 0 & \text{if } i > y \end{cases}$$
for $i \in (1/2)\mathbb{Z}$ with $i \equiv x \bmod \mathbb{Z}$;
3. $\langle e(\rho_1, 1, 2i+1), \pi_0 \rangle_{\psi_0} = -\langle e(\rho_1, 1, 2i-1), \pi_0 \rangle_{\psi_0}$ *for $x < i \leq y$ with $i \equiv x \bmod \mathbb{Z}$;*
4. *one of the following holds:*

(a) $\rho_1 \boxtimes S_{2x-1} \not\subset \phi_0$, *or* $\rho_1 \boxtimes S_{2x-1} \subset \phi_0$ *and* $\langle e(\rho_1, 1, 2x+1), \pi_0 \rangle_{\psi_0} = \langle e(\rho_1, 1, 2x-1), \pi_0 \rangle_{\psi_0}$;
(b) $\rho_1 \boxtimes S_{2x-1} \subset \phi_0$, $\langle e(\rho_1, 1, 2x+1), \pi_0 \rangle_{\psi_0} = -\langle e(\rho_1, 1, 2x-1), \pi_0 \rangle_{\psi_0}$ *and* $m = m_{\phi_0}(\rho_1, 2x+1) \geq 1$ *is odd;*
(c) $\rho_1 \boxtimes S_{2x-1} \subset \phi_0$, $\langle e(\rho_1, 1, 2x+1), \pi_0 \rangle_{\psi_0} = -\langle e(\rho_1, 1, 2x-1), \pi_0 \rangle_{\psi_0}$ *and* $m = m_{\phi_0}(\rho_1, 2x+1) > 1$ *is even.*

Proof. Since π_0 is not almost supercuspidal, in the appropriate Grothendieck group, $\mathrm{Jac}_P(\pi_0)$ contains a representation of the form $\rho_1|\cdot|^{-x} \otimes \pi_0'$ for some parabolic subgroup P, some irreducible bounded representation ρ_1 of W_E and $x > 0$. Since ϕ_0 is of good parity, ρ_1 must be conjugate-self-dual. Fix such a ρ_1, and take the maximal x with this condition. Then by Theorems C.3.3 and C.3.4, we have $\rho_1 \boxtimes S_{2x+1} \subset \phi_0$. Moreover, if we set

$$2y + 1 = \max\{d \mid \rho_1 \boxtimes S_d \subset \phi_0\},$$

the same theorems together with Lemma 7.1.2 imply the desired conditions (1)–(4). $\square$

We will treat the cases (a), (b) and (c) in Sections 7.4, 7.5 and 7.6, respectively. In the cases (a) and (b), the new A-parameter ψ_0' is also co-tempered. However, in the case (c), ψ_0' is no longer co-tempered, and it is more difficult to compute $\langle s_u, \pi' \rangle_{\psi'}$. This is where Corollary 4.5.3 will be used, and where we have to separate (**A-LIR**) and (**LIR**).

7.3. **The initial case.** Let $\psi_M = \widehat{\phi}_M$ be a co-tempered A-parameter for M, where $\phi_M = \phi_{\mathrm{GL}} \oplus \phi_0$. Fix $\pi_0 \in \Pi_{\psi_0}$. In this subsection, we assume that $\pi_0 \in \Pi_{\psi_0}$ is almost supercuspidal (see Definition 7.2.1). Write

$$\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}, \quad \phi_0 = \bigoplus_{i=1}^{t} \rho_i \boxtimes S_{2\beta_i+1},$$

where ρ_i is an irreducible conjugate-self-dual representation of W_E. As in Lemma 7.2.2, write $\{\rho_1', \dots, \rho_r'\}$ for the subset of $\{\rho_1, \dots, \rho_t\}$ consisting of ρ_i's which appear in ϕ_0 with even multiplicities, and set $2y_i + 1 = \max\{d \geq 1 \mid \rho_i' \boxtimes S_d \subset \phi_0\}$.

Let us check the equation $(\star)$ for π. By Lemma 7.2.2 (2), the left-hand side of $(\star)$ is

$$\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)} \cdot \prod_{i=1}^{r} \frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes (\rho_i' \boxtimes (S_1 \oplus S_{2y_i+1})), \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes (\rho_i'(|\cdot|^{\frac{y_i}{2}} \oplus |\cdot|^{-\frac{y_i}{2}}) \boxtimes S_{y_i+1}), \psi_E)}\Bigg|_{s=0}.$$

In the rest of this section, we write $\prod_{-\alpha \leq a \leq \alpha}$ for the product with respect to $a = -\alpha, -\alpha+1, \dots, \alpha$ (even if $\alpha \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$). First, we compute the quotient involving ρ_i'. To simplify the notation, we drop the subscript i. Then by the formulas for local factors in Section A.1, we have

$$\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes (\rho' \boxtimes (S_1 \oplus S_{2y+1})), \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes (\rho'(|\cdot|^{\frac{y}{2}} \oplus |\cdot|^{-\frac{y}{2}}) \boxtimes S_{y+1}), \psi_E)}$$

$$= \prod_{-\alpha \le a \le \alpha} \frac{\gamma_A(s, ({}^c\rho_{\mathrm{GL}} \otimes \rho')|\cdot|^a \boxtimes (S_1 \oplus S_{2y+1}), \psi_E)}{\gamma_A(s, ({}^c\rho_{\mathrm{GL}} \otimes \rho')(|\cdot|^{a+\frac{y}{2}} \oplus |\cdot|^{a-\frac{y}{2}}) \boxtimes S_{y+1}, \psi_E)} = 1.$$

On the other hand, using the notations and the formulas for local factors in Section A.1, we have

$$\begin{aligned}
&\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)} \\
&= \prod_{i=1}^{t} \prod_{-\alpha \le a \le \alpha} \frac{\gamma_A(s, {}^c\rho_{\mathrm{GL}}|\cdot|^a \otimes (\rho_i|\cdot|^{\beta_i} \oplus \cdots \oplus \rho_i|\cdot|^{-\beta_i}), \psi_E)}{\gamma_A(s, ({}^c\rho_{\mathrm{GL}}|\cdot|^a \otimes \rho_i) \boxtimes S_{2\beta_i+1}, \psi_E)} \\
&= \prod_{i=1}^{t} \prod_{-\alpha \le a \le \alpha} \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)} (-q_E^{-(\frac{1}{2}-s-\mu-a)})^{2\beta_i} \prod_{-\beta_i \le b \le \beta_i - 1} \frac{\zeta_E(s+\mu+a+b+1)}{\zeta_E(s+\mu+a+b)} \\
&= \prod_{i=1}^{t} \prod_{-\alpha \le a \le \alpha} \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)} (-q_E^{-(\frac{1}{2}-s-\mu-a)})^{2\beta_i} \frac{\zeta_E(s+\mu+a+\beta_i)}{\zeta_E(s+\mu+a-\beta_i)} \\
&= \prod_{i=1}^{t} \prod_{-\alpha \le a \le \alpha} \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)} (-q_E^{-(\frac{1}{2}-s-\mu-a)})^{2\beta_i} \frac{\zeta_E(s+\mu-a+\beta_i)}{\zeta_E(s+\mu+a-\beta_i)}.
\end{aligned}$$

Lemma 7.3.1. *For $\mu \in \mathbb{C}/2\pi\sqrt{-1}(\log q_E)^{-1}\mathbb{Z}$ and $a, \beta \in (1/2)\mathbb{Z}$, set*

$$f_{\mu,a,\beta}(s) = (-q_E^{-(\frac{1}{2}-s-\mu-a)})^{2\beta} \frac{\zeta_E(s+\mu-a+\beta)}{\zeta_E(s+\mu+a-\beta)}.$$

(1) *If $\mathrm{Re}(\mu) = 0$ and $\mu \ne -\mu$, then*

$$f_{\mu,a,\beta}(s) f_{-\mu,a,\beta}(s)|_{s=0} = q_E^{2a(2\beta-1)}.$$

(2) *Let $\mu_0 \in \mathbb{C}/2\pi\sqrt{-1}(\log q_E)^{-1}\mathbb{Z}$ be the unique nonzero element such that $\mu_0 = -\mu_0$. Then*

$$f_{\mu_0,a,\beta}(s)|_{s=0} = q_E^{a(2\beta-1)}.$$

(3) *If $\mu = 0$, then*

$$f_{0,a,\beta}(s)|_{s=0} = \begin{cases} (-1)^{2\beta} q_E^{a(2\beta-1)} & \text{if } a = \beta, \\ (-1)^{2\beta+1} q_E^{a(2\beta-1)} & \text{if } a \ne \beta. \end{cases}$$

Proof. If $\mathrm{Re}(\mu) = 0$ and $\mu \ne -\mu$, then

$$\begin{aligned}
f_{\mu,a,\beta}(s) f_{-\mu,a,\beta}(s)|_{s=0} &= q_E^{-2(1-2a)\beta} \frac{1 - q_E^{-\mu-a+\beta}}{1 - q_E^{\mu+a-\beta}} \frac{1 - q_E^{\mu-a+\beta}}{1 - q_E^{-\mu+a-\beta}} \\
&= q_E^{-2(1-2a)\beta} \cdot (-q_E^{-\mu-a+\beta}) \cdot (-q_E^{\mu-a+\beta}) = q_E^{2a(2\beta-1)}.
\end{aligned}$$

Similarly, if $\mu = \mu_0$, then $q_E^{-\mu_0} = -1$ so that

$$f_{\mu_0,a,\beta}(s)|_{s=0} = q_E^{-(1-2a)\beta} \frac{1+q_E^{-a+\beta}}{1+q_E^{a-\beta}} = q_E^{a(2\beta-1)}.$$

Finally, suppose that $\mu = 0$. If $a \neq \beta$, then

$$f_{0,a,\beta}(s)|_{s=0} = (-1)^{2\beta} q_E^{-(1-2a)\beta} \frac{1-q_E^{-a+\beta}}{1-q_E^{a-\beta}} = (-1)^{2\beta+1} q_E^{a(2\beta-1)}.$$

On the other hand, if $a = \beta$, then

$$f_{0,a,\beta}(s)|_{s=0} = (-1)^{2\beta} q_E^{-(1-2s-2a)a} \frac{1-q_E^{-s}}{1-q_E^{-s}}\bigg|_{s=0} = (-1)^{2\beta} q_E^{a(2\beta-1)}.$$

This completes the proof. □

We have proven that

$$\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)}\bigg|_{s=0} = \prod_{i=1}^{t} \prod_{-\alpha \leq a \leq \alpha} \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)} f_{\mu,a,\beta_i}(s)|_{s=0}.$$

Note that for $\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)$, if $\mu \neq -\mu$, then $-\mu$ appears in $X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)$ with the same multiplicity as μ. Since $\prod_{-\alpha \leq a \leq \alpha} q_E^{a(2\beta_i-1)} = 1$, by Lemma 7.3.1, only $\mu = 0$ can contribute. Note that $0 \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)$ if and only if $\rho_i \cong {}^c\rho_{\mathrm{GL}}^\vee \cong \rho_{\mathrm{GL}}$. In this case, 0 appears in $X({}^c\rho_{\mathrm{GL}} \otimes \rho_i)$ with multiplicity one. Hence

$$\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)}\bigg|_{s=0} = \prod_{\substack{1 \leq i \leq t \\ \rho_i \cong \rho_{\mathrm{GL}}}} \prod_{-\alpha \leq a \leq \alpha} f_{0,a,\beta_i}(s)|_{s=0},$$

and by Lemma 7.3.1, we have

$$\prod_{-\alpha \leq a \leq \alpha} f_{0,a,\beta_i}(s)|_{s=0} = \begin{cases} (-1)^{(2\alpha+1)(2\beta_i+1)-1} & \text{if } \beta_i \leq \alpha,\ \beta_i \equiv \alpha \bmod \mathbb{Z}, \\ (-1)^{(2\alpha+1)(2\beta_i+1)} & \text{otherwise.} \end{cases}$$

Suppose first that $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ is not of the same type as ϕ_0. Then if $\rho_i \cong \rho_{\mathrm{GL}}$, then $\beta_i \not\equiv \alpha \bmod \mathbb{Z}$ so that $(2\alpha+1)(2\beta_i+1) \in 2\mathbb{Z}$. Hence

$$\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)}\bigg|_{s=0} = \prod_{\substack{1 \leq i \leq t \\ \rho_i \cong \rho_{\mathrm{GL}}}} (-1)^{(2\alpha+1)(2\beta_i+1)} = 1.$$

Since $\langle s_u, \pi \rangle_\psi = 1$ in this case, we obtain the equation $(\star)$. In the rest of this subsection, we assume that $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ is of the same type as ϕ_0 so that $\beta_i \equiv \alpha \bmod \mathbb{Z}$ if $\rho_i \cong \rho_{\mathrm{GL}}$.

Suppose that $2\alpha + 1$ is even. Then we conclude that

$$\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)}\bigg|_{s=0} = (-1)^{|\{i \,|\, \rho_i \cong \rho_{\mathrm{GL}},\, \beta_i \leq \alpha\}|}.$$

If $\rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1} \subset \phi_0$, then

- $\rho_{\mathrm{GL}} \boxtimes (S_2 \oplus S_4 \oplus \cdots \oplus S_{2\alpha+1}) \subset \phi_0$; and
- $m_{\phi_0}(\rho_{\mathrm{GL}} \boxtimes S_{2x}) = 1$ and $\langle e(\rho_{\mathrm{GL}}, 1, 2x), \pi_0 \rangle_{\psi_0} = (-1)^x$ for $1 \leq x \leq \alpha + \frac{1}{2}$.

In particular,

$$\langle s_u, \pi \rangle_\psi = \langle e(\rho_{\mathrm{GL}}, 1, 2\alpha+1), \pi_0 \rangle_{\psi_0} = (-1)^{\alpha+\frac{1}{2}} = (-1)^{|\{i \,|\, \rho_i \cong \rho_{\mathrm{GL}},\, \beta_i \leq \alpha\}|}.$$

On the other hand, if $\rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1} \not\subset \phi_0$, then setting $d_- = \max\{d \geq 0 \,|\, \rho_{\mathrm{GL}} \boxtimes S_d \subset \phi_0\}$, we have

$$\langle s_u, \pi \rangle_\psi = \langle e(\rho_{\mathrm{GL}}, 1, d_-), \pi_0 \rangle_{\psi_0} = (-1)^{\frac{d_-}{2}} = (-1)^{|\{i \,|\, \rho_i \cong \rho_{\mathrm{GL}},\, \beta_i \leq \alpha\}|}.$$

Therefore, we obtain the equation $(\star)$ when $2\alpha + 1$ is even.

Next, we suppose that $2\alpha+1$ is odd. If $\rho_{\mathrm{GL}} \boxtimes S_d \not\subset \phi_0$ for any $d \geq 1$, then both sides of $(\star)$ are equal to 1. Hence we assume that $\rho_{\mathrm{GL}} \boxtimes S_d \subset \phi_0$ for some $d \geq 1$. Set

$$d_0 = \max\{d \geq 1 \,|\, \rho_{\mathrm{GL}} \boxtimes S_d \subset \phi_0\},$$

and write $d_0 = 2\beta_0 + 1$. If $\alpha > \beta_0$, then since $\langle s_u, \pi \rangle_\psi = \langle e(\rho_{\mathrm{GL}}, 1, d_0), \pi_0 \rangle_{\psi_0} = \langle e(\rho_{\mathrm{GL}}, 1, 1), \tau \rangle_{\phi_\tau}$, we have

$$\left.\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)}\right|_{s=0} = 1 = \frac{\langle s_u, \pi \rangle_\psi}{\langle e(\rho_{\mathrm{GL}}, 1, 1), \tau \rangle_{\phi_\tau}}.$$

If $\alpha \leq \beta_0$, then

$$\left.\frac{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \widehat{\phi}_0, \psi_E)}{\gamma_A(s, {}^c\widehat{\phi}_{\mathrm{GL}} \otimes \phi_0, \psi_E)}\right|_{s=0} = (-1)^{\beta_0 - \alpha}.$$

On the other hand, we note that

- $\rho_{\mathrm{GL}} \boxtimes (S_1 \oplus S_3 \oplus \cdots \oplus S_{2\beta_0+1}) \subset \phi_0$; and
- $\langle e(\rho_{\mathrm{GL}}, 1, 2x+1), \pi_0 \rangle_{\psi_0} = -\langle e(\rho, 1, 2x-1), \pi_0 \rangle_{\psi_0}$ for $1 \leq x \leq \beta_0$.

Hence

$$\frac{\langle s_u, \pi \rangle_\psi}{\langle e(\rho, 1, 1), \tau \rangle_{\phi_\tau}} = \frac{\langle e(\rho_{\mathrm{GL}}, 1, 2\alpha+1), \pi_0 \rangle_{\psi_0}}{\langle e(\rho_{\mathrm{GL}}, 1, 2\beta_0+1), \pi_0 \rangle_{\psi_0}} = (-1)^{\beta_0 - \alpha}.$$

Therefore, we obtain the equation $(\star)$ when $2\alpha + 1$ is odd. This completes the proof of $(\star)$ when $\pi_0 \in \Pi_{\psi_0}$ is almost supercuspidal.

7.4. **The inductive case (a).** Let ϕ_0 be a tempered L-parameter for G_0 of good parity, and set $\psi_0 = \widehat{\phi}_0$. Fix $\pi_0 \in \Pi_{\psi_0}$. In this subsection, we assume the conditions (1)–(3) and (4a) in Lemma 7.2.3.

In this case, by applying Theorem C.3.3 repeatedly, we see that

$$\Delta([x, y]_{\rho_1}) \times (\rho_1 |\cdot|^x)^{m-1} \rtimes \pi_0' \twoheadrightarrow \pi_0,$$

where $m = m_{\phi_0}(\rho_1, 2x+1) > 0$, and $\pi_0' \in \Pi_{\psi_0'}$ is characterized such that

- $\psi_0' = \widehat{\phi}_0'$ is a co-tempered A-parameter with

$$\phi_0' = \phi_0 - \rho_1 \boxtimes (S_{2x+1}^{\oplus m-1} \oplus S_{2y+1}) \oplus \rho_1 \boxtimes S_{2x-1}^{\oplus m};$$

- the character $\langle \cdot, \pi'_0 \rangle_{\psi'_0}$ is given by

$$\langle e(\rho, 1, 2i-1), \pi'_0 \rangle_{\psi'_0} = \begin{cases} \langle e(\rho_1, 1, 2i+1), \pi_0 \rangle_{\psi_0} & \text{if } \rho \cong \rho_1,\ x \leq i \leq y, \\ \langle e(\rho, 1, 2i-1), \pi_0 \rangle_{\psi_0} & \text{otherwise.} \end{cases}$$

Moreover, the L-parameters ϕ_{π_0} and $\phi_{\pi'_0}$ are related by

$$\phi_{\pi_0} = \phi_{\pi'_0} \oplus \rho_1(|\cdot|^{\frac{x+y}{2}} \oplus |\cdot|^{-\frac{x+y}{2}}) \boxtimes S_{y-x+1} \oplus (\rho_1(|\cdot|^{x} \oplus |\cdot|^{-x}) \boxtimes S_1)^{\oplus m-1}.$$

Set $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ and $\psi_{\mathrm{GL}} = \widehat{\phi}_{\mathrm{GL}}$. We denote the irreducible representation of $\mathrm{GL}_k(E)$ corresponding to ψ_{GL} by π_{GL}. Taking a suitable classical group G' (with $\mathrm{rank}(G') < \mathrm{rank}(G)$) and its maximal parabolic subgroup $P' = M'N'$, write $\psi'_{M'} = \psi_{\mathrm{GL}} \oplus \psi'_0$. By the induction hypothesis, we assume that the equation $(\star)$ holds for a highly non-tempered representation $\pi' \subset I_{P'}(\pi'_{M'})$ with $\pi'_{M'} = \pi_{\mathrm{GL}} \boxtimes \pi'_0$. Take a highly non-tempered representation $\pi \subset I_P(\pi_M)$ such that the tempered part τ of the Langlands data of π coincides with the one for π'.

By Lemma 7.1.3, we see that

$$\frac{\langle s_u, \pi \rangle_{\psi}}{\langle s_u, \pi' \rangle_{\psi'}} = \begin{cases} -1 & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x \leq \alpha < y, \alpha \equiv x \bmod \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

We will check the left-hand side of $(\star\star)$ is equal to the right-hand side of this equation.

Since $\gamma_A(s, \phi, \psi_E)$ is multiplicative, we have

$$\begin{aligned}
&\left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)} \right) \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi'_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi'_0}, \psi_E)} \right)^{-1} \\
&= \frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x+1}), \psi_E)^{m-1} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2y+1}), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x-1}), \psi_E)^m} \\
&\quad \times \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1 |\cdot|^{\epsilon \frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1 |\cdot|^{\epsilon x}, \psi_E)^{-(m-1)} \\
&= \prod_{-\alpha \leq a \leq \alpha} \prod_{-x \leq b \leq x} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^{a} \otimes \rho_1 |\cdot|^{b}, \psi_E)^{m-1} \\
&\quad \times \prod_{-\alpha \leq a \leq \alpha} \prod_{-y \leq b \leq y} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^{a} \otimes \rho_1 |\cdot|^{b}, \psi_E) \\
&\quad \times \prod_{-\alpha \leq a \leq \alpha} \prod_{-x+1 \leq b \leq x-1} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^{a} \otimes \rho_1 |\cdot|^{b}, \psi_E)^{-m} \\
&\quad \times \prod_{-\alpha \leq a \leq \alpha} \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^{a} \otimes \rho_1 |\cdot|^{\epsilon \frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1} \\
&\quad \times \prod_{-\alpha \leq a \leq \alpha} \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^{a} \otimes \rho_1 |\cdot|^{\epsilon x}, \psi_E)^{-(m-1)}
\end{aligned}$$

$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} q_E^{-(1-2s-2\mu-2a)(y-x)} \prod_{\substack{-y \leq b \leq -x-1 \\ \text{or} \\ x \leq b \leq y-1}} \frac{\zeta_E(s+\mu+a+b+1)}{\zeta_E(s+\mu+a+b)}$$

$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} q_E^{-(1-2s-2\mu-2a)(y-x)} \frac{\zeta_E(s+\mu+a-x)}{\zeta_E(s+\mu+a-y)} \frac{\zeta_E(s+\mu+a+y)}{\zeta_E(s+\mu+a+x)}$$

$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} q_E^{-(1-2s-2\mu-2a)(y-x)} \frac{\zeta_E(s+\mu+a-x)}{\zeta_E(s+\mu+a-y)} \frac{\zeta_E(s+\mu-a+y)}{\zeta_E(s+\mu-a+x)}$$

$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \otimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} \frac{f_{\mu,a,y}(s)}{f_{\mu,a,x}(s)}.$$

As in the previous subsection, Lemma 7.3.1 implies that only $\mu = 0$ can contribute to this product after evaluating at $s = 0$. Hence it is 1 unless $\rho_{\mathrm{GL}} \cong \rho_1$. Moreover, since

$$\left. \frac{f_{0,a,y}(s)}{f_{0,a,x}(s)} \right|_{s=0} = \begin{cases} -q_E^{2a(y-x)} & \text{if } a = x \neq y \text{ or } a = y \neq x, \\ q_E^{2a(y-x)} & \text{otherwise} \end{cases}$$

by Lemma 7.3.1, we conclude that

$$\left. \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)} \right) \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0', \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0'}, \psi_E)} \right)^{-1} \right|_{s=0}$$
$$= \begin{cases} -1 & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x \leq \alpha < y,\ \alpha \equiv x \bmod \mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases}$$

as desired.

7.5. **The inductive case (b).** We use the same notation as in the previous subsection. In this subsection, we assume (1)–(3) and (4b) in Lemma 7.2.3.

In this case, by Theorems C.3.3 and C.3.4 (1), we see that

$$(\rho_1 |\cdot|^x)^{m-1} \rtimes \pi_0' \twoheadrightarrow \pi_0,$$

where $\pi_0' \in \Pi_{\psi_0'}$ is characterized such that

- $\psi_0' = \widehat{\phi_0'}$ is a co-tempered A-parameter with
$$\phi_0' = \phi_0 - \rho_1 \boxtimes S_{2x+1}^{\oplus m-1} \oplus \rho_1 \boxtimes S_{2x-1}^{\oplus m-1};$$
- $\langle \cdot, \pi_0' \rangle_{\psi_0'} = \langle \cdot, \pi_0 \rangle_{\psi_0}$ via the canonical identification $\mathcal{A}_{\psi_0'} \cong \mathcal{A}_{\psi_0}$.

Moreover, the L-parameters ϕ_{π_0} and $\phi_{\pi_0'}$ are related by

$$\phi_{\pi_0} = \phi_{\pi_0'} \oplus (\rho_1(|\cdot|^x \oplus |\cdot|^{-x}) \boxtimes S_1)^{\oplus m-1}.$$

We take highly non-tempered summands $\pi \subset I_P(\pi_M)$ and $\pi' \subset I_{P'}(\pi'_{M'})$ as in the previous subsection. By Lemma 7.1.3, we have $\langle s_u, \pi \rangle_\psi = \langle s_u, \pi' \rangle_{\psi'}$. On the other hand, by the multiplicativity of γ_A-factors, we have

$$\begin{aligned}
&\left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)}\right)\left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0', \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0'}, \psi_E)}\right)^{-1} \\
&= \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x+1}), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x-1}), \psi_E)} \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1|\cdot|^{\epsilon x}, \psi_E)^{-1}\right)^{m-1} \\
&= 1.
\end{aligned}$$

Therefore, we conclude that

$$\left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)}\right)\left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0', \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0'}, \psi_E)}\right)^{-1}\Bigg|_{s=0} = \frac{\langle s_u, \pi\rangle_\psi}{\langle s_u, \pi'\rangle_{\psi'}}.$$

7.6. **The inductive case (c).** We continue to consider the inductive case. Let ϕ_0 be a tempered L-parameter for G_0 of good parity, and set $\psi_0 = \widehat{\phi}_0$. Fix $\pi_0 \in \Pi_{\psi_0}$. In this subsection, we assume the conditions (1)–(3) and (4c) in Lemma 7.2.3.

In this case, by Theorems C.3.3 and C.3.4 (2), we see that

$$\hat{\pi}_0 \hookrightarrow Z([x, y]_{\rho_1}) \times (\rho_1|\cdot|^x)^{m-2} \rtimes \hat{\pi}_0'$$

for some irreducible representation $\hat{\pi}_0'$. This is no longer tempered, and

$$\Delta([-(x-1), x]_{\rho_1}) \rtimes \sigma_0' \twoheadrightarrow \hat{\pi}_0',$$

where σ_0' is tempered, and its L-parameter is given by

$$\phi_{\sigma_0'} = \phi_0 - \rho_1 \boxtimes (S_{2x+1}^{\oplus m-1} \oplus S_{2y+1}) \oplus \rho_1 \boxtimes S_{2x-1}^{\oplus m-2}$$

and

$$\langle e(\rho, 2i-1, 1), \sigma_0'\rangle_{\phi_{\sigma_0'}} = \begin{cases} \langle e(\rho_1, 2i+1, 1), \hat{\pi}_0\rangle_{\phi_0} & \text{if } \rho \cong \rho_1,\ x < i \leq y, \\ \langle e(\rho, 2i-1, 1), \hat{\pi}_0\rangle_{\phi_0} & \text{otherwise.} \end{cases}$$

In particular, $\hat{\pi}_0'$ belongs to the L-packet associated to the A-parameter

$$\widehat{\psi}_0' = \phi_{\sigma_0'} \oplus \rho_1 \boxtimes S_{2x} \boxtimes S_2,$$

and the character $\langle \cdot, \hat{\pi}_0'\rangle_{\widehat{\psi}_0'}$ is given by

$$\begin{aligned}
&\langle e(\rho, 2i-1, 1), \hat{\pi}_0'\rangle_{\widehat{\psi}_0'} = \begin{cases} \langle e(\rho_1, 2i+1, 1), \hat{\pi}_0\rangle_{\widehat{\psi}_0} & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x < i \leq y, \\ \langle e(\rho_1, 2i-1, 1), \hat{\pi}_0\rangle_{\widehat{\psi}_0} & \text{otherwise,} \end{cases} \\
&\langle e(\rho_1, 2x, 2), \hat{\pi}_0'\rangle_{\widehat{\psi}_0'} = 1.
\end{aligned}$$

Now $\pi_0' \in \Pi_{\psi_0'}$ where ψ_0' is defined such that $\psi_0'(w, g_1, g_2) = \widehat{\psi}_0'(w, g_2, g_1)$. Hence

$$\psi_0' = \psi_0 - \rho_1 \boxtimes S_1 \boxtimes (S_{2x+1}^{\oplus m-1} \oplus S_{2y+1}) \oplus \rho_1 \boxtimes S_1 \boxtimes S_{2x-1}^{\oplus m-2} \oplus \rho_1 \boxtimes S_2 \boxtimes S_{2x}.$$

We take $\pi \subset I_P(\pi_M)$ and $\pi' \subset I_{P'}(\pi'_{M'})$ as in the previous subsections. By Hypothesis 6.1.1, we know the local intertwining relation for $\psi_{M'} = \psi_{\mathrm{GL}} \oplus \psi_0'$. However, it is

(**A-LIR**). As explained in Lemma 1.10.2, we need the following to deduce our (**LIR**) for $\pi' \subset I_{P'}(\pi'_{M'})$.

Lemma 7.6.1. *Let $\psi_{M'}$ be as above. For $\pi'_{M'} \in \Pi_{\psi_{M'}}$, the induced representation $I_{P'}(\pi'_{M'})$ is multiplicity-free. Moreover, for $\pi'_{M'}, \pi''_{M'} \in \Pi_{\psi_{M'}}$, if $I_{P'}(\pi'_{M'})$ and $I_{P'}(\pi''_{M'})$ have a common irreducible summand, then $\pi'_{M'} \cong \pi''_{M'}$.*

Proof. This is a special case of Mœglin's multiplicity one theorem (see [X2, Theorem 8.12]). However, her proof relies on (**ECR1**) and (**ECR2**) for all classical groups, and hence we cannot use this result.

We shall give another proof. By taking Aubert duality, it is enough to show that $I_{P'}(\hat{\pi}'_{M'})$ is multiplicity-free, where $\hat{\pi}'_{M'} = \hat{\pi}_{\mathrm{GL}} \boxtimes \hat{\pi}'_0$ is the Aubert dual of $\pi'_{M'} = \pi_{\mathrm{GL}} \boxtimes \pi'_0$. Recall that $\hat{\pi}_{\mathrm{GL}}$ is tempered and

$$\Delta([-(x-1), x]_{\rho_1}) \rtimes \sigma'_0 \twoheadrightarrow \hat{\pi}'_0$$

with σ'_0 tempered. By Casselman's criterion ([Kon2, Lemma 2.4]), we see that all irreducible constituents of $\Delta([-(x-1), x]_{\rho_1}) \rtimes \sigma'_0$ other than $\hat{\pi}'_0$ are tempered. Since $\hat{\pi}_{\mathrm{GL}} \times \Delta([-(x-1), x]_{\rho_1}) \cong \Delta([-(x-1), x]_{\rho_1}) \times \hat{\pi}_{\mathrm{GL}}$ by [Z, Theorem 9.7], we have

$$\Delta([-(x-1), x]_{\rho_1}) \times \hat{\pi}_{\mathrm{GL}} \rtimes \sigma'_0 \twoheadrightarrow I_{P'}(\hat{\pi}'_{M'}).$$

Since $\hat{\pi}_{\mathrm{GL}} \rtimes \sigma'_0$ is a multiplicity-free sum of irreducible tempered representations, $\Delta([-(x-1), x]_{\rho_1}) \times \hat{\pi}_{\mathrm{GL}} \rtimes \sigma'_0$ is a sum of standard modules of the form $\Delta([-(x-1), x]_{\rho_1}) \rtimes \pi_i$ for tempered representations π_i which are not isomorphic to each other. Therefore, $I_{P'}(\hat{\pi}'_{M'})$ is a semisimple quotient of a sum of distinct standard modules, and hence $I_{P'}(\hat{\pi}'_{M'})$ is multiplicity-free.

Similarly, write $\hat{\pi}''_{M'} = \hat{\pi}_{\mathrm{GL}} \boxtimes \hat{\pi}''_0$ and $\Delta([-(x-1), x]_{\rho_1}) \rtimes \sigma''_0 \twoheadrightarrow \hat{\pi}''_0$. Suppose that $\pi'_{M'} \not\cong \pi''_{M'}$. Then $\sigma'_0 \not\cong \sigma''_0$. Since tempered L-packets are multiplicity-free ([Ar2, Theorem 1.5.1], [Mok, Theorem 2.5.1]), we see that $\hat{\pi}_{\mathrm{GL}} \rtimes \sigma'_0$ and $\hat{\pi}_{\mathrm{GL}} \rtimes \sigma''_0$ have no common irreducible summand. Hence $I_{P'}(\pi'_{M'})$ and $I_{P'}(\pi''_{M'})$ are semisimple quotients of sums of standard modules which have no common standard module. Therefore, $I_{P'}(\pi'_{M'})$ and $I_{P'}(\pi''_{M'})$ have no common irreducible summand. □

Set

$$\psi' = \psi_{\mathrm{GL}} \oplus \psi'_0 \oplus {}^c\psi_{\mathrm{GL}}^\vee \in \Psi(G'),$$

where G' is a classical group such that $\mathrm{rank}(G') < \mathrm{rank}(G)$. Using Corollary 4.5.3, we compare $\langle s_u, \pi \rangle_\psi$ with $\langle s'_u, \pi' \rangle_{\psi'}$.

Lemma 7.6.2. *Write $\phi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes S_{2\alpha+1}$ and $\psi_{\mathrm{GL}} = \widehat{\phi}_{\mathrm{GL}}$. If ψ_{GL} is of the same type as ψ_0, then we have*

$$\frac{\langle s_u, \pi \rangle_\psi}{\langle s_u, \pi' \rangle_{\psi'}} = \left. \frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_2 \boxtimes S_{2x}), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes (S_{2x-1} \oplus S_{2x+1})), \psi_E)} \right|_{s=0} \times \begin{cases} -1 & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x \leq \alpha < y, \alpha \equiv x \bmod \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

Otherwise, $\langle s_u, \pi \rangle_\psi = \langle s_u, \pi' \rangle_{\psi'} = 1$.

Proof. Since $\phi = \widehat{\psi}$ is tempered, we can apply Corollary 4.4.5 and get

$$\frac{\langle s_u, \pi\rangle_\psi}{\langle \widehat{s_u}, \hat{\pi}\rangle_\phi} = (-1)^{r(\phi)-r(\phi_+)-r(\phi_-)}.$$

Note that $\widehat{\psi}'$ satisfies the assumption of Corollary 4.5.3. Since $\dim(\widehat{\psi}') < \dim(\phi)$, we can use (**ECR1**) and (**ECR2**) for $\widehat{\psi}'$, and hence we can apply Corollary 4.5.3 to $\widehat{\psi}'$. Since $\psi_{\mathrm{GL}} = \widehat{\phi}_{\mathrm{GL}} = \phi^A_{\mathrm{GL}}$, we have

$$\frac{\langle s_u, \pi'\rangle_{\psi'}}{\langle \widehat{s_u}, \hat{\pi}'\rangle_{\widehat{\psi}'}} = (-1)^{r(\widehat{\psi}')-r(\widehat{\psi}'_+)-r(\widehat{\psi}'_-)} \left.\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes (S_{2x-1} \oplus S_{2x+1})), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_2 \boxtimes S_{2x}), \psi_E)}\right|_{s=0}.$$

By similar arguments to the proofs of Lemmas 7.1.2 and 7.1.3, we have

$$(-1)^{r(\phi)-r(\phi_+)-r(\phi_-)} = (-1)^{r(\widehat{\psi}')-r(\widehat{\psi}'_+)-r(\widehat{\psi}'_-)}$$

and

$$\frac{\langle \widehat{s_u}, \hat{\pi}\rangle_\phi}{\langle \widehat{s_u}, \hat{\pi}'\rangle_{\widehat{\psi}'}} = \begin{cases} -1 & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x \le \alpha < y, \alpha \equiv x \bmod \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

Putting together, we obtain the assertion. $\square$

Since

$$\Delta([x,y]_{\rho_1}) \times (\rho_1|\cdot|^x)^{m-2} \rtimes \pi'_0 \twoheadrightarrow \pi_0,$$

the L-parameters ϕ_{π_0} and $\phi_{\pi'_0}$ are related by

$$\phi_{\pi_0} = \phi_{\pi'_0} \oplus \rho_1(|\cdot|^{\frac{x+y}{2}} \oplus |\cdot|^{-\frac{x+y}{2}}) \boxtimes S_{y-x+1} \oplus (\rho_1(|\cdot|^x \oplus |\cdot|^{-x}) \boxtimes S_1)^{\oplus m-2}.$$

Now, what we have to show is that

$$\left.\frac{\langle s_u, \pi\rangle_\psi}{\langle s_u, \pi'\rangle_{\psi'}} \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)}\right) \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi'_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi'_0}, \psi_E)}\right)^{-1}\right|_{s=0} = 1.$$

Since

$$\begin{aligned} &\left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_E)}\right) \left(\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \psi'_0, \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \phi_{\pi'_0}, \psi_E)}\right)^{-1} \\ &= \frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x+1}), \psi_E)^{m-1} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2y+1}), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x-1}), \psi_E)^{m-2} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_2 \boxtimes S_{2x}), \psi_E)} \\ &\quad \times \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1|\cdot|^{\epsilon\frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1|\cdot|^{\epsilon x}, \psi_E)^{-(m-2)}, \end{aligned}$$

it is equivalent to checking that

$$\begin{aligned} &\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x+1}), \psi_E)^{m-2} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2y+1}), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x-1}), \psi_E)^{m-1}} \\ &\left.\times \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1|\cdot|^{\epsilon\frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1|\cdot|^{\epsilon x}, \psi_E)^{-(m-2)}\right|_{s=0} \end{aligned}$$

$$= \begin{cases} -1 & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x \leq \alpha < y, \alpha \equiv x \bmod \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

Let us check this equation. We have

$$\frac{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x+1}), \psi_E)^{m-2} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2y+1}), \psi_E)}{\gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes (\rho_1 \boxtimes S_1 \boxtimes S_{2x-1}), \psi_E)^{m-1}}$$
$$\times \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1 |\cdot|^{\epsilon \frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1} \gamma_A(s, {}^c\psi_{\mathrm{GL}} \otimes \rho_1 |\cdot|^{\epsilon x}, \psi_E)^{-(m-2)}$$
$$= \prod_{-\alpha \leq a \leq \alpha} \prod_{-x \leq b \leq x} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^b, \psi_E)^{m-2}$$
$$\times \prod_{-\alpha \leq a \leq \alpha} \prod_{-y \leq b \leq y} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^b, \psi_E)$$
$$\times \prod_{-\alpha \leq a \leq \alpha} \prod_{-x+1 \leq b \leq x-1} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^b, \psi_E)^{-(m-1)}$$
$$\times \prod_{-\alpha \leq a \leq \alpha} \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^{\epsilon \frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1}$$
$$\times \prod_{-\alpha \leq a \leq \alpha} \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^{\epsilon x}, \psi_E)^{-(m-2)}$$
$$= \prod_{-\alpha \leq a \leq \alpha} \prod_{-y \leq b \leq y} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^b, \psi_E)$$
$$\times \prod_{-\alpha \leq a \leq \alpha} \prod_{-x+1 \leq b \leq x-1} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^b, \psi_E)^{-1}$$
$$\times \prod_{-\alpha \leq a \leq \alpha} \prod_{\epsilon \in \{\pm 1\}} \gamma_A(s, {}^c\rho_{\mathrm{GL}} |\cdot|^a \otimes \rho_1 |\cdot|^{\epsilon \frac{x+y}{2}} \boxtimes S_{y-x+1}, \psi_E)^{-1}$$
$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \boxtimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} q_E^{-(1-2s-2\mu-2a)(y-x)} \prod_{\substack{-y \leq b \leq -x-1 \\ \text{or} \\ x \leq b \leq y-1}} \frac{\zeta_E(s+\mu+a+b+1)}{\zeta_E(s+\mu+a+b)}$$
$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \boxtimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} q_E^{-(1-2s-2\mu-2a)(y-x)} \frac{\zeta_E(s+\mu+a-x)}{\zeta_E(s+\mu+a-y)} \frac{\zeta_E(s+\mu-a+y)}{\zeta_E(s+\mu-a+x)}$$
$$= \prod_{\mu \in X({}^c\rho_{\mathrm{GL}} \boxtimes \rho_1)} \prod_{-\alpha \leq a \leq \alpha} \frac{f_{\mu,a,y}(s)}{f_{\mu,a,x}(s)}.$$

By the same argument as in previous subsections, this is equal to

$$\begin{cases} -1 & \text{if } \rho_{\mathrm{GL}} \cong \rho_1,\ x \leq \alpha < y, \alpha \equiv x \bmod \mathbb{Z}, \\ 1 & \text{otherwise} \end{cases}$$

after evaluating at $s = 0$. This completes the case (c), and the proof of Theorem 1.10.5 (2).

Appendix A. Local factors

In this appendix, we recall some facts about local factors. In particular, we shall show that the local Langlands correspondence for classical groups G_0 identifies the standard Shahidi local factors for irreducible generic representations of $\mathrm{GL}_k(E) \times G_0$ with the tensor product Artin local factors of the corresponding L-parameters. This result was used in Lemma 2.2.2, which is for the first main theorem (Theorem 1.8.1). We also give a proof of Proposition 1.7.2.

A.1. **Formulas for Artin local factors.** We use the notation in Section 1.1. Assume that E is non-archimedean, and write q_E for the cardinality of the residue field of E.

Let I_E be the inertia subgroup of the Weil group W_E. For a representation (ϕ, V) of W_E, we write

$$\phi^{I_E} = \{v \in V \mid \phi(w)v = v, \, \forall w \in I_E\}.$$

This is a subrepresentation of ϕ. Moreover, any irreducible component of ϕ^{I_E} is unramified so that there is a finite multi-set

$$X(\phi) \subset \mathbb{C}/2\pi\sqrt{-1}(\log q_E)^{-1}\mathbb{Z}$$

such that

$$\phi^{I_E} \cong \bigoplus_{\mu \in X(\phi)} |\cdot|_E^{\mu}.$$

Note that

- $X(\phi|\cdot|_E^{s_0}) = \{\mu + s_0 \mid \mu \in X(\phi)\}$;
- if $\phi(W_E)$ is bounded, then $\mathrm{Re}(\mu) = 0$ for any $\mu \in X(\phi)$;
- if ϕ is conjugate-self-dual, then $X(\phi)$ is invariant under $\mu \mapsto -\mu$.

We denote by $\mu_0 \in \mathbb{C}/2\pi\sqrt{-1}(\log q_E)^{-1}\mathbb{Z}$ the unique nonzero element such that $\mu_0 = -\mu_0$. It satisfies that $q_E^{-\mu_0} = -1$.

We recall the formulas for the local factors. Let ϕ be a representation of $W_E \times \mathrm{SL}_2(\mathbb{C})$, and decompose it as

$$\phi \cong \bigoplus_{d \geq 1} \phi_d \boxtimes S_d$$

with ϕ_d a representation of W_E. Let $\zeta_E(s) = (1 - q_E^{-s})^{-1}$ be the local zeta function associated to E. Then there are constants $\varepsilon(\phi_d) \in \mathbb{C}^\times$ and $c(\phi_d) = c(\phi_d, \psi_E) \in \mathbb{Z}$ such that

$$L(s, \phi) = \prod_{d \geq 1} \prod_{\mu \in X(\phi_d)} \zeta_E\left(s + \mu + \frac{d-1}{2}\right),$$

$$\varepsilon(s, \phi, \psi_E) = \prod_{d \geq 1} \left(\varepsilon(\phi_d) q_E^{c(\phi_d)(\frac{1}{2}-s)}\right)^d \prod_{\mu \in X(\phi_d)} (-q_E^{\frac{1}{2}-s-\mu})^{d-1}.$$

In particular, we have

$$\gamma_A(s,\phi,\psi_E) = \varepsilon(s,\phi,\psi_E)\frac{L(1+s,\phi)}{L(s,\phi)}$$
$$= \prod_{d\geq 1}\left(\varepsilon(\phi_d)q_E^{c(\phi_d)(\frac{1}{2}-s)}\right)^d \prod_{\mu\in X(\phi_d)}(-q_E^{(\frac{1}{2}-s-\mu)})^{d-1}\frac{\zeta_E(s+\mu+\frac{d+1}{2})}{\zeta_E(s+\mu+\frac{d-1}{2})}.$$

Moreover, $L(s,\phi|\cdot|^{s_0}) = L(s+s_0,\phi)$ and $\varepsilon(s,\phi|\cdot|^{s_0},\psi_E) = \varepsilon(s+s_0,\phi,\psi_E)$ hold. Hence we have $c(\phi_d|\cdot|^{s_0}) = c(\phi_d)$ and $\varepsilon(\phi_d|\cdot|^{s_0}) = \varepsilon(\phi_d)q_E^{-c(\phi_d)s_0}$.

When $[E:F]=2$, we denote by ${}^c\phi$ the conjugate of ϕ. We note that $X({}^c\phi) = X(\phi)$ and $c({}^c\phi) = c(\phi)$ for any representation ϕ of W_E.

Lemma A.1.1. *Suppose that $[E:F]=2$. Let ψ'_E be a non-trivial additive character of E which is trivial on F. If ϕ be a conjugate-orthogonal representation of $W_E\times \mathrm{SL}_2(\mathbb{C})$, then we have*

$$\gamma_A(s,\phi,\psi'_E) = \gamma_A(s,{}^c\phi,\psi'_E).$$

Proof. Since $X({}^c\phi) = X(\phi)$, we have $L(s,{}^c\phi) = L(s,\phi)$. On the other hand, since

$$\varepsilon(s,\phi^\vee,\psi'_E)\varepsilon(1-s,\phi,\psi'_E) = \det\phi^\vee(-1) = \det\phi(-1),$$

by using ${}^c\phi\cong\phi^\vee$, we see that

$$\frac{\varepsilon(s,{}^c\phi,\psi'_E)}{\varepsilon(s,\phi,\psi'_E)} = \frac{\varepsilon(s,{}^c\phi,\psi'_E)\varepsilon(1-s,\phi,\psi'_E)}{\varepsilon(s,\phi,\psi'_E)\varepsilon(1-s,\phi,\psi'_E)} = \frac{\det\phi(-1)}{\varepsilon(\frac{1}{2},\phi,\psi'_E)^2}.$$

Since ϕ is conjugate-orthogonal, its determinant $\det\phi$ is a character of $E^\times/F^\times$. Hence $\det\phi(-1)=1$. In addition, by [GGP, Proposition 5.1 (2)], we know that $\varepsilon(\frac{1}{2},\phi,\psi'_E)^2 = 1$. Therefore, we have $\gamma_A(s,\phi,\psi'_E) = \gamma_A(s,{}^c\phi,\psi'_E)$. □

Proposition A.1.2. *Suppose that $[E:F]=2$. Let ϕ_1 and ϕ_2 be two conjugate-orthogonal representations of $W_E\times\mathrm{SL}_2(\mathbb{C})$. If $\det\phi_1 = \det\phi_2$, then we have*

$$\frac{\gamma_A(s,\phi_1,\psi_E)}{\gamma_A(s,\phi_2,\psi_E)} = \frac{\gamma_A(s,{}^c\phi_1,\psi_E)}{\gamma_A(s,{}^c\phi_2,\psi_E)}.$$

Proof. For $a\in E^\times$, define an additive character $a\psi_E$ of E by $(a\psi_E)(x) = \psi_E(ax)$. Then

$$\varepsilon(s,\phi_i,a\psi_E) = \det(\phi_i)(a)|a|_E^{\dim(\phi_i)(s-\frac{1}{2})}\varepsilon(s,\phi_i,\psi_E)$$

holds. Hence we may replace ψ_E by any non-trivial additive character ψ'_E of E. If we use ψ'_E such that $\psi'_E|_F = \mathbf{1}$, then the assertion follows from the previous lemma. □

A.2. **Comparison of γ-factors.** Now we drop the assumption that E is non-archimedean. Let G° be a connected quasi-split classical group. Fix a Whittaker datum $\mathfrak{w}$ for G°. Let $P^\circ = M^\circ N_P$ be a maximal parabolic subgroup of G° so that $M^\circ = \mathrm{GL}_k(E)\times G_0^\circ$. Consider irreducible $\mathfrak{w}$-generic representations τ and σ of $\mathrm{GL}_k(E)$ and G_0°, respectively.

(By abuse of language, we say that τ or σ is $\mathfrak{w}$-generic if it is generic with respect to the Whittaker datum induced by $\mathfrak{w}$.) We denote by

$$\gamma^{\mathrm{Sh}}(s,\tau\times\sigma,\psi_E)=\varepsilon^{\mathrm{Sh}}(s,\tau\times\sigma,\psi_E)\frac{L^{\mathrm{Sh}}(1-s,\tau^\vee\times\sigma^\vee)}{L^{\mathrm{Sh}}(s,\tau\times\sigma)}$$

the associated γ-factor of Shahidi [Sha7]. On the other hand, let ϕ_τ and ϕ_σ be the L-parameters associated to τ and σ, respectively, and define the Artin γ-factor over E associated to $\phi_\tau\otimes\phi_\sigma$ by

$$\gamma(s,\phi_\tau\otimes\phi_\sigma,\psi_E)=\varepsilon(s,\phi_\tau\otimes\phi_\sigma,\psi_E)\frac{L(1-s,\phi_\tau^\vee\otimes\phi_\sigma^\vee)}{L(s,\phi_\tau\otimes\phi_\sigma)}.$$

Here, we assume the local classification theorem for G_0° by the induction hypothesis, so that there is a commutative diagram

$$\begin{array}{ccc}\mathrm{Irr}(G_0^\circ) & \longrightarrow & \Phi(G_0^\circ)\\ \big\downarrow & & \big\downarrow\\ \mathrm{Irr}(\mathrm{GL}_{N_0}(E)) & \longrightarrow & \Phi(\mathrm{GL}_{N_0}(E))\end{array}$$

where the horizontal arrows are the local Langlands correspondence, the left vertical arrow is the twisted endoscopic transfer (for tempered representations), and the right vertical arrow is the natural map. In other words, if π is the functorial lift of σ to $\mathrm{GL}_{N_0}(E)$ (in terms of the twisted endoscopic character relations), then ϕ_σ (regarded as a representation of L_E) is defined as the L-parameter ϕ_π of π. In particular, we have

$$\gamma(s,\phi_\tau\otimes\phi_\sigma,\psi_E)=\gamma(s,\phi_\tau\otimes\phi_\pi,\psi_E).$$

Proposition A.2.1. *For any irreducible $\mathfrak{w}$-generic representation $\tau\boxtimes\sigma$ of M°, we have*

$$\gamma^{\mathrm{Sh}}(s,\tau\times\sigma,\psi_E)=\gamma(s,\phi_\tau\otimes\phi_\sigma,\psi_E).$$

In particular, if τ and σ are tempered, then the equalities

$$L^{\mathrm{Sh}}(s,\tau\times\sigma)=L(s,\phi_\tau\otimes\phi_\sigma),\quad \varepsilon^{\mathrm{Sh}}(s,\tau\times\sigma,\psi_E)=\varepsilon(s,\phi_\tau\otimes\phi_\sigma,\psi_E)$$

also hold.

The desired equality of γ-factors in Proposition A.2.1 seems to be well-known to experts, but we briefly review the argument in Section A.4 below. Before it, we give realizations of our quasi-split groups and splittings.

A.3. **Groups and splittings.** We denote by $E_{i,j}$ the square matrix (of a certain size) with 1 at the (i,j)-th entry and 0 elsewhere. Define an $n\times n$ anti-diagonal matrix J_n by

$$J_n=\begin{pmatrix} & & 1\\ & ⋰ & \\ 1 & & \end{pmatrix}.$$

For reductive algebraic groups $G,T,\dots$ over F, we use the corresponding Gothic letters $\mathfrak{g},\mathfrak{t},\dots$ for the associated Lie algebras. We consider the following quasi-split reductive algebraic group G over F and the F-splitting $\boldsymbol{spl}=(B^\circ,T^\circ,\{X_\alpha\})$ of G°.

Symplectic groups: Suppose that $E = F$. Let $G = \mathrm{Sp}_{2n}(F)$ be the symplectic group defined by

$$\mathrm{Sp}_{2n}(F) = \{g \in \mathrm{GL}_{2n}(F) \,|\, {}^t g J'_{2n} g = J'_{2n}\}, \quad J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}.$$

Take the Borel subgroup B of G consisting of upper triangular matrices and the maximal torus T of G consisting of diagonal matrices. Then the corresponding positive simple roots are given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = 2e_n$, where $\{e_1, \dots, e_n\}$ is the standard basis of $X^*(T)$. Take the root vectors given by $X_{\alpha_i} = E_{i,i+1} - E_{2n-i,2n+1-i}$ for $1 \leq i \leq n-1$ and $X_{\alpha_n} = E_{n,n+1}$.

Odd special orthogonal groups: Suppose that $E = F$. For an extension K of F, let $\mathrm{O}_N(K)$ be the orthogonal group defined by

$$\mathrm{O}_N(K) = \{g \in \mathrm{GL}_N(K) \,|\, {}^t g J_N g = J_N\}.$$

Take the Borel subgroup B° of $\mathrm{SO}_N(\overline{F})$ consisting of upper triangular matrices and the maximal torus T° of $\mathrm{SO}_N(\overline{F})$ consisting of diagonal matrices.

Suppose that $N = 2n+1$ and consider $G = \mathrm{SO}_{2n+1}(F)$. Then the corresponding positive simple roots are given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = e_n$, where $\{e_1, \dots, e_n\}$ is the standard basis of $X^*(T)$. Take the root vectors given by $X_{\alpha_i} = E_{i,i+1} - E_{2n+1-i,2n+2-i}$ for $1 \leq i \leq n$.

Even special orthogonal groups: Suppose that $E = F$. Let $\mathrm{O}_N(\overline{F})$ be the orthogonal group as above. Suppose that $N = 2n$ and consider $\mathrm{SO}_{2n}(\overline{F})$. Then the corresponding positive simple roots are given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = e_{n-1} + e_n$, where $\{e_1, \dots, e_n\}$ is the standard basis of $X^*(T)$. Take the root vectors given by $X_{\alpha_i} = E_{i,i+1} - E_{2n-i,2n+1-i}$ for $1 \leq i \leq n-1$ and $X_{\alpha_n} = E_{n-1,n+1} - E_{n,n+2}$. For example, if $n = 2$, then

$$X_{\alpha_{n-1}} = \left(\begin{array}{c|cc|c} 0 & 1 & & \\ \hline & 0 & & \\ & & 0 & -1 \\ \hline & & & 0 \end{array}\right), \quad X_{\alpha_n} = \left(\begin{array}{c|cc|c} 0 & & 1 & \\ \hline & 0 & & -1 \\ & & 0 & \\ \hline & & & 0 \end{array}\right).$$

Now for a (possibly trivial) quadratic character η of $F^\times$, we define a form $G = \mathrm{O}_{2n}^\eta(F)$ as follows. When η is trivial, we put $\mathrm{O}_{2n}^\eta(F) = \mathrm{O}_{2n}(F)$ as above. When η is non-trivial, we denote by K the quadratic extension of F associated to η by local class field theory. Then we define $\mathrm{O}_{2n}^\eta(F)$ as the subgroup $\mathrm{O}_{2n}(K)$ consisting of matrices g such that

$$g = \epsilon g^\rho \epsilon^{-1}, \quad \epsilon = \begin{pmatrix} \mathbf{1}_{n-1} & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & \mathbf{1}_{n-1} \end{pmatrix},$$

where ρ is the non-trivial element in $\mathrm{Gal}(K/F)$. Note that B° and T° are defined over F, and $\mathrm{Ad}(\epsilon)(X_{\alpha_i}) = X_{\alpha_i}$ for $1 \leq i \leq n-2$ and $\mathrm{Ad}(\epsilon)(X_{\alpha_{n-1}}) = X_{\alpha_n}$. In particular, ϵ fixes the F-splitting $\boldsymbol{spl} = (B^\circ, T^\circ, \{X_\alpha\})$.

We remark that any F-splitting of G° is conjugate to the splitting $\boldsymbol{spl}' = (B^\circ, T^\circ, \{X'_\alpha\})$, where $X'_{\alpha_i} = X_{\alpha_i}$ for $1 \leq i \leq n-2$ and

$$(X'_{\alpha_{n-1}}, X'_{\alpha_n}) = \begin{cases} (X_{\alpha_{n-1}}, aX_{\alpha_n}) & \text{if } \eta = \mathbf{1}, \\ (bX_{\alpha_{n-1}}, b^\rho X_{\alpha_n}) & \text{if } \eta \neq \mathbf{1} \end{cases}$$

for some $a \in F^\times$ and $b \in K^\times$. Then $\boldsymbol{spl}'$ is fixed by $\epsilon' = \epsilon t_0$ with

$$t_0 = \begin{cases} \mathrm{diag}(\mathbf{1}_{n-1}, a^{-1}, a, \mathbf{1}_{n-1}) & \text{if } \eta = \mathbf{1}, \\ \mathrm{diag}(\mathbf{1}_{n-1}, b(b^\rho)^{-1}, b^{-1}b^\rho, \mathbf{1}_{n-1}) & \text{if } \eta \neq \mathbf{1}. \end{cases}$$

Finally, we notice that $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(G^\circ)$.

Unitary groups: Suppose that $[E : F] = 2$. Let $G = \mathrm{U}_n$ be the unitary group defined by

$$\mathrm{U}_n = \{g \in \mathrm{GL}_n(E) \,|\, {}^t\overline{g} J''_n g = J''_n\}, \quad J''_n = \mathrm{diag}(1, -1, \dots, (-1)^{n-1}) \cdot J_n.$$

Take the Borel subgroup B of G consisting of upper triangular matrices and the maximal torus T of G consisting of diagonal matrices. Then the corresponding positive simple roots are given $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$, where $\{e_1, \dots, e_n\}$ is the standard basis of $X^*(T)$. Take the root vectors given by $X_{\alpha_i} = E_{i,i+1}$ for $1 \leq i \leq n-1$.

A.4. **Proof of Proposition A.2.1.** Now we shall prove Proposition A.2.1.

Proof of Proposition A.2.1. If τ and σ are tempered, then the L-factors and the ε-factors are uniquely determined by the corresponding γ-factors. Hence the equations for the L-factors and the ε-factors are derived by the one for the γ-factors. Moreover, since $\phi_\sigma = \phi_\pi$ as representations of L_E, where π is the functorial lift of σ, it suffices to show that

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E) = \gamma(s, \phi_\tau \otimes \phi_\pi, \psi_E).$$

The equation for the γ-factors easily follows from the characterizing properties of $\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E)$, proved in [Sha7, Theorem 3.5], which are some of the "Ten Commandments" given in [LR]. For the convenience of the readers, we recall the properties we need.

(1) (unramified twisting) For $s_0 \in \mathbb{C}$, we have

$$\gamma^{\mathrm{Sh}}(s, \tau|\det|_E^{s_0} \times \sigma, \psi_E) = \gamma^{\mathrm{Sh}}(s + s_0, \tau \times \sigma, \psi_E).$$

(2) (dependence on ψ_F) Let ψ'_F be another non-trivial additive character of F, so that $\psi'_F(x) = \psi_F(ax)$ for some $a \in F^\times$, and put $\psi'_E = \psi'_F \circ \mathrm{tr}_{E/F}$. Then we have

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi'_E) = \eta(a)^k \omega_\tau(a)^{N_0} |a|_E^{kN_0(s-\frac{1}{2})} \gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E),$$

where η is the trivial character of $F^\times$ unless G° is an even special orthogonal group, in which case η is the (possibly trivial) quadratic character of $F^\times$ associated to the splitting field of G°, ω_τ is the central character of τ, and $N_0 = \dim(\phi_\sigma)$. See Section A.5.

(3) (multiplicativity) Assume that τ is a subrepresentation of $I_{P_1}(\tau_1 \boxtimes \tau_2)$, where P_1 is a standard parabolic subgroup of $\mathrm{GL}_k(E)$ with Levi component $\mathrm{GL}_{k_1}(E) \times \mathrm{GL}_{k_2}(E)$, and τ_1 and τ_2 are irreducible $\mathfrak{w}$-generic representations of $\mathrm{GL}_{k_1}(E)$ and $\mathrm{GL}_{k_2}(E)$, respectively. Then we have

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E) = \gamma^{\mathrm{Sh}}(s, \tau_1 \times \sigma, \psi_E) \cdot \gamma^{\mathrm{Sh}}(s, \tau_2 \times \sigma, \psi_E).$$

Similarly, assume that σ is a subrepresentation of $I_{P_0}(\tau_0 \boxtimes \sigma_0)$, where P_0 is a standard parabolic subgroup of G_0° with Levi component $\mathrm{GL}_{k_0}(E) \times G_{00}^\circ$, and τ_0 and σ_0 are irreducible $\mathfrak{w}$-generic representations of $\mathrm{GL}_{k_0}(E)$ and G_{00}°, respectively. Then we have

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E) = \gamma(s, \tau \times \tau_0, \psi_E) \cdot \gamma(s, \tau \times {}^c\tau_0^\vee, \psi_E) \cdot \gamma^{\mathrm{Sh}}(s, \tau \times \sigma_0, \psi_E).$$

Here, $\gamma(s, \tau \times \tau_0, \psi_E)$ is the Rankin–Selberg γ-factor which is equal to the Artin γ-factor $\gamma(s, \phi_\tau \otimes \phi_{\tau_0}, \psi_E)$.

(4) (unramified factors) Assume that F is non-archimedean, and τ and σ are spherical (in the sense that they have nonzero fixed vectors under good special maximal compact subgroups). Then we have

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E) = \gamma(s, \phi_\tau \otimes \phi_\sigma, \psi_E).$$

(5) (archimedean property) Assume that F is archimedean. Then we have

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E) = \gamma(s, \phi_\tau \otimes \phi_\sigma, \psi_E).$$

(6) (global property) Let $\dot{F}$ be a number field with ring of adèles $\dot{\mathbb{A}} = \dot{\mathbb{A}}_{\dot{F}}$, and let $\dot{E}$ be either $\dot{F}$ or a quadratic field extension of $\dot{F}$. Fix a non-trivial additive character $\psi_{\dot{F}}$ of $\dot{\mathbb{A}}/\dot{F}$ and put $\psi_{\dot{E}} = \psi_{\dot{F}} \circ \mathrm{tr}_{\dot{E}/\dot{F}}$. Let $\dot{G}^\circ$ be a classical group defined over $\dot{F}$, and let $\dot{M}^\circ = \mathrm{Res}_{\dot{E}/\dot{F}}\mathrm{GL}_k \times \dot{G}_0^\circ$ be a maximal semi-standard Levi subgroup of $\dot{G}^\circ$. We denote by $\dot{\mathfrak{w}}$ the Whittaker datum induced by the $\dot{F}$-splitting of $\dot{G}^\circ$ and $\psi_{\dot{F}}$. Let $\dot{\tau}$ and $\dot{\sigma}$ be irreducible globally $\dot{\mathfrak{w}}$-generic cuspidal automorphic representations of $\mathrm{GL}_k(\dot{\mathbb{A}}_{\dot{E}})$ and $\dot{G}_0^\circ(\dot{\mathbb{A}})$, respectively. Then we have

$$L^S(s, \dot{\tau} \times \dot{\sigma}) = \prod_{v \in S} \gamma^{\mathrm{Sh}}(s, \dot{\tau}_v \times \dot{\sigma}_v, \psi_{\dot{E},v}) \cdot L^S(1-s, \dot{\tau}^\vee \times \dot{\sigma}^\vee),$$

where S is a sufficiently large finite set of places of $\dot{F}$ and $L^S(s, \dot{\tau} \times \dot{\sigma}) = \prod_{v \notin S} L(s, \phi_{\dot{\tau}_v} \otimes \phi_{\dot{\sigma}_v})$ is the partial L-function (for $\mathrm{Re}(s)$ sufficiently large).

By (5), we may assume that F is non-archimedean. By the Langlands classification, we may write τ and σ as unique irreducible subrepresentations of the duals of standard modules. Since τ and σ are $\mathfrak{w}$-generic, the inducing data of these standard modules are also $\mathfrak{w}$-generic. (See cf., [AG2, Lemma 2.2].) Hence, by (1), (3), and the definition of L-parameters, we may assume that τ and σ are tempered. In this case, we may write τ and σ as subrepresentations of parabolic inductions of square-integrable representations. By the same argument, we may assume that τ and σ are square-integrable.

First we treat the case when τ and σ are supercuspidal. Choose $\dot{F}, \dot{E}, \dot{G}^\circ, \dot{M}^\circ, \psi_{\dot{F}}$ as in (6) such that $\dot{F}_{v_0} = F, \dot{E}_{v_0} = E, \dot{G}^\circ_{v_0} = G^\circ, \dot{M}^\circ_{v_0} = M^\circ$ for some finite place v_0 of $\dot{F}$. Note that it is not always possible to find $\psi_{\dot{F}}$ such that $\psi_{\dot{F},v_0} = \psi_F$. By the Poincaré series argument [He1, Appendice 1], [Sha7, Proposition 5.1], we can find $\dot{\tau}$ and $\dot{\sigma}$ as in (6) such that $\dot{\tau}_{v_0} \cong \tau$ and $\dot{\sigma}_{v_0} \cong \sigma$, and such that $\dot{\tau}_v$ and $\dot{\sigma}_v$ are spherical for all finite places $v \neq v_0$. Moreover, by [CKPSS1, CKPSS2], [KK1, KK2] and [CPSS], the functorial lift $\dot{\pi}$ of $\dot{\sigma}$ is cuspidal. In other words, by the global classification theorem for $\dot{G}^\circ_0$ (which we assume by the induction hypothesis), the global A-parameter of $\dot{\sigma}$ is generic and $\dot{\pi}_v$ is the functorial lift of $\dot{\sigma}_v$ for all places v. Then the global functional equation says that

$$L^S(s, \dot{\tau} \times \dot{\pi}) = \prod_{v \in S} \gamma(s, \dot{\tau}_v \times \dot{\pi}_v, \psi_{\dot{E},v}) \cdot L^S(1-s, \dot{\tau}^\vee \times \dot{\pi}^\vee),$$

where S is a sufficiently large finite set of places of $\dot{F}$, $L^S(s, \dot{\tau} \times \dot{\pi}) = \prod_{v \notin S} L(s, \phi_{\dot{\tau}_v} \otimes \phi_{\dot{\pi}_v})$ is the partial L-function (for $\mathrm{Re}(s)$ sufficiently large), and $\gamma(s, \dot{\tau}_v \times \dot{\pi}_v, \psi_{\dot{E},v})$ is the Rankin–Selberg γ-factor. Since $L^S(s, \dot{\tau} \times \dot{\pi}) = L^S(s, \dot{\tau} \times \dot{\sigma})$ and $\gamma(s, \dot{\tau}_v \times \dot{\pi}_v, \psi_{\dot{E},v}) = \gamma(s, \phi_{\dot{\tau}_v} \otimes \phi_{\dot{\pi}_v}, \psi_{\dot{E},v})$ (which is a desideratum of the local Langlands correspondence for general linear groups), we may write this equality as

$$L^S(s, \dot{\tau} \times \dot{\sigma}) = \prod_{v \in S} \gamma(s, \phi_{\dot{\tau}_v} \otimes \phi_{\dot{\pi}_v}, \psi_{\dot{E},v}) \cdot L^S(1-s, \dot{\tau}^\vee \times \dot{\sigma}^\vee).$$

On the other hand, by (4), (5), we have

$$\gamma^{\mathrm{Sh}}(s, \dot{\tau}_v \times \dot{\sigma}_v, \psi_{\dot{E},v}) = \gamma(s, \phi_{\dot{\tau}_v} \otimes \phi_{\dot{\sigma}_v}, \psi_{\dot{E},v}) = \gamma(s, \phi_{\dot{\tau}_v} \otimes \phi_{\dot{\pi}_v}, \psi_{\dot{E},v})$$

for all places $v \neq v_0$. From this and (6), we can deduce that

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \dot{\psi}_{E,v_0}) = \gamma(s, \phi_\tau \otimes \phi_\pi, \dot{\psi}_{E,v_0}).$$

If we write $\psi_{\dot{F},v_0}(x) = \psi_F(ax)$ for some $a \in F^\times$, then the left-hand side is equal to

$$\eta(a)^k \omega_\tau(a)^{N_0} |a|_E^{kN_0(s-\frac{1}{2})} \gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E)$$

by (2), whereas the right-hand side is equal to

$$\det(\phi_\tau \otimes \phi_\pi)(a) |a|_E^{\dim(\phi_\tau \otimes \phi_\pi)(s-\frac{1}{2})} \gamma(s, \phi_\tau \otimes \phi_\pi, \psi_E)$$

(see [Tate, Section 3.6]). This implies the desired equation for the γ-factors.

Now suppose that τ and σ are square-integrable. Choose $\dot{F}, \dot{E}, \dot{G}^\circ, \dot{M}^\circ, \psi_{\dot{F}}$ as above and fix an auxiliary finite place $v_1 \neq v_0$ of $\dot{F}$ such that v_1 does not split in $\dot{E}$ when $[\dot{E} : \dot{F}] = 2$. By the Poincaré series argument [He1, Appendice 1] for a central division algebra over $\dot{E}$ of degree k ramified precisely at v_0, v_1 (where we regard v_i for $i = 0, 1$ as the unique place of $\dot{E}$ lying over v_i when $[\dot{E} : \dot{F}] = 2$), and the global Jacquet–Langlands correspondence [Bad], we can find $\dot{\tau}$ as in (6) such that $\dot{\tau}_{v_0} \simeq \tau$, $\dot{\tau}_{v_1}$ is supercuspidal, and $\dot{\tau}_v$ is spherical for all finite places $v \neq v_0, v_1$. Also, by [ILM, Appendix A], [GI3, Appendix A], we can find $\dot{\sigma}$ as in (6) such that $\dot{\sigma}_{v_0} \simeq \sigma$, $\dot{\sigma}_{v_1}$ is supercuspidal, and $\dot{\sigma}_v$

is spherical for all finite places $v \neq v_0, v_1$. (Note that [GI3] only treats the case of metaplectic groups, but the same argument goes through for other classical groups.) Since we already know that

$$\gamma^{\mathrm{Sh}}(s, \dot{\tau}_{v_1} \times \dot{\sigma}_{v_1}, \psi_{\dot{E},v_1}) = \gamma(s, \phi_{\dot{\tau}_{v_1}} \otimes \phi_{\dot{\pi}_{v_1}}, \psi_{\dot{E},v_1}),$$

where $\dot{\pi}_{v_1}$ is the functorial lift of $\dot{\sigma}_{v_1}$, the above argument proves the analogous equation for v_0 and hence the desired equation for the γ-factors. □

Remark A.4.1. Recently, Cai–Friedberg–Ginzburg–Kaplan [CFGK], [Cai], [CFK1] introduced a new family of zeta integrals, which generalizes the doubling zeta integrals of Piatetski-Shapiro–Rallis [PSR], [LR], and established an analytic theory of γ-factors

$$\gamma^{\mathrm{CFGK}}(s, \tau \times \sigma, \psi_E)$$

for all irreducible representations τ and σ of $\mathrm{GL}_k(E)$ and G_0°, respectively. In particular, when G_0° is a split special orthogonal group or a symplectic group, the "Ten Commandments" were proved in [CFK1, Theorem 4.2]. In this case, we can modify the above argument and show that

$$\gamma^{\mathrm{CFGK}}(s, \tau \times \sigma, \psi_E) = \gamma(s, \phi_\tau \otimes \phi_\sigma, \psi_E)$$

as follows: To globalize an irreducible square-integrable representation σ of G_0°, we can use [Ar2, Lemma 6.2.2] to find an irreducible cuspidal automorphic representation $\dot{\sigma}$ of $\dot{G}_0^\circ(\dot{\mathbb{A}})$ with generic global A-parameter such that $\dot{\sigma}_{v_0} \cong \sigma$ and $\dot{\sigma}_v$ is spherical for all finite places $v \neq v_0$.

Remark A.4.2. The second and third authors would like to take this opportunity to remark that the various desiderata of the local Langlands correspondence used in [GI1, GI2] are now supplied by the results of this paper in the case of quasi-split classical groups. Namely, in the proofs of [GI1, Theorem C.5] and [GI2, Proposition B.1], they assumed the following hypothesis:

(1) the equality between the local γ-factors of Shahidi and the corresponding Artin γ-factors;
(2) the equality between the local γ-factors of Piatetski-Shapiro–Rallis and the corresponding Artin γ-factors;
(3) the formula for the Plancherel measures in terms of Artin γ-factors.

(See [GI1, Section C.2] and [GI2, Section B.2] for details.) Now (1) follows from Proposition A.2.1 and [He3], [CST], [Shan], [He4], (2) can be verified as in Remark A.4.1 (where the "Ten Commandments" in this case were proved in [LR, Theorem 4]), and (3) is a consequence of the multiplicative property of the normalized intertwining operators (see Proposition 1.7.2). They would also like to point out that the reason they gave for the correct formulation of Shahidi's formula at the end of the proof of [GI2, Lemma B.2] is not accurate: the proper justifications for the reformulation of Shahidi's results are given in Section 2.6 of this paper.

A.5. **Dependence on ψ_F for Shahidi's gamma factor.** The property (2) in the proof of Proposition A.2.1 is not stated in this form in [Sha7] but can be derived as follows. Suppose first that we are not in the case where $G = \mathrm{O}_{2k}^{\eta}(F)$ with $\eta = 1$, $M = \mathrm{GL}_k(F)$, and $k > 1$ is odd. Put $\pi = \tau \boxtimes \sigma_0^\vee$ (regarded as a representation of M) and write $\pi_\lambda = \tau|\det|_E^s \boxtimes \sigma_0^\vee$ with $s \in \mathbb{C}$. We also write

$$C_P(w, \pi_\lambda) = C_P(w, \pi_\lambda, \psi_F), \quad J_P(w, \pi_\lambda) = J_P(w, \pi_\lambda, \psi_F), \quad \Omega(\pi_\lambda) = \Omega_\chi(\pi_\lambda)$$

to indicate the dependence on ψ_F or χ. They are related by

$$\Omega_\chi(\pi_\lambda) = C_P(w, \pi_\lambda, \psi_F) \cdot \Omega_\chi(w\pi_\lambda) \circ J_P(w, \pi_\lambda, \psi_F)$$

as in Section 2.1. Similarly, we denote by du (resp. $d'u$) the Haar measure on N_P given by ***spl*** and ψ_F (resp. ψ'_F). If we set $l = \dim(N_P)$, then

$$d'u = |a|_F^{\frac{l}{2}}\, du.$$

Let χ' be the non-degenerate character of U determined by ***spl*** and ψ'_F. Then we have

$$\chi'(u) = \chi(\mathrm{Ad}(t_0)u)$$

for all $u \in U$, where $t_0 \in A_T(\overline{F})$ is given by

$$t_0 = \begin{cases} \mathrm{diag}(a^{n-\frac{1}{2}}, a^{n-\frac{3}{2}}, \dots, a^{\frac{1}{2}}, a^{-\frac{1}{2}}, \dots, a^{-n+\frac{3}{2}}, a^{-n+\frac{1}{2}}) & \text{if } G = \mathrm{Sp}_{2n}(F), \\ \mathrm{diag}(a^{n}, a^{n-1}, \dots, a, 1, a^{-1}, \dots, a^{-n+1}, a^{-n}) & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \mathrm{diag}(a^{n-1}, a^{n-2}, \dots, a, 1, 1, a^{-1}, \dots, a^{-n+2}, a^{-n+1}) & \text{if } G = \mathrm{O}_{2n}^{\eta}(F), \\ \mathrm{diag}(a^{\frac{n-1}{2}}, a^{\frac{n-3}{2}}, \dots, a^{-\frac{n-3}{2}}, a^{-\frac{n-1}{2}}) & \text{if } G = \mathrm{U}_n. \end{cases}$$

Note that $\mathrm{Ad}(t_0)$ is an automorphism of G defined over F. Set $z_0 = \mathrm{Ad}(t_0)(\widetilde{w}^{-1}) \cdot \widetilde{w}$. Recall that $\widetilde{w}$ is a representative of $w_T \in N(M^\circ, M^\circ)/T^\circ$. It is easy to see that a representative of w_T is given by

$$\begin{pmatrix} & & \mathbf{1}_k \\ & \mathbf{1}_{G_0} & \\ \pm\mathbf{1}_k & & \end{pmatrix}$$

for some sign. Hence $z_0 = \mathrm{diag}(a^{N_0+k+\delta}\mathbf{1}_k, \mathbf{1}_{G_0}, a^{-(N_0+k+\delta)}\mathbf{1}_k)$ with

$$\delta = \begin{cases} 1 & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ -1 & \text{if } G = \mathrm{Sp}_{2n}(F),\ \mathrm{O}_{2n}^{\eta}(F), \\ 0 & \text{if } G = \mathrm{U}_n. \end{cases}$$

Recall that $N_0 = \dim(\phi_\sigma)$ with $\phi_\sigma \in \Phi(G_0)$. In particular, z_0 belongs to the center of M.

Put $\pi'_\lambda = \pi_\lambda \circ \mathrm{Ad}(t_0)$. For $f \in I_P(\pi_\lambda)$, we define $f' \in I_P(\pi'_\lambda)$ by $f' = f \circ \mathrm{Ad}(t_0)$. Then we claim that

$$J_P(w, \pi'_\lambda, \psi'_F)f' = \omega_{\pi_\lambda}(z_0) \cdot |a|_F^{\frac{l}{2}}(J_P(w, \pi_\lambda, \psi_F)f)',$$

$$\Omega_{\chi'}(\pi'_\lambda)f' = \omega_{\pi_\lambda}(z_0) \cdot |a|_F^{\frac{l}{2}} \Omega_\chi(\pi_\lambda)f.$$

Indeed, if we set $u' = \mathrm{Ad}(t_0)(u)$ and $g' = \mathrm{Ad}(t_0)(g)$, then we have

$$\begin{aligned}
(J_P(w,\pi'_\lambda,\psi'_F)f')(g) &= \int_{N_P} f'(\widetilde{w}^{-1}ug)d'u \\
&= \int_{N_P} f(z_0 \cdot \widetilde{w}^{-1}u'g')d'u \\
&= \delta_P(t_0)^{-1} \int_{N_P} f(z_0 \cdot \widetilde{w}^{-1}u'g')d'u' \\
&= \delta_P(t_0)^{-1} \int_{N_P} \delta_P(z_0)^{\frac{1}{2}} \pi_\lambda(z_0) f(\widetilde{w}^{-1}u'g')d'u' \\
&= \delta_P(t_0)^{-1} \cdot \delta_P(z_0)^{\frac{1}{2}} \omega_{\pi_\lambda}(z_0) \cdot |a|_F^{\frac{l}{2}} \int_{N_P} f(\widetilde{w}^{-1}u'g')du'.
\end{aligned}$$

Since $\delta_P(t_0) = \delta_P(z_0)^{\frac{1}{2}}$, we obtain that

$$(J_P(w,\pi'_\lambda,\psi'_F)f')(g) = \omega_{\pi_\lambda}(z_0) \cdot |a|_F^{\frac{l}{2}} \cdot (J_P(w,\pi_\lambda,\psi_F)f)(g').$$

Similarly, we have

$$\begin{aligned}
\Omega_{\chi'}(\pi'_\lambda)f' &= \int_{N_P} \omega(f'(\widetilde{w}^{-1}u))\chi'(u)^{-1}d'u \\
&= \int_{N_P} \omega(f(z_0\widetilde{w}^{-1}u'))\chi(u')^{-1}d'u \\
&= \delta_P(t_0)^{-1} \cdot \delta_P(z_0)^{\frac{1}{2}} \omega_{\pi_\lambda}(z_0) \cdot |a|_F^{\frac{l}{2}} \Omega_\chi(\pi_\lambda)f.
\end{aligned}$$

Hence

$$\begin{aligned}
\Omega_{\chi'}(\pi'_\lambda)f' &= C_P(w,\pi'_\lambda,\psi'_F) \cdot \Omega_{\chi'}(w\pi'_\lambda) \circ J_P(w,\pi'_\lambda,\psi'_F)f' \\
&= \omega_{\pi_\lambda}(z_0)|a|_F^{\frac{l}{2}} \cdot C_P(w,\pi'_\lambda,\psi'_F) \cdot \Omega_{\chi'}(w\pi'_\lambda)(J_P(w,\pi_\lambda,\psi_F)f)'
\end{aligned}$$

so that

$$\Omega_\chi(\pi_\lambda)f = \omega_{w\pi_\lambda}(z_0)|a|_F^{\frac{l}{2}} \cdot C_P(w,\pi'_\lambda,\psi'_F) \cdot \Omega_\chi(w\pi_\lambda) \circ J_P(w,\pi_\lambda,\psi_F)f.$$

Since $\pi'_\lambda \cong \pi_\lambda$, it implies that

$$C_P(w,\pi_\lambda,\psi'_F) = \omega_{w\pi_\lambda}(z_0)^{-1}|a|_F^{-\frac{l}{2}} \cdot C_P(w,\pi_\lambda,\psi_F).$$

Recall that

$$C_P(w,\pi_\lambda,\psi_F) = \lambda(w,\psi_F)^{-1}\lambda(E/F,\psi_F)^{kN_0}\gamma^{\mathrm{Sh}}(s,\tau\times\sigma,\psi_E)\gamma^{\mathrm{Sh}}(2s,\tau,R,\psi_F)$$

with

$$R = \begin{cases} \mathrm{Sym}^2 & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \wedge^2 & \text{if } G = \mathrm{Sp}_{2n}(F),\ \mathrm{O}_{2n}^{\eta}(F), \\ \mathrm{Asai}^+ & \text{if } G = \mathrm{U}_n,\ n \equiv 0 \bmod 2, \\ \mathrm{Asai}^- & \text{if } G = \mathrm{U}_n,\ n \equiv 1 \bmod 2. \end{cases}$$

Note that

$$\lambda(w, \psi_F) = \begin{cases} \lambda(K/F, \psi_F)^k & \text{if } G = \mathrm{O}_{2n}^{\eta}(F), \\ \lambda(E/F, \psi_F)^{kN_0 + \frac{k(k-1)}{2}} & \text{if } G = \mathrm{U}_n,\ n \equiv 0 \bmod 2, \\ \lambda(E/F, \psi_F)^{kN_0 + \frac{k(k+1)}{2}} & \text{if } G = \mathrm{U}_n,\ n \equiv 1 \bmod 2, \\ 1 & \text{otherwise} \end{cases}$$

where K/F is the abelian extension corresponding to η. Since

$$\lambda(K/F, \psi_F) = \frac{\varepsilon(s, \mathrm{Ind}_{W_K}^{W_F}(\mathbf{1}_{W_K}), \psi_F)}{\varepsilon(s, \mathbf{1}_{W_K}, \psi_F \circ \mathrm{tr}_{K/F})}$$

which does not depend on s, we have

$$\frac{\lambda(K/F, \psi'_F)}{\lambda(K/F, \psi_F)} = \det(\mathrm{Ind}_{W_K}^{W_F}(\mathbf{1}_K))(a) = \eta(a).$$

Hence

$$\frac{\lambda(w, \psi'_F)}{\lambda(w, \psi_F)} = \begin{cases} \eta(a)^k & \text{if } G = \mathrm{O}_{2n}^{\eta}(F), \\ \eta'(a)^{kN_0 + \frac{k(k-1)}{2}} & \text{if } G = \mathrm{U}_n,\ n \equiv 0 \bmod 2, \\ \eta'(a)^{kN_0 + \frac{k(k+1)}{2}} & \text{if } G = \mathrm{U}_n,\ n \equiv 1 \bmod 2, \\ 1 & \text{otherwise,} \end{cases}$$

where, if $G = \mathrm{U}_n$, we set η' to be the quadratic character of $F^\times$ associated to E/F by the class field theory. If $G \neq \mathrm{O}_{2n}^{\eta}(F)$ (resp. $G \neq \mathrm{U}_n$), we simply set $\eta = \mathbf{1}$ (resp. $\eta' = \mathbf{1}$). Since $\omega_{w\pi_\lambda}(z_0)^{-1} = \omega_\tau(a^{N_0+k+\delta})|a|_E^{(N_0+k+\delta)ks}$, we obtain

$$\begin{aligned} \gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi'_E)\gamma^{\mathrm{Sh}}(2s, \tau, R, \psi'_F) = \eta(a)^k \eta'(a)^{\frac{k(k\pm 1)}{2}} \cdot \omega_\tau(a^{N_0+k+\delta})|a|_E^{(N_0+k+\delta)ks} \cdot |a|_F^{-\frac{l}{2}} \\ \times \gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E)\gamma^{\mathrm{Sh}}(2s, \tau, R, \psi_F), \end{aligned}$$

where the exponent of $\eta'(a)$ is $\frac{k(k-1)}{2}$ (resp. $\frac{k(k+1)}{2}$) for $G = \mathrm{U}_n$ with n even (resp. n odd).

If $G = \mathrm{O}_{2k}^{\eta}(F)$ with $\eta = 1$, $M = \mathrm{GL}_k(F)$, and $k > 1$ is odd, then we take $w \in W(M^\circ, \epsilon M^\circ \epsilon^{-1})$ such that $\det(w) = 1$. Since t_0 is commutative with ϵ, the above computation works after replacing $\omega_{w\pi_\lambda}(z_0)$ with $\omega_{w\pi_\lambda}(\epsilon z_0 \epsilon^{-1})$. Hence we obtain the same formula for $\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi'_E)\gamma^{\mathrm{Sh}}(2s, \tau, R, \psi'_F)$.

In particular, if we take $(G, M) = (\mathrm{SO}_{2k+1}(F), \mathrm{GL}_k(F))$, $(\mathrm{O}_{2k}^\eta(F), \mathrm{GL}_k(F))$ with $\eta = 1$, and $(\mathrm{U}_{2k}, \mathrm{GL}_k(E))$, then we obtain

$$\gamma^{\mathrm{Sh}}(2s, \tau, R, \psi'_F) = \eta'(a)^{\frac{k(k-1)}{2}} \omega_\tau(a)^{k+\delta(R)} |a|_F^{(2s-\frac{1}{2})d(R)} \gamma^{\mathrm{Sh}}(2s, \tau, R, \psi_F),$$

where

$$\delta(R) = \begin{cases} 1 & \text{if } R = \mathrm{Sym}^2, \\ -1 & \text{if } R = \wedge^2, \\ 0 & \text{if } R = \mathrm{Asai}^+, \end{cases} \qquad d(R) = \begin{cases} \dfrac{k(k+1)}{2} & \text{if } R = \mathrm{Sym}^2, \\ \dfrac{k(k-1)}{2} & \text{if } R = \wedge^2, \\ k^2 & \text{if } R = \mathrm{Asai}^+. \end{cases}$$

Moreover, in the last case, it follows from the definition (see [Sha7, p. 304]) that

$$\gamma^{\mathrm{Sh}}(2s, \tau, \mathrm{Asai}^-, \psi_F) = \gamma^{\mathrm{Sh}}(2s, \tau \otimes (\mu \circ \det), \mathrm{Asai}^+, \psi_F),$$

where μ is a character of $E^\times$ such that $\mu|_{F^\times} = \eta'$. Since $k^2 \equiv k \bmod 2$, we obtain that

$$\gamma(2s, \tau, \mathrm{Asai}^-, \psi'_F) = \eta'(a)^{\frac{k(k+1)}{2}} \omega_\tau(a)^k |a|_F^{(2s-\frac{1}{2})k^2} \gamma^{\mathrm{Sh}}(2s, \tau, \mathrm{Asai}^-, \psi_F).$$

This implies that

$$\gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi'_E) = \eta(a)^k \omega_\tau(a)^{N_0} |a|_E^{kN_0(s-\frac{1}{2})} \gamma^{\mathrm{Sh}}(s, \tau \times \sigma, \psi_E),$$

as desired.

A.6. **Proof of Proposition 1.7.2.** Here, we give a proof of Proposition 1.7.2 for classical groups in general. Recall the setting. Let G be a quasi-split classical group. For $i = 1, 2, 3$, consider a standard parabolic subgroup $P_i = M_i N_i$ of G. Assume that $W(M_1^\circ, M_2^\circ) \neq \emptyset$ and $W(M_2^\circ, M_3^\circ) \neq \emptyset$. Then for $w_1 \in W(M_1^\circ, M_2^\circ)$ and $w_2 \in W(M_2^\circ, M_3^\circ)$, and for an irreducible tempered representation π of M_1, Proposition 1.7.2 asserts that

$$R_{P_1}(w_2 w_1, \pi_\lambda) = R_{P_2}(w_2, w_1 \pi_\lambda) \circ R_{P_1}(w_1, \pi_\lambda).$$

Set $w_1^{-1} P_2 = \widetilde{w}_1^{-1} P_2 \widetilde{w}_1$. Following [Ar2, Section 2.3], we decompose $R_{P_1}(w_1, \pi_\lambda) \colon I_{P_1}(\pi_\lambda) \to I_{P_2}(w_1 \pi_\lambda)$ as

$$R_{P_1}(w_1, \pi_\lambda) = \ell(w_1, \pi_\lambda) \circ R_{w_1^{-1} P_2 | P_1}(\pi_\lambda),$$

where $R_{w_1^{-1} P_2 | P_1}(\pi_\lambda) \colon I_{P_1}(\pi_\lambda) \to I_{w_1^{-1} P_2}(\pi_\lambda)$ is given by (the meromorphic continuation of) the integral

$$(R_{w_1^{-1} P_2 | P_1}(\pi_\lambda) f_\lambda)(g) = \frac{\gamma_A(0, \pi_\lambda, \rho^\vee_{w_1^{-1} P_2 | P_1}, \psi_F)}{\varepsilon(1/2, \pi_\lambda, \rho^\vee_{w_1^{-1} P_2 | P_1}, \psi_F)} \int_{(N_1 \cap \widetilde{w}_1^{-1} N_2 \widetilde{w}_1) \backslash \widetilde{w}_1^{-1} N_2 \widetilde{w}_1} f_\lambda(ug) du$$

and $\ell(w_1, \pi_\lambda) \colon I_{w_1^{-1} P_2}(\pi_\lambda) \to I_{P_2}(w_1 \pi_\lambda)$ is defined by

$$\ell(w_1, \pi_\lambda) = \lambda(w_1)^{-1} \varepsilon(1/2, \pi_\lambda, \rho^\vee_{w_1^{-1} P_2 | P_1}, \psi_F) L(\widetilde{w}_1)$$

with $L(\widetilde{w}_1) f'_\lambda(g) = f'_\lambda(\widetilde{w}_1^{-1} g)$. The key property of $\ell(w_1, \pi_\lambda)$ is as follows.

Lemma A.6.1. *The operator $\ell(w_1, \pi_\lambda)$ satisfies the condition*

$$\ell(w_2 w_1, \pi_\lambda) = \ell(w_2, w_1\pi_\lambda) \circ \ell(w_1, \pi_\lambda).$$

Before showing this lemma, we prove Proposition 1.7.2.

Proof of Proposition 1.7.2. By [Ar2, Proposition 2.3.1], [Mok, Proposition 3.3.1] and Lemma A.6.1, we have

$$\begin{aligned}
&R_{P_2}(w_2, w_1\pi_\lambda) \circ R_{P_1}(w_1, \pi_\lambda)\\
&= \Big(\ell(w_2, w_1\pi_\lambda) \circ R_{w_2^{-1}P_3|P_2}(w_1\pi_\lambda)\Big) \circ \Big(\ell(w_1, \pi_\lambda) \circ R_{w_1^{-1}P_2|P_1}(\pi_\lambda)\Big)\\
&= \ell(w_2, w_1\pi_\lambda) \circ \ell(w_1, \pi_\lambda) \circ R_{w_1^{-1}w_2^{-1}P_3|w_1^{-1}P_2}(\pi_\lambda) \circ R_{w_1^{-1}P_2|P_1}(\pi_\lambda)\\
&= \ell(w_2w_1, \pi_\lambda) \circ R_{(w_2w_1)^{-1}P_3|P_1}(\pi_\lambda) = R_{P_1}(w_2w_1, \pi_\lambda).
\end{aligned}$$

This completes the proof of Proposition 1.7.2. □

Therefore, Lemma A.6.1 is the missing part for Proposition 1.7.2. For the proof of Lemma A.6.1, we need the following elementary fact.

Lemma A.6.2. *Let V be a finite dimensional real vector space. For $i = 1, 2, 3$, let $P_i \subset V$ be a finite subset such that the union $R_i = P_i \cup -P_i$ is disjoint. For $\beta_i \in R_i$, write $\beta_i > 0$ (resp. $\beta_i < 0$) if $\beta_i \in P_i$ (resp. $-\beta_i \in P_i$). Fix two automorphisms w_1 and w_2 on V such that $w_1(R_1) = R_2$ and $w_2(R_2) = R_3$. Then for a function $f\colon R_1 \to \mathbb{C}^\times$, we have*

$$\prod_{\substack{\beta_1>0\\ w_1\beta_1<0}} f(\beta_1) \cdot \prod_{\substack{w_1\beta_1>0\\ w_2w_1\beta_1<0}} f(\beta_1) \cdot \prod_{\substack{\beta_1>0\\ w_2w_1\beta_1<0}} f(\beta_1)^{-1} = \prod_{\substack{\beta_1>0\\ w_1\beta_1<0,\, w_2w_1\beta_1>0}} f(\beta_1)f(-\beta_1).$$

Proof. We write $R_1 = \sqcup_{k=1}^8 I_k$, where

$$\begin{aligned}
I_1 &= \{\beta_1 \in R_1 \,|\, \beta_1 > 0,\ w_1\beta_1 > 0,\ w_2w_1\beta_1 > 0\},\\
I_2 &= \{\beta_1 \in R_1 \,|\, \beta_1 > 0,\ w_1\beta_1 > 0,\ w_2w_1\beta_1 < 0\},\\
I_3 &= \{\beta_1 \in R_1 \,|\, \beta_1 > 0,\ w_1\beta_1 < 0,\ w_2w_1\beta_1 > 0\},\\
I_4 &= \{\beta_1 \in R_1 \,|\, \beta_1 > 0,\ w_1\beta_1 < 0,\ w_2w_1\beta_1 < 0\},\\
I_5 &= \{\beta_1 \in R_1 \,|\, \beta_1 < 0,\ w_1\beta_1 > 0,\ w_2w_1\beta_1 > 0\},\\
I_6 &= \{\beta_1 \in R_1 \,|\, \beta_1 < 0,\ w_1\beta_1 > 0,\ w_2w_1\beta_1 < 0\},\\
I_7 &= \{\beta_1 \in R_1 \,|\, \beta_1 < 0,\ w_1\beta_1 < 0,\ w_2w_1\beta_1 > 0\},\\
I_8 &= \{\beta_1 \in R_1 \,|\, \beta_1 < 0,\ w_1\beta_1 < 0,\ w_2w_1\beta_1 < 0\}.
\end{aligned}$$

Then

$$\prod_{\substack{\beta_1>0\\ w_1\beta_1<0}} f(\beta_1) = \prod_{\beta_1\in I_3\sqcup I_4} f(\beta_1), \qquad \prod_{\substack{w_1\beta_1>0\\ w_2w_1\beta_1<0}} f(\beta_1) = \prod_{\beta_1\in I_2\sqcup I_6} f(\beta_1),$$

$$\prod_{\substack{\beta_1>0 \\ w_2w_1\beta_1<0}} f(\beta_1) = \prod_{\beta_1\in I_2\sqcup I_4} f(\beta_1), \qquad \prod_{\substack{\beta_1>0 \\ w_1\beta_1<0,\, w_2w_1\beta_1>0}} f(\beta_1)f(-\beta_1) = \prod_{\beta_1\in I_3\sqcup I_6} f(\beta_1).$$

This implies the lemma. □

Now we prove Lemma A.6.1.

Proof of Lemma A.6.1. When $G = G^\circ$ and $M_1 = M_2 = M_3$, the lemma is [Ar2, Lemma 2.3.4] and [Mok, Lemma 3.3.4]. The proof of the general case is essentially the same as these lemmas.

Since $\widetilde{w}_2\widetilde{w}_1$ and $\widetilde{w_2w_1}$ are two representatives of $w_2w_1 \in W(M_1^\circ, M_3^\circ)$, we can write

$$\widetilde{w}_2\widetilde{w}_1 = \widetilde{w_2w_1} \cdot z(w_2, w_1)$$

for some $z(w_2, w_1) \in M_1^\circ$. By Lemma 1.7.1, we see that $z(w_2, w_1)$ is in the center of M_1 so that

$$L(\widetilde{w_2w_1}) = \omega_{\pi_\lambda}(z(w_2, w_1))L(\widetilde{w}_2) \circ L(\widetilde{w}_1),$$

where ω_{π_λ} is the central character of π_λ. Our goal is to show that $\omega_{\pi_\lambda}(z(w_2, w_1))$ is equal to the product of

$$\varepsilon(1/2, \pi_\lambda, \rho^\vee_{w_1^{-1}P_2|P_1}, \psi_F)\varepsilon(1/2, w_1\pi_\lambda, \rho^\vee_{w_2^{-1}P_3|P_2}, \psi_F)\varepsilon(1/2, \pi_\lambda, \rho^\vee_{(w_2w_1)^{-1}P_3|P_1}, \psi_F)^{-1}$$

and

$$\lambda(w_1)^{-1}\lambda(w_2)^{-1}\lambda(w_2w_1).$$

First, we consider the central character. Let $R(T^\circ, G^\circ)$ be the set of roots of T° in G°. As in the proofs of [Ar2, Lemma 2.3.4] and [Mok, Lemma 3.3.4], by [LSh, Lemma 2.1.A], we have

$$z(w_2, w_1) = (-1)^{\lambda^\vee(w_2, w_1)},$$

where $\lambda^\vee(w_2, w_1) = \sum_{\alpha\in R_B(w_2,w_1)} \alpha^\vee$ with

$$R_B(w_2, w_1) = \{\alpha \in R(T^\circ, G^\circ) \,|\, \alpha > 0,\ w_1\alpha < 0,\ w_2w_1\alpha > 0\}.$$

Moreover, $z(w_2, w_1)$ is in the split part $A_{M_1^\circ}$ of the center of M_1°, i.e., $\lambda^\vee(w_2, w_1) \in X_*(A_{M_1^\circ})$.

For $i = 1, 2, 3$, we set $R_i = R(A_{\widehat{M_i^\circ}}, \widehat{G^\circ})$. For $\beta_i \in R_i$, we denote by $\beta_i > 0$ if the weight space $\widehat{\mathfrak{g}}_{\beta_i}$ lies in $\widehat{\mathfrak{n}}_i$. Via the isomorphism $X_*(A_{M_1^\circ}) \cong X^*(\widehat{M_1^\circ})_F$ in [Ar2, Section 2.3], we can identify $\lambda^\vee(w_2, w_1) \in X_*(A_{M^\circ})$ with a character of ${}^LM_1^\circ$ which is trivial on the semi-direct factor W_F of ${}^LM_1^\circ$. Then we can write

$$\lambda^\vee(w_2, w_1) = \sum_{\substack{\beta_1>0 \\ w_1\beta_1<0,\, w_2w_1\beta_1>0}} \lambda^\vee_{\beta_1},$$

where β_1 runs over the elements of R_1 satisfying the specified conditions, and

$$\lambda^\vee_{\beta_1} = \sum_{\alpha\in R_{\beta_1}} \alpha^\vee$$

with R_{β_1} being the set of roots α of T° in G° such that $\alpha^\vee|_{A_{M_1^\circ}}$ is a positive multiple of β. Hence, fixing an arbitrary element $u \in W_F$ whose image in $F^\times \cong W_F^{\mathrm{ab}}$ is equal to -1, we have

$$\omega_{\pi_\lambda}(z(w_2, w_1)) = \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0,\, w_2w_1\beta_1 > 0}} \lambda_\beta^\vee(\phi_\lambda(u)) = \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0,\, w_2w_1\beta_1 > 0}} \lambda_\beta^\vee(m(u))$$

where ϕ_λ is the L-parameter for π_λ, and we write $\phi_\lambda(u) = m(u) \rtimes u$ with $m(u) \in \widehat{M_1^\circ}$.

Next, we consider the ε-factors. For simplicity, we write $\varepsilon(\pi_\lambda, \rho)$ for $\varepsilon(1/2, \pi_\lambda, \rho, \psi_F)$. The adjoint representation $\rho_{w_1^{-1}P_2|P_1}$ of ${}^L M_1^\circ$ on

$$\widetilde{w}_1^{-1}\widehat{\mathfrak{n}}_2\widetilde{w}_1/(\widetilde{w}_1^{-1}\widehat{\mathfrak{n}}_2\widetilde{w}_1 \cap \widehat{\mathfrak{n}}_1) \cong \widetilde{w}_1^{-1}\widehat{\mathfrak{n}}_2\widetilde{w}_1 \cap \widehat{\overline{\mathfrak{n}}}_1$$

decomposes as the direct sum

$$\bigoplus_{\substack{\beta_1 < 0 \\ w_1\beta_1 > 0}} \widehat{\mathfrak{g}}_{\beta_1},$$

where β_1 runs over the elements of R_1 satisfying the specified conditions. If we denote the adjoint action of ${}^L M_1^\circ$ on $\widehat{\mathfrak{g}}_{\beta_1}$ by ρ_{β_1}, then we obtain that

$$\varepsilon(\pi_\lambda, \rho^\vee_{w_1^{-1}P_2|P_1}) = \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0}} \varepsilon(\pi_\lambda, \rho_{\beta_1}).$$

In particular, by applying Lemma A.6.2 to the function $f(\beta_1) = \varepsilon(\pi_\lambda, \rho_{\beta_1})$ together with [Tate, (3.6.8)], we see that

$$\begin{aligned}
&\varepsilon(\pi_\lambda, \rho^\vee_{w_1^{-1}P_2|P_1})\varepsilon(w_1\pi_\lambda, \rho^\vee_{w_2^{-1}P_3|P_2})\varepsilon(\pi_\lambda, \rho^\vee_{(w_2w_1)^{-1}P_3|P_1})^{-1} \\
&= \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0}} f(\beta_1) \cdot \prod_{\substack{w_1\beta_1 > 0 \\ w_2w_1\beta_1 < 0}} f(\beta_1) \cdot \prod_{\substack{\beta_1 > 0 \\ w_2w_1\beta_1 < 0}} f(\beta_1)^{-1} \\
&= \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0,\, w_2w_1\beta_1 > 0}} f(\beta_1)f(-\beta_1) \\
&= \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0,\, w_2w_1\beta_1 > 0}} \varepsilon(\pi_\lambda, \rho_{\beta_1})\varepsilon(\pi_\lambda, \rho^\vee_{\beta_1}) \\
&= \prod_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0,\, w_2w_1\beta_1 > 0}} \det(\rho_{\beta_1} \circ \phi_\lambda(u)).
\end{aligned}$$

Writing $\phi_\lambda(u) = m(u) \rtimes u$ with $m(u) \in \widehat{M_1^\circ}$, this product is equal to

$$\det\left(\mathrm{Ad}(m(u) \rtimes u); \bigoplus_{\substack{\beta_1 > 0 \\ w_1\beta_1 < 0,\, w_2w_1\beta_1 > 0}} \widehat{\mathfrak{g}}_{\beta_1}\right).$$

Since $1 \times u$ acts on $A_{\widehat{M_1^\circ}}$ trivially, the direct sum is stable under the adjoint action of $1 \rtimes u$, and we may compute this determinant as the product of the ones of $\mathrm{Ad}(m(u))$ and $\mathrm{Ad}(1 \rtimes u)$. For the determinant of $\mathrm{Ad}(m(u))$, we may assume that $m(u) \in \widehat{M_1^\circ}$ is in $\widehat{T^\circ}$. Then we conclude that

$$\det\left(\mathrm{Ad}(m(u)); \bigoplus_{\substack{\beta_1>0 \\ w_1\beta_1<0,\, w_2w_1\beta_1>0}} \widehat{\mathfrak{g}}_{\beta_1}\right) = \prod_{\substack{\beta_1>0 \\ w_1\beta_1<0,\, w_2w_1\beta_1>0}} \lambda_{\beta_1}^\vee(m(u)),$$

which is equal to $\omega_{\pi_\lambda}(z(w_2, w_1))$ as we have seen above.

Finally, we show that

$$\det\left(\mathrm{Ad}(1 \rtimes u); \bigoplus_{\substack{\beta_1>0 \\ w_1\beta_1<0,\, w_2w_1\beta_1>0}} \widehat{\mathfrak{g}}_{\beta_1}\right) = \lambda(w_1)\lambda(w_2)\lambda(w_2w_1)^{-1}.$$

Recall that if we set Δ_1 (resp. Δ_2) to be the set of reduced roots $a \in R(A_{T^\circ}, G^\circ)$ with $G_{a,sc} \cong \mathrm{Res}_{F_a/F}\mathrm{SL}_2$ (resp. $\mathrm{Res}_{F_a/F}\mathrm{SU}_{E_a/F_a}(2,1)$), and if we define $f_1(a) = \lambda(F_a/F, \psi_F)$ (resp. $f_2(a) = \lambda(E_a/F, \psi_F)^2\lambda(F_a/F, \psi_F)^{-1}$), then

$$\begin{aligned}\lambda(w_1)\lambda(w_2)\lambda(w_2w_1)^{-1} = &\prod_{\substack{a\in\Delta_1 \\ a>0, w_1a<0}} f_1(a) \cdot \prod_{\substack{a\in\Delta_1 \\ w_1a>0, w_2w_1a<0}} f_1(w_1a) \cdot \prod_{\substack{a\in\Delta_1 \\ a>0, w_2w_1a<0}} f_1(a)^{-1} \\ &\times \prod_{\substack{a\in\Delta_2 \\ a>0, w_1a<0}} f_2(a) \cdot \prod_{\substack{a\in\Delta_2 \\ w_1a>0, w_2w_1a<0}} f_2(w_1a) \cdot \prod_{\substack{a\in\Delta_2 \\ a>0, w_2w_1a<0}} f_2(a)^{-1}.\end{aligned}$$

Here, we notice that w_1 induces a bijection on Δ_i, and $f_i(w_1a) = f_i(a)$ for $i = 1, 2$. Applying Lemma A.6.2 twice, we have

$$\lambda(w_1)\lambda(w_2)\lambda(w_2w_1)^{-1} = \prod_{\substack{a\in\Delta_1 \\ a>0, w_1a<0, w_2w_1a>0}} f_1(a)^2 \cdot \prod_{\substack{a\in\Delta_2 \\ a>0, w_1a<0, w_2w_1a>0}} f_2(a)^2$$

since $f_i(-a) = f_i(a)$. Therefore, what we need to show is that $\det(\mathrm{Ad}(1 \rtimes u); \widehat{\mathfrak{g}}_{(a^\vee)}) = f_i(a)^2$ if $a \in \Delta_i$ for $i = 1, 2$, where we set $\widehat{\mathfrak{g}}_{(a^\vee)} = \widehat{\mathfrak{g}}_{a^\vee} \oplus \widehat{\mathfrak{g}}_{2a^\vee}$ and

$$\widehat{\mathfrak{g}}_{a^\vee} = \bigoplus_{\substack{\alpha\in R(T^\circ, G^\circ) \\ \alpha|_{A_T^\circ}=a}} \widehat{\mathfrak{g}}_{\alpha^\vee}.$$

When $a \in \Delta_1$, the W_F-module $\widehat{\mathfrak{g}}_{(a^\vee)}$ is isomorphic to $\mathrm{Ind}_{W_{F_a}}^{W_F}(\mathbf{1}_{F_a^\times})$ and hence

$$\det(\mathrm{Ad}(1 \rtimes u); \widehat{\mathfrak{g}}_{(a^\vee)}) = \det\left(\mathrm{Ind}_{W_{F_a}}^{W_F}(\mathbf{1}_{F_a^\times})\right)(-1) = \lambda(F_a/F, \psi_F)^2.$$

If $a \in \Delta_2$, then the W_F-module $\widehat{\mathfrak{g}}_{(a^\vee)}$ is isomorphic to $\mathrm{Ind}_{W_{F_a}}^{W_F}(\widehat{\mathfrak{n}}_{\mathrm{SU}(3)})$, where $\widehat{\mathfrak{n}}_{\mathrm{SU}(3)}$ is the space of the sum of positive root spaces in the Lie algebra of the dual group of

$\mathrm{SU}_{E_a/F_a}(3)$. It is isomorphic to

$$\mathrm{Ind}_{W_{E_a}}^{W_{F_a}}(\mathbf{1}_{E_a^\times}) \oplus \eta_{E_a/F_a} \cong \mathbf{1}_{F_a^\times} \oplus \eta_{E_a/F_a} \oplus \eta_{E_a/F_a},$$

where η_{E_a/F_a} is the quadratic character of $F_a^\times$ associated to E_a/F_a by the local class field theory. Hence

$$\begin{aligned}\det(\mathrm{Ad}(1 \rtimes u); \widehat{\mathfrak{g}}_{(a^\vee)}) &= \det\left(\mathrm{Ind}_{W_{F_a}}^{W_F}(\mathbf{1}_{F_a^\times})\right)(-1) \cdot \det\left(\mathrm{Ind}_{W_{F_a}}^{W_F}(\eta_{E_a/F_a})\right)(-1)^2 \\ &= \det\left(\mathrm{Ind}_{W_{F_a}}^{W_F}(\mathbf{1}_{F_a^\times})\right)(-1) = \lambda(F_a/F, \psi_F)^2.\end{aligned}$$

Since $\lambda(E_a/F, \psi_F)^4 = 1$, we have $f_2(a)^2 = \lambda(E_a/F, \psi_F)^4 \lambda(F_a/F, \psi_F)^{-2} = \lambda(F_a/F, \psi_F)^2$. This completes the proof of Lemma A.6.1. □

This result, along with others on the LIR will be extended to general disconnected groups in a forthcoming paper.

Appendix B. Review of Aubert duality

The purpose of this appendix is to review several properties of Aubert duality, which we used in the main body. In particular, we prove the commutativity of the normalized intertwining operators with the Aubert involution up to scalars. It is a crucial result to prove the local intertwining relations for co-tempered A-packets. This appendix is an adaptation of an appendix in an arXiv version of [KMSW].

We also define the twisted Aubert dual, and establish some properties that we need in this paper. A more detailed study of the twisted Aubert dual will appear in a forthcoming paper.

B.1. **Definition of a complex.** Let F be a non-archimedean local field and let G be a connected reductive group over F. We identify G with the group of F-points $G(F)$. We denote by $\mathrm{Rep}(G)$ the category of smooth representations of G of finite length.

For a parabolic subgroup $P = MN_P$ of G, where M is a Levi component of P and N_P is the unipotent radical of P, we have the normalized parabolic induction functor

$$\mathrm{Ind}_P^G \colon \mathrm{Rep}(M) \to \mathrm{Rep}(G),\ (\sigma, V_\sigma) \mapsto (\pi, V_\pi),$$

where V_π is the space of locally constant functions $f \colon G \to V_\sigma$ such that

$$f(nmg) = \delta_P(m)^{\frac{1}{2}} \sigma(m) f(g)$$

for $n \in N_P$, $m \in M$ and $g \in G$ with δ_P the modulus character of P, and $(\pi(x)f)(g) = f(gx)$ for $x, g \in G$. We also have the normalized Jacquet functor

$$\mathrm{Jac}_P \colon \mathrm{Rep}(G) \to \mathrm{Rep}(M),\ (\pi, V_\pi) \mapsto (\sigma, V_\sigma),$$

where

$$V_\sigma = (V_\pi)_{N_P} = V_\pi / \langle \pi(n)v - v \mid n \in N_P,\ v \in V_\pi \rangle,$$

and $\sigma(m)\overline{v} = \delta_P(m)^{-\frac{1}{2}}\overline{\pi(m)v}$ for $m \in M$ and $v \in V_\pi$ with the image $\overline{v} \in V_\sigma$. These functors are adjoint, i.e., there is an isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \cong \mathrm{Hom}_M(\mathrm{Jac}_P(\pi), \sigma)$$

for $\pi \in \mathrm{Rep}(G)$ and $\sigma \in \mathrm{Rep}(M)$.

For $(\pi, V_\pi) \in \mathrm{Rep}(G)$ and for a parabolic subgroup $P = MN_P$ of G, we set $X_P(\pi) = \mathrm{Ind}_P^G(\mathrm{Jac}_P(\pi))$. This is the space of locally constant functions $\overline{f}\colon G \to (V_\pi)_{N_P}$ such that

$$\overline{f(nmg)} = \overline{\pi(m)f(g)}$$

for $n \in N_P$, $m \in M$ and $g \in G$ with $f(g) \in V_\pi$ a representative of $\overline{f(g)} \in (V_\pi)_{N_P}$. If Q is another parabolic subgroup of G with $Q \supset P$, since $N_Q \subset N_P$, we have a projection map $(V_\pi)_{N_Q} \twoheadrightarrow (V_\pi)_{N_P}$. We define a map $\varphi_P^Q\colon X_Q(\pi) \to X_P(\pi)$ by the composition of functions $G \to (V_\pi)_{N_Q}$ with this projection $(V_\pi)_{N_Q} \twoheadrightarrow (V_\pi)_{N_P}$.

Fix a minimal parabolic subgroup $P_0 = M_0 N_{P_0}$ of G. We denote the maximal split central torus in a Levi subgroup M by A_M, and set $A_0 = A_{M_0}$. Let $S \subset X^*(A_0)$ be the set of (relative) simple roots corresponding to P_0, and set $r = |S| = \dim(A_0/A_G)$. We say that a parabolic subgroup P of G is *standard* if $P \supset P_0$. Then there is a bijection

$$\{J \subset S\} \to \{\text{standard parabolic subgroups of } G\},\ J \mapsto P_J,$$

where $P_J = M_J N_{P_J}$ is such that $\mathrm{Lie}(P_J)$ is the sum of A_0-weight spaces for all weights that are $\mathbb{Z}$-linear combinations of S with non-negative contributions of $S \setminus J$. We write $X_J(\pi) = X_{P_J}(\pi)$ for short. Note that if $I \subset J$, then $P_I \subset P_J$. Hence we have a map $\varphi_I^J = \varphi_{P_I}^{P_J}\colon X_J(\pi) \to X_I(\pi)$ for $\pi \in \mathrm{Rep}(G)$.

For $J \subset S$, we consider the 1-dimensional vector space

$$\Lambda_J = \bigwedge^{|S \setminus J|} (\mathbb{C}^{|S \setminus J|}),$$

which is regarded as the trivial representation of G. Let $\{e_i\}_{i \in S \setminus J}$ be the standard basis of $\mathbb{C}^{|S \setminus J|}$. For $I \subset J \subset S$ with $|J \setminus I| = 1$, letting $J \setminus I = \{j\}$, we define the isomorphism

$$\epsilon_I^J\colon \Lambda_J \to \Lambda_I,\ \omega \mapsto \omega \wedge e_j.$$

Consider a functor

$$\widetilde{X}_J\colon \mathrm{Rep}(G) \to \mathrm{Rep}(G)$$

given by $\widetilde{X}_J(\pi) = X_J(\pi) \otimes_{\mathbb{C}} \Lambda_J$ for $\pi \in \mathrm{Rep}(G)$. Then we define

$$\widetilde{\varphi}_I^J\colon \widetilde{X}_J(\pi) \to \widetilde{X}_I(\pi)$$

by $\widetilde{\varphi}_I^J = \varphi_I^J \otimes \epsilon_I^J$.

For $\pi \in \mathrm{Rep}(G)$ and for $0 \leq t \leq r$, set

$$\widetilde{X}_t(\pi) = \bigoplus_{\substack{J \subset S \\ |J| = t}} \widetilde{X}_J(\pi).$$

In particular, $\widetilde{X}_r(\pi) = \pi$. For $1 \leq t \leq r$, define $d_t \colon \widetilde{X}_t(\pi) \to \widetilde{X}_{t-1}(\pi)$ by

$$d_t \left(\sum_{\substack{J \subset S \\ |J|=t}} x_J \right) = \sum_{\substack{I \subset S \\ |I|=t-1}} \sum_{\substack{I \subset J \subset S \\ |J|=t}} \widetilde{\varphi}_I^J(x_J),$$

where $x_J \in \widetilde{X}_J(\pi)$. Then we have a sequence

$$0 \longrightarrow \widetilde{X}_r(\pi) \longrightarrow \widetilde{X}_{r-1}(\pi) \longrightarrow \cdots \longrightarrow \widetilde{X}_0(\pi)$$

of representations of G.

B.2. **The Aubert involution.** For $0 \leq t \leq r$, we denote by $\mathrm{Rep}(G)_t$ the full subcategory of $\mathrm{Rep}(G)$ consisting of representations π such that for every irreducible subquotient π' of π, there is $J \subset S$ with $|J| = t$ such that π' is a subquotient of $\mathrm{Ind}_{P_J}^G(\sigma)$ for an irreducible supercuspidal representation of M_J. Bernstein's decomposition implies the block decomposition

$$\mathrm{Rep}(G) = \prod_{t=0}^{r} \mathrm{Rep}(G)_t.$$

Note that the factor $\mathrm{Rep}(G)_r$ consists of direct sums of supercuspidal representations of G, and every supercuspidal representation of G lies in $\mathrm{Rep}(G)_r$.

Theorem B.2.1 ([Au, Théorème 3.6])**.** *For $\pi \in \mathrm{Rep}(G)_t$, we have $\widetilde{X}_0(\pi) = \cdots = \widetilde{X}_{t-1}(\pi) = 0$. Moreover, the sequence*

$$0 \longrightarrow \widetilde{X}_r(\pi) \longrightarrow \widetilde{X}_{r-1}(\pi) \longrightarrow \cdots \longrightarrow \widetilde{X}_t(\pi)$$

is exact.

Definition B.2.2. *For $\pi \in \mathrm{Rep}(G)_t$, set*

$$\hat{\pi} = \widetilde{X}_t(\pi)/d_{t+1}(\widetilde{X}_{t+1}(\pi))$$

and call $\hat{\pi}$ the Aubert dual *of π.*

Theorem B.2.3. *Aubert duality $\pi \mapsto \hat{\pi}$ satisfies the following properties.*

(1) *The map $\mathrm{Rep}(G) \ni \pi \to \hat{\pi} \in \mathrm{Rep}(G)$ is an exact covariant functor.*
(2) *For $\pi \in \mathrm{Rep}(G)_t$, we have*

$$[\hat{\pi}] = (-1)^{r-t} \sum_{P=MN_P} (-1)^{\dim(A_M/A_G)} [\mathrm{Ind}_P^G(\mathrm{Jac}_P(\pi))]$$

in the Grothendieck group $\mathcal{R}(G)$, where P runs over the set of standard parabolic subgroups of G. Here, $[\Pi]$ denotes the element in $\mathcal{R}(G)$ corresponding to a representation Π of G of finite length.
(3) *If π is irreducible, then $\hat{\pi}$ is also irreducible.*
(4) *Aubert dual of $\hat{\pi}$ is isomorphic to π as representations of G.*

(5) *Let P be a parabolic subgroup of G with Levi component M, and denote by $\overline{P}$ the parabolic subgroup of G opposite to P. Then for $\pi \in \mathrm{Rep}(M)$ of finite length, Aubert dual of $\mathrm{Ind}_P^G(\pi)$ is isomorphic to $\mathrm{Ind}_{\overline{P}}^G(\hat{\pi})$.*

Proof. For (1), see [SS, III.3]. The assertions (2), (3) and (4) are [Au, Corollaire 3.9]. Finally, (5) is [Ber2, Theorem 31 (4)]. □

B.3. **Intertwining operators and Aubert duality.** In this subsection, let G be one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{O}_{2n}(F), \quad \mathrm{U}_n.$$

Let $P = MN$ be a maximal parabolic subgroup of G so that $M \cong \mathrm{GL}_k(E) \times G_0$ for some classical group G_0 of the same type as G. We denote by $\overline{P} = M\overline{N}$ the parabolic subgroup of G opposite to P. We fix $\psi \in \Psi(M)$.

In this subsection, we consider Aubert dualities for G° and M°, the connected components of the identity of G and M, respectively. To avoid Aubert duality for non-connected groups explained in the next subsections, we assume the following.

Hypothesis B.3.1. There are A-packets Π_ψ and $\Pi_{\widehat{\psi}}$ which are (multi-)sets over $\mathrm{Irr}_{\mathrm{unit}}(M)$. Moreover, there is a bijection $\Pi_\psi \ni \pi \mapsto \pi' \in \Pi_{\widehat{\psi}}$ such that $\pi'|_{M^\circ}$ is the Aubert dual of $\pi|_{M^\circ}$.

Write $\psi = \psi_{\mathrm{GL}} \oplus \psi_0$ with $\psi_0 \in \Psi(G_0)$ and $\pi = \tau \boxtimes \sigma$ for $\tau \in \mathrm{Irr}(\mathrm{GL}_k(E))$ corresponding to ψ_{GL} and $\sigma \in \Pi_{\psi_0}$. We set $\psi_s = \psi_{\mathrm{GL}}|\cdot|_E^s \oplus \psi_0$ and $\pi_s = \tau|\det|_E^s \boxtimes \sigma$ for $s \in \mathbb{C}$.

As in Section 1.7, for $w \in W(M^\circ)$ and $s \in \mathbb{C}$, we have the normalized intertwining operator

$$R_P(w, \pi_s, \psi_s) \colon I_P(\pi_s) \to I_P(w\pi_s),$$

which is a meromorphic family of operators. Since π is unitary, $R_P(w, \pi_s, \psi_s)$ is regular at $s = 0$, and we obtain a well-defined operator $R_P(w, \pi, \psi) = R_P(w, \pi_s, \psi_s)|_{s=0}$.

Note that $\widehat{I_P(\pi_s)} \cong I_{\overline{P}}(\hat{\pi}_s)$ as representations of G° by Theorem B.2.3 (5). Since $\widehat{w\pi_s} \cong w\hat{\pi}_s$, by the functoriality of Aubert duality, we have

$$R_P\widehat{(w, \pi_s}, \psi_s) \colon I_{\overline{P}}(\hat{\pi}_s) \to I_{\overline{P}}(w\hat{\pi}_s).$$

On the other hand, we can define a normalized intertwining operator

$$R_{\overline{P}}(w, \hat{\pi}_s, \widehat{\psi}_s) \colon I_{\overline{P}}(\hat{\pi}_s) \to I_{\overline{P}}(w\hat{\pi}_s),$$

which is regular at $s = 0$.

Proposition B.3.2. *Assume Hypothesis B.3.1. We further assume that $R_{\overline{P}}(w, \hat{\pi}, \widehat{\psi})$ is bijective.*

(1) *If $G = G^\circ$, then there is $c \in \mathbb{C}^\times$ such that*

$$R_P\widehat{(w, \pi}, \psi) = c \cdot R_{\overline{P}}(w, \hat{\pi}, \widehat{\psi}).$$

(2) *Suppose that $G = \mathrm{O}_{2n}(F)$.*

(a) *If $P \neq P^\circ$ and if $\sigma|_{G_0^\circ}$ is irreducible, then there is $c \in \mathbb{C}^\times$ such that*

$$R_P\widehat{(w,\pi,\psi)} = c \cdot R_{\overline{P}}(w,\hat{\pi},\widehat{\psi}).$$

(b) *Otherwise, there is $\pi_s^\circ \in \mathrm{Irr}(M^\circ)$ such that $I_P(\pi_s) = I_{P^\circ}(\pi_s^\circ) = \mathrm{Ind}_{P^\circ}^G(\pi_s^\circ)$. If we denote by $I_{P^\circ}^+(\pi_s^\circ)$ (resp. $I_{P^\circ}^-(\pi_s^\circ)$) the subspace of $I_{P^\circ}(\pi_s^\circ)$ consisting of functions f_s on G whose supports are contained in G° (resp. $G \setminus G^\circ$), then $I_{P^\circ}(\pi_s^\circ) = I_{P^\circ}^+(\pi_s^\circ) \oplus I_{P^\circ}^-(\pi_s^\circ)$. Moreover, there are two constants $c_+, c_- \in \mathbb{C}^\times$ such that*

$$R_P\widehat{(w,\pi,\psi)} f_\pm = c_\pm \cdot R_{\overline{P}}(w,\hat{\pi},\widehat{\psi}) f_\pm$$

for all $f_\pm$ in the Aubert dual of $I_{P^\circ}^\pm(\pi_s^\circ)$.

Proof. Suppose that $G = G^\circ$. Then since $I_{\overline{P}}(\hat{\pi}_s)$ is irreducible for almost all q_E^{-s}, we have the inverse map

$$R_{\overline{P}}(w,\hat{\pi}_s,\widehat{\psi}_s)^{-1} \colon I_{\overline{P}}(w\hat{\pi}_s) \to I_{\overline{P}}(\hat{\pi}_s),$$

for almost all q_E^{-s}. It is regular at $s=0$ since $R_{\overline{P}}(w,\hat{\pi},\widehat{\psi})$ is bijective. We consider the composition

$$R_{\overline{P}}(w,\hat{\pi}_s,\widehat{\psi}_s)^{-1} \circ R_P\widehat{(w,\pi_s,\psi_s)}.$$

This is a meromorphic family of self-intertwining operators on $I_{\overline{P}}(\hat{\pi}_s)$. Again by the irreducibility of $I_{\overline{P}}(\hat{\pi}_s)$ for almost all q_E^{-s}, we can find a meromorphic function $c(s)$ such that

$$R_{\overline{P}}(w,\hat{\pi}_s,\widehat{\psi}_s)^{-1} \circ R_P\widehat{(w,\pi_s,\psi_s)} = c(s) \cdot \mathrm{id}.$$

Since the left-hand side is regular at $s=0$, we can define $c = c(0) \in \mathbb{C}$. Then

$$R_P\widehat{(w,\pi,\psi)} = c \cdot R_{\overline{P}}(w,\hat{\pi},\widehat{\psi}).$$

Since $R_P(w,\pi,\psi)$ is not identically zero, so is its Aubert dual. Hence $c \neq 0$. This completes the proof of (1).

When $G = \mathrm{O}_{2n}(F)$, the proof is essentially the same, but we need to consider the restrictions to the connected components of the identity. First, we consider (2a). Then $\pi_s|_{M^\circ} = (\pi|_{M^\circ})_s$ is irreducible, and $I_P(\pi_s)|_{G^\circ} \cong \mathrm{Ind}_{P^\circ}^{G^\circ}((\pi|_{M^\circ})_s)$ by Lemma 2.1.3 (a). By regarding $R_P(w,\pi,\psi)$ and $R_{\overline{P}}(w,\hat{\pi},\hat{\psi})$ as G°-homomorphisms, the same argument as in (1) shows that

$$R_P\widehat{(w,\pi,\psi)} = c \cdot R_{\overline{P}}(w,\hat{\pi},\widehat{\psi})$$

for some $c \in \mathbb{C}^\times$.

Next, we assume that $P = P^\circ$ or $\pi|_{M^\circ}$ is reducible. Then by Lemma 2.1.3 (b), we have $I_P(\pi_s) = I_{P^\circ}(\pi_s^\circ)$ for any irreducible component π° of $\pi|_{M^\circ}$. Moreover, by the proof of that lemma, we have $I_{P^\circ}(\pi_s^\circ) = I_{P^\circ}^+(\pi_s^\circ) \oplus I_{P^\circ}^-(\pi_s^\circ)$. As a G°-homomorphism, $R_P(w,\pi_s,\psi_s)$ can be decomposed to the direct sum of

$$R_P(w,\pi_s,\psi_s) \colon I_{P^\circ}^\pm(\pi_s^\circ) \to I_{P^\circ}^{\pm\delta}(\pi_s^\circ),$$

where $\delta = \det(w) \in \{\pm 1\}$. By the same argument as in (1), we can find $c_\pm \in \mathbb{C}^\times$ such that

$$R_P\widehat{(w, \pi, \psi)} = c_\pm \cdot R_{\overline{P}}(w, \hat{\pi}, \widehat{\psi})$$

holds on the Aubert dual of $I^\pm_{P^\circ}(\pi_s^\circ)$. This completes the proof of (2). □

Now suppose that τ is conjugate-self-dual. Then as in Section 1.10, we can define the normalized self-intertwining operators

$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}, \psi) \colon I_P(\pi) \to I_P(\pi),$$
$$\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_{\overline{P}}(w_u, \widetilde{\hat{\pi}}, \widehat{\psi}) \colon I_{\overline{P}}(\hat{\pi}) \to I_{\overline{P}}(\hat{\pi}).$$

Now we have the following corollary, which is a key result to prove Theorem 1.10.5.

Corollary B.3.3. *Assume Hypothesis B.3.1. We further assume that $R_{\overline{P}}(w, \hat{\pi}, \widehat{\psi})$ is bijective, and that τ is conjugate-self-dual.*

(1) *If we are in the case (1) or (2a) of Proposition B.3.2, then there is $c \in \mathbb{C}^\times$ such that*
$$(\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}, \psi))\widehat{\ } = c \cdot \langle \widetilde{u}, \widetilde{\pi}_M \rangle R_{\overline{P}}(w_u, \widetilde{\hat{\pi}}, \widehat{\psi}).$$
(2) *If we are in the case (2b) of Proposition B.3.2, then there are two constants $c_+, c_- \in \mathbb{C}^\times$ such that*
$$(\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}, \psi))\widehat{\ } = c_\pm \cdot \langle \widetilde{u}, \widetilde{\pi}_M \rangle R_{\overline{P}}(w_u, \widetilde{\hat{\pi}}, \widehat{\psi})$$
holds on the Aubert dual of $I^\pm_{P^\circ}(\pi_s^\circ)$.

Proof. Recall that the normalized intertwining operator $\langle \widetilde{u}, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}, \psi)$ is defined by

$$\langle \widetilde{u}, \widetilde{\pi} \rangle R_P(w_u, \widetilde{\pi}, \psi) f(g) = \langle \widetilde{u}, \widetilde{\pi} \rangle \widetilde{\pi}(w_u) \left(R_P(w_u, \pi, \psi) f(g) \right)$$

for an isomorphism $\langle \widetilde{u}, \widetilde{\pi} \rangle \widetilde{\pi}(w_u) \colon w_u \pi \xrightarrow{\sim} \pi$. By Schur's lemma, its Aubert dual is equal to $\langle \widetilde{u}, \widetilde{\hat{\pi}} \rangle \widetilde{\hat{\pi}}(w_u)$ up to a nonzero constant. Then the claim follows from Theorem B.2.3 (1), (5) and Proposition B.3.2. □

Remark B.3.4. In the next subsections, we introduce Aubert duality for non-connected groups, in particular for $\mathrm{O}_{2n}(F)$. However, an analogue of Theorem B.2.3 (5) will not be established, and hence a direct approach as in Proposition B.3.2 (1) cannot be applied.

B.4. **Twisted Aubert duality as a functor.** Now we define the twisted Aubert dual, and establish some properties that we need in this paper. We use the same notations in Sections B.1 and B.2.

Let θ be an involution of G which preserves P_0 and M_0. Then θ also preserves A_0 and acts on the set S of (relative) simple roots. We consider the disconnected group $\widetilde{G} = G \rtimes \langle \theta \rangle$. Let $\mathrm{Rep}(\widetilde{G})$ be the category of smooth representations of $\widetilde{G}$ of finite length. For $0 \le t \le r$, we denote by $\mathrm{Rep}(\widetilde{G})_t$ the inverse image of $\mathrm{Rep}(G)_t$ under the restriction map $\mathrm{Res} \colon \mathrm{Rep}(\widetilde{G}) \to \mathrm{Rep}(G)$.

Lemma B.4.1. *The category $\mathrm{Rep}(\widetilde{G})$ has a block decomposition $\prod_t \mathrm{Rep}(\widetilde{G})_t$.*

Proof. The automorphism θ acts on $\mathrm{Rep}(G)$ by $\pi \mapsto \pi \circ \theta$. This action preserves $\mathrm{Rep}(G)_t$. Hence the Bernstein decomposition for $\mathrm{Rep}(G)$ gives the decomposition for $\mathrm{Rep}(\widetilde{G})$. Since the functor $\mathrm{Res}\colon \mathrm{Rep}(\widetilde{G}) \to \mathrm{Rep}(G)$ is faithful, the orthogonality of the factors follows. □

Let $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})_t$. Set $\pi = \mathrm{Res}(\widetilde{\pi}) \in \mathrm{Rep}(G)_t$ and denote the space of π by V_π. Recall that for $J \subset S$, we have a representation $X_J(\pi)$ of G. This is a space of functions $f\colon G \to (V_\pi)_{N_{P_J}}$. We have a map

$$X_J(\theta)\colon X_J(\pi) \to X_{\theta(J)}(\pi),\ X_J(\theta)(f)(g) = \widetilde{\pi}(\theta)(f(\theta(g))).$$

On the other hand, the bijection $\theta\colon (S \setminus J) \to (S \setminus \theta(J))$ induces an isomorphism $\lambda_J(\theta)\colon \Lambda_J \to \Lambda_{\theta(J)}$.

Lemma B.4.2. *Let $I \subset J \subset S$ be two subsets.*

(1) *We have $X_I(\theta) \circ \varphi_I^J = \varphi_{\theta(I)}^{\theta(J)} \circ X_J(\theta)$.*

(2) *If $|J| = |I| + 1$, then $\lambda_I(\theta) \circ \epsilon_I^J = \epsilon_{\theta(I)}^{\theta(J)} \circ \lambda_J(\theta)$.*

Proof. For $f \in X_J(\pi)$, we have a commutative diagram

$$\begin{array}{ccccccc} G & \xrightarrow{\theta} & G & \xrightarrow{f} & (V_\pi)_{N_{P_J}} & \twoheadrightarrow & (V_\pi)_{N_{P_I}} \\ & & & & \downarrow{\scriptstyle \widetilde{\pi}(\theta)} & & \downarrow{\scriptstyle \widetilde{\pi}(\theta)} \\ & & & & (V_\pi)_{N_{P_{\theta(J)}}} & \twoheadrightarrow & (V_\pi)_{N_{P_{\theta(I)}}}. \end{array}$$

This shows that $X_I(\theta) \circ \varphi_I^J(f) = \varphi_{\theta(I)}^{\theta(J)} \circ X_J(\theta)(f)$.

The second assertion is obvious. □

By taking the tensor product $X_J(\theta) \otimes \lambda_J(\theta)$, we obtain a map

$$\widetilde{X}_J(\theta)\colon \widetilde{X}_J(\pi) \to \widetilde{X}_{\theta(J)}(\pi).$$

By Lemma B.4.2, it satisfies that $\widetilde{X}_I(\theta) \circ \widetilde{\varphi}_I^J = \widetilde{\varphi}_{\theta(I)}^{\theta(J)} \circ \widetilde{X}_J(\theta)$ if $|J| = |I| + 1$. Putting these together for all $J \subset S$ with $|J| = t$, we can define an isomorphism

$$\widetilde{X}_t(\theta)\colon \widetilde{X}_t(\pi) \to \widetilde{X}_t(\pi)$$

by requiring the diagram

$$\begin{array}{ccc} \widetilde{X}_t(\pi) & \xrightarrow{\widetilde{X}_t(\theta)} & \widetilde{X}_t(\pi) \\ \downarrow & & \downarrow \\ \widetilde{X}_J(\pi) & \xrightarrow{\widetilde{X}_J(\theta)} & \widetilde{X}_{\theta(J)}(\pi) \end{array}$$

is commutative for all $J \subset S$ with $|J| = t$, where the vertical maps are the canonical projections.

Proposition B.4.3. *We have the following.*

(1) *The isomorphism* $\widetilde{X}_t(\theta)$ *satisfies that* $\widetilde{X}_t(\pi(g)) \circ \widetilde{X}_t(\theta) = \widetilde{X}_t(\theta) \circ \widetilde{X}_t(\pi(\theta(g)))$ *for* $g \in G$.

(2) *The differential* $d_t \colon \widetilde{X}_t(\pi) \to \widetilde{X}_{t-1}(\pi)$ *satisfies that* $d_t \circ \widetilde{X}_t(\theta) = \widetilde{X}_{t-1}(\theta) \circ d_t$.

(3) *For* $\widetilde{\pi}, \widetilde{\pi}' \in \mathrm{Rep}(\widetilde{G})$ *with* $\pi = \widetilde{\pi}|_G$ *and* $\pi' = \widetilde{\pi}'|_G$, *if* $\varphi \colon \widetilde{\pi} \to \widetilde{\pi}'$ *is a* $\widetilde{G}$*-equivariant map, then the induced homomorphism* $\widetilde{X}_t(\varphi) \colon \widetilde{X}_t(\pi) \to \widetilde{X}_t(\pi')$ *satisfies that* $\widetilde{X}_t(\varphi) \circ \widetilde{X}_t(\theta) = \widetilde{X}_t(\theta) \circ \widetilde{X}_t(\varphi)$.

Proof. If $f \in X_J(\pi)$, then

$$\begin{aligned} X_J(\pi(g)) \circ X_J(\theta)(f)(x) &= X_J(\theta)(f)(xg) \\ &= \widetilde{\pi}(\theta)\Big(f(\theta(xg))\Big) \\ &= \widetilde{\pi}(\theta)\Big(X_J(\pi(\theta(g)))f(\theta(x))\Big) \\ &= X_J(\theta) \circ X_J(\pi(\theta(g)))(f)(x). \end{aligned}$$

Since G acts on Λ_J trivially, this action commutes with $\lambda_J(\theta)$. Hence $\widetilde{X}_J(\pi(g)) \circ \widetilde{X}_J(\theta) = \widetilde{X}_J(\theta) \circ \widetilde{X}_J(\pi(\theta(g)))$ for $g \in G$. This implies (1).

Next, we have

$$\begin{aligned} d_t \circ \widetilde{X}_t(\theta) \left(\sum_{\substack{J \subset S \\ |J|=t}} f_J \right) &= d_t \left(\sum_{\substack{J \subset S \\ |J|=t}} \widetilde{X}_{\theta(J)}(\theta)(f_{\theta(J)}) \right) \\ &= \sum_{\substack{I \subset S \\ |I|=t-1}} \sum_{\substack{I \subset J \subset S \\ |J|=t}} \widetilde{\varphi}_I^J \circ \widetilde{X}_{\theta(J)}(\theta)(f_{\theta(J)}) \\ &= \sum_{\substack{I \subset S \\ |I|=t-1}} \sum_{\substack{I \subset J \subset S \\ |J|=t}} \widetilde{X}_{\theta(I)}(\theta) \circ \widetilde{\varphi}_{\theta(I)}^{\theta(J)}(f_{\theta(J)}) \\ &= \widetilde{X}_{t-1}(\theta) \left(\sum_{\substack{I \subset S \\ |I|=t-1}} \sum_{\substack{I \subset J \subset S \\ |J|=t}} \widetilde{\varphi}_I^J(f_J) \right). \end{aligned}$$

Hence we obtain (2).

Finally, in the situation of (3), if $f \in X_J(\pi)$, then

$$\begin{aligned} X_J(\varphi) \circ X_J(\theta)(f)(x) &= \varphi \circ \widetilde{\pi}(\theta)\Big(f(\theta(x))\Big) \\ &= \widetilde{\pi}'(\theta)\Big(\varphi(f(\theta(x)))\Big) \\ &= X_J(\theta) \circ X_J(\varphi)(f)(x). \end{aligned}$$

This implies (3). ∎

On $\widetilde{X}_t(\pi)$, we have operators $\widetilde{X}_t(\theta)$ and $\widetilde{X}_t(\pi(g))$ for $g \in G$. For $\widetilde{g} \in \widetilde{G}$, we write

$$\widetilde{X}_t(\widetilde{\pi}(\widetilde{g})) = \begin{cases} \widetilde{X}_t(\pi(g)) & \text{if } \widetilde{g} = g \in G, \\ \widetilde{X}_t(\pi(g)) \circ \widetilde{X}_t(\theta) & \text{if } \widetilde{g} = g \rtimes \theta \in G \rtimes \theta. \end{cases}$$

Then by Proposition B.4.3 (1), we see that

$$\widetilde{X}_t(\widetilde{\pi}(\widetilde{g} \cdot \widetilde{g}')) = \widetilde{X}_t(\widetilde{\pi}(\widetilde{g})) \circ \widetilde{X}_t(\widetilde{\pi}(\widetilde{g}'))$$

for $\widetilde{g}, \widetilde{g}' \in G$. This gives the $\mathbb{C}$-vector space $\widetilde{X}_t(\pi)$ the structure of a representation of $\widetilde{G}$. We denote this representation by $\widetilde{X}_t(\widetilde{\pi})$. Moreover, the automorphism $\widetilde{X}_t(\widetilde{\pi}(\widetilde{g}))$ is functorial in $\widetilde{\pi}$ by Proposition B.4.3 (3).

Namely, we have a functor

$$\widetilde{X}_t\colon \mathrm{Rep}(\widetilde{G}) \to \mathrm{Rep}(\widetilde{G}).$$

Lemma B.4.4. *For $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})$, we have*

$$\widetilde{X}_t(\widetilde{\pi})|_G = \widetilde{X}_t(\widetilde{\pi}|_G).$$

Proof. This is obvious from the construction. □

Proposition B.4.5. *If $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})_t$, then $\widetilde{X}_0(\widetilde{\pi}) = \cdots = \widetilde{X}_{t-1}(\widetilde{\pi}) = 0$, and the sequence*

$$0 \longrightarrow \widetilde{X}_r(\widetilde{\pi}) \longrightarrow \widetilde{X}_{r-1}(\widetilde{\pi}) \longrightarrow \cdots \longrightarrow \widetilde{X}_t(\widetilde{\pi})$$

is an exact sequence of representations of $\widetilde{G}$.

Proof. By Theorem B.2.1 and Lemma B.4.4, we have the first assertion and the exactness of the sequence. On the other hand, by Proposition B.4.3 (2), this sequence consists of $\widetilde{G}$-equivariant maps. □

Now we can define twisted Aubert duality.

Definition B.4.6. *For $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})_t$, set*

$$\hat{\widetilde{\pi}} = \widetilde{X}_t(\widetilde{\pi})/d_{t+1}(\widetilde{X}_{t+1}(\widetilde{\pi}))$$

and call $\hat{\widetilde{\pi}}$ the twisted Aubert dual *of $\widetilde{\pi}$.*

Proposition B.4.7. *Twisted Aubert duality functor $\widetilde{\pi} \mapsto \hat{\widetilde{\pi}}$ satisfies the following properties.*

1. *The map $\mathrm{Rep}(\widetilde{G}) \ni \widetilde{\pi} \to \hat{\widetilde{\pi}} \in \mathrm{Rep}(\widetilde{G})$ is an exact covariant functor.*
2. *If we write $\pi = \widetilde{\pi}|_G$, then $\hat{\widetilde{\pi}}|_G = \hat{\pi}$.*
3. *If $\widetilde{\pi}$ is irreducible, then $\hat{\widetilde{\pi}}$ is also irreducible.*

Proof. (1) follows from the construction together with Theorem B.2.3 and Lemma B.4.4. (2) is a direct consequence from Lemma B.4.4.

To show (3), let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$. Set $\pi = \widetilde{\pi}|_G$ so that $\hat{\pi} = \hat{\widetilde{\pi}}|_G$ by (2). Note that π is a direct sum of at most two irreducible representations of G since $(\widetilde{G} : G) = 2$. If π is irreducible, then so is $\hat{\pi}$ and hence $\hat{\widetilde{\pi}}$ must be an irreducible representation of $\widetilde{G}$. Suppose that $\pi = \pi_1 \oplus \pi_2$ with π_i irreducible. Then $\hat{\pi} = \hat{\pi}_1 \oplus \hat{\pi}_2$. Moreover, $\widetilde{\pi}(\theta)$ gives a linear isomorphism $\pi_1 \xrightarrow{\sim} \pi_2$ as $\mathbb{C}$-vector spaces. By the construction, it induces a linear isomorphism $\hat{\pi}_1 \xrightarrow{\sim} \hat{\pi}_2$. Since this is nothing but $\hat{\widetilde{\pi}}(\theta)$, we conclude that $\hat{\widetilde{\pi}}$ is irreducible as a representation of $\widetilde{G}$. □

Remark B.4.8. We do not know whether the twisted Aubert duality functor $\widetilde{\pi} \mapsto \hat{\widetilde{\pi}}$ is really an involution. This and the commutativites of the twisted Aubert duality functor with the contragredient functor, the parabolic induction functors, and Jacquet functors would be solved in a forthcoming paper. In this paper, we do not use these expected properties.

If η is a character of G satisfying $\eta \circ \theta = \eta$, then we can extend η to a character of $\widetilde{G}$ by setting $\eta(\theta) = 1$. Hence for $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})$, one can consider the twist $\widetilde{\pi} \otimes \eta$.

Lemma B.4.9. *Let η be a character of G such that $\eta \circ \theta = \eta$. Then for $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})$, the twisted Aubert dual of $\widetilde{\pi} \otimes \eta$ is equal to $\hat{\widetilde{\pi}} \otimes \eta$.*

Proof. This follows from the construction. □

B.5. **Twisted Aubert duality at the level of Grothendieck groups.** Let $P = MN_P$ be a standard parabolic subgroup of G. If P is θ-stable, we may assume that M is also θ-stable. In this case, write $\widetilde{P} = P \rtimes \langle\theta\rangle$ and $\widetilde{M} = M \rtimes \langle\theta\rangle$. Then the normalized parabolic induction functor

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}} \colon \mathrm{Rep}(\widetilde{M}) \to \mathrm{Rep}(\widetilde{G})$$

and the normalized Jacquet functor

$$\mathrm{Jac}_{\widetilde{P}} \colon \mathrm{Rep}(\widetilde{G}) \to \mathrm{Rep}(\widetilde{M})$$

can be defined as in the connected case. Note that for $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})$, we have

$$\widetilde{X}_P(\widetilde{\pi}) \cong \mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\mathrm{Jac}_{\widetilde{P}}(\widetilde{\pi})) \otimes \Lambda_J$$

as representations of $\widetilde{G}$, where $J \subset S$ is such that $P = P_J$.

Let $\mathcal{R}(\widetilde{G})$ be the Grothendieck group of $\mathrm{Rep}(\widetilde{G})$. When $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})$, we denote by $[\widetilde{\pi}] \in \mathcal{R}(\widetilde{G})$ the corresponding element. The character $\Theta_{\widetilde{\pi}}$ of $\widetilde{\pi}$, which is a linear functional on $C_c^\infty(\widetilde{G})$, depends only on $[\widetilde{\pi}]$. Moreover, it gives a map

$$\mathcal{R}(\widetilde{G}) \ni [\widetilde{\pi}] \mapsto \Theta_{\widetilde{\pi}} \in C_c^\infty(\widetilde{G})^*.$$

For $[\widetilde{\pi}_1], [\widetilde{\pi}_2] \in \mathcal{R}(\widetilde{G})$, we write

$$[\widetilde{\pi}_1] \stackrel{\theta}{=} [\widetilde{\pi}_2]$$

if $\Theta_{\widetilde{\pi}_1}(f) = \Theta_{\widetilde{\pi}_2}(f)$ for any $f \in C_c^\infty(G \rtimes \theta)$. For example, for $\pi \in \mathrm{Irr}(G)$, if $\mathrm{Ind}_G^{\widetilde{G}}(\pi) = \widetilde{\pi}_1 \oplus \widetilde{\pi}_2$ is reducible, then $[\widetilde{\pi}_2] \stackrel{\theta}{=} -[\widetilde{\pi}_1]$.

In this subsection, we prove the following.

Proposition B.5.1. *Let $\widetilde{\pi} \in \mathrm{Rep}(\widetilde{G})_t$. Then*

$$[\widehat{\widetilde{\pi}}] \stackrel{\theta}{=} (-1)^{r-t} \sum_{P=MN_P} (-1)^{\dim((A_M/A_G)^\theta)} \left[\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\mathrm{Jac}_{\widetilde{P}}(\widetilde{\pi}))\right],$$

where P runs over the set of θ-stable standard parabolic subgroups of G, and $(A_M/A_G)^\theta$ is the subgroup of A_M/A_G fixed by θ.

Proof. By construction, we have the exact sequence

$$0 \longrightarrow \widetilde{X}_r(\widetilde{\pi}) \longrightarrow \widetilde{X}_{r-1}(\widetilde{\pi}) \longrightarrow \cdots \longrightarrow \widetilde{X}_t(\widetilde{\pi}) \longrightarrow \widehat{\widetilde{\pi}} \longrightarrow 0$$

of representations of $\widetilde{G}$. Hence

$$[\widehat{\widetilde{\pi}}] = (-1)^t \sum_{j=t}^{r} (-1)^j \left[\widetilde{X}_j(\widetilde{\pi})\right].$$

Since $\widetilde{X}_j(\widetilde{\pi}) = 0$ for $0 \leq j \leq t-1$, we may extend the sum over all $0 \leq j \leq r$.

Recall that if we write $\pi = \widetilde{\pi}|_G$, as representations of G, we have

$$\widetilde{X}_j(\widetilde{\pi}) = \bigoplus_{\substack{J \subset S \\ |J|=j}} X_J(\pi) \otimes \Lambda_J.$$

Moreover, the action of $g \rtimes \theta$ on $\widetilde{X}_j(\widetilde{\pi})$ sends the summand $X_J(\pi) \otimes \Lambda_J$ to $X_{\theta(J)}(\pi) \otimes \Lambda_{\theta(J)}$. This means that if $\theta(J) \neq J$, then

$$\left[(X_J(\pi) \otimes \Lambda_J) \oplus \left(X_{\theta(J)}(\pi) \otimes \Lambda_{\theta(J)}\right)\right] \stackrel{\theta}{=} 0,$$

and hence

$$[\widetilde{X}_j(\widetilde{\pi})] \stackrel{\theta}{=} \sum_{\substack{J \subset S \\ |J|=j,\, \theta(J)=J}} [X_J(\pi) \otimes \Lambda_J] = \sum_{\substack{J \subset S \\ |J|=j,\, \theta(J)=J}} \left[\mathrm{Ind}_{\widetilde{P}_J}^{\widetilde{G}}(\mathrm{Jac}_{\widetilde{P}_J}(\widetilde{\pi})) \otimes \Lambda_J\right].$$

Note that Λ_J is a 1-dimensional $\mathbb{C}$-vector space, and $g \rtimes \theta$ acts on it by a scalar $\lambda_J(\theta)$, which is equal to the sign of the action of θ on $S \setminus J$. Hence we have

$$[\widehat{\widetilde{\pi}}] \stackrel{\theta}{=} (-1)^t \sum_{\substack{J \subset S \\ \theta(J)=J}} (-1)^{|J|} \lambda_J(\theta) \left[\mathrm{Ind}_{\widetilde{P}_J}^{\widetilde{G}}(\mathrm{Jac}_{\widetilde{P}_J}(\widetilde{\pi}))\right].$$

Therefore, what we have to show is the equation

$$\lambda_J(\theta) = (-1)^{\dim(A_M/A_G)} \cdot (-1)^{\dim((A_M/A_G)^\theta)}$$

for $J = \theta(J) \subset S$ with $P = P_J = MN_P$ since $\dim(A_M/A_G) = |S \setminus J| = r - |J|$. Note that $S \setminus J$ forms a basis of the $\mathbb{Q}$-vector space $X^*(A_M/A_G) \otimes \mathbb{Q}$. Moreover, θ acts on this space and its co-invariant space $X^*(A_M/A_G)_\theta \otimes \mathbb{Q}$ is isomorphic to $X^*((A_M/A_G)^\theta) \otimes \mathbb{Q}$. As a basis of this space, one can take the set of θ-orbits in $S \setminus J$. If we denote the number of θ-orbits in $S \setminus J$ of order n by m_n, we see that

$$m_1 + 2m_2 = \dim(A_M/A_G), \quad m_1 + m_2 = \dim((A_M/A_G)^\theta).$$

Since $\lambda_J(\theta) = (-1)^{m_2}$ by definition, we obtain the claim. □

Appendix C. Derivatives of tempered representations

We use the notations in Sections 1.5 and 1.6. In particular, let G be one of the following quasi-split classical groups

$$\mathrm{SO}_{2n+1}(F), \quad \mathrm{Sp}_{2n}(F), \quad \mathrm{O}_{2n}(F), \quad \mathrm{U}_n.$$

Through this appendix, we assume Hypothesis 5.1.1. For convenience, we restate this hypothesis again.

Hypothesis C.0.1. For any quasi-split classical group G' with $\mathrm{rank}(G') \leq \mathrm{rank}(G)$, and for any tempered L-parameter ϕ' for G', there exists a subset $\Pi_{\phi'}$ of $\mathrm{Irr}_{\mathrm{temp}}(G')$ equipped with $\langle \cdot, \pi' \rangle_{\phi'}$ satisfying (**ECR1**) and (**ECR2**) in Section 1.6.

The purpose of this appendix is to extend some results in [Mœ], [X1] and [At] to G. Note that in [At], the author used Mœglin's construction of tempered L-packets, which relies on (**ECR1**) and (**ECR2**) for all classical groups G'. Hence it is not trivial that the arguments in [At] can work under our weaker hypothesis. To avoid Mœglin's construction, we will apply the argument in [X1] directly to tempered L-parameters.

C.1. **Derivatives.** Let G be a quasi-split classical group as above. We denote by G° the connected component of $\mathbf{1} \in G$. Note that $G^\circ = G$ unless $G = \mathrm{O}_{2n}(F)$, in which case $G^\circ = \mathrm{SO}_{2n}(F)$. Fix a rational Borel subgroup $B^\circ = T^\circ U$ of G°, and we denote the normalizer of (T°, B°) in G by T. Let $P^\circ = M^\circ N_P$ be the standard parabolic subgroup of G° with Levi subgroup M° isomorphic to $\mathrm{GL}_{d_1}(E) \times \cdots \times \mathrm{GL}_{d_r}(E) \times G_0^\circ$, where G_0° is a classical group of the same type as G°. If P° is stable under the adjoint action of T, we set $P = P^\circ \cdot T$ and $M = M^\circ \cdot T$ so that $M \cong \mathrm{GL}_{d_1}(E) \times \cdots \times \mathrm{GL}_{d_r}(E) \times G_0$. Otherwise, we put $P = P^\circ$ and $M = M^\circ$.

For $\pi_0 \in \mathrm{Rep}(G_0)$ and $\tau_i \in \mathrm{Rep}(\mathrm{GL}_{d_i}(E))$ for $1 \leq i \leq r$, we denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 = \mathrm{Ind}_P^G(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0).$$

On the other hand, for $\pi \in \mathrm{Rep}(G)$, we have the normalized Jacquet module

$$\mathrm{Jac}_P(\pi) \in \mathrm{Rep}(M)$$

along P. They are related by Frobenius reciprocity

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \cong \mathrm{Hom}_M(\mathrm{Jac}_P(\pi), \sigma)$$

for $\pi \in \mathrm{Rep}(G)$ and $\sigma \in \mathrm{Rep}(M)$.

Recall that $\mathcal{R}(G)$ is the Grothendieck group of $\mathrm{Rep}(G)$. If $P = MN_P$ is as above, the normalized parabolic induction and the normalized Jacquet functor induce linear maps

$$\mathrm{Ind}_P^G \colon \mathcal{R}(M) \to \mathcal{R}(G),$$
$$\mathrm{Jac}_P \colon \mathcal{R}(G) \to \mathcal{R}(M).$$

It is easy to see that these maps are invariant if we replace (P, M) with its conjugate by elements in T even when P is not stable under the adjoint action of T.

Definition C.1.1. *Fix an irreducible supercuspidal representation ρ of $\mathrm{GL}_d(E)$.*

(1) *Suppose that G° has a standard parabolic subgroup $P^\circ = M^\circ N_P$ such that $M \cong \mathrm{GL}_d(E) \times G_-$. For $\pi \in \mathcal{R}(G)$, if we write*

$$\mathrm{Jac}_P(\pi) = \sum_{i \in I} \tau_i \otimes \sigma_i \in \mathcal{R}(\mathrm{GL}_d(E)) \otimes \mathcal{R}(G_-)$$

*with $\tau_i \in \mathrm{Irr}(\mathrm{GL}_d(E))$ and $\sigma_i \in \mathrm{Irr}(G_-)$, we define the ρ-*derivative *$D_\rho(\pi)$ by*

$$D_\rho(\pi) = \sum_{\substack{i \in I \\ \tau_i \cong \rho}} \sigma_i \in \mathcal{R}(G_-).$$

If such a parabolic subgroup P° does not exist, we set $D_\rho(\pi) = 0$.

(2) *For $k \geq 0$, we define the k-*th *ρ-*derivative *$D_\rho^{(k)}(\pi)$ by*

$$D_\rho^{(k)}(\pi) = \frac{1}{k!} \underbrace{D_\rho \circ \cdots \circ D_\rho}_{k}(\pi).$$

In particular, $D_\rho^{(0)}(\pi) = \pi$.

(3) *If $D_\rho^{(k)}(\pi) \neq 0$ but $D_\rho^{(k+1)}(\pi) = 0$, we say that $D_\rho^{(k)}(\pi)$ is the* highest ρ-derivative, *and denote it by $D_\rho^{\max}(\pi)$.*

(4) *We say that π is ρ-*reduced *if $D_\rho(\pi) = 0$.*

Note that this notion of derivatives differs from the Bernstein–Zelevinsky derivatives in [BZ].

We write $\rho^{\times k} = \rho \times \cdots \times \rho$ (k times) for short. Note that for $\pi \in \mathcal{R}(G)$, the k-th derivative $D_\rho^{(k)}(\pi)$ is a linear combination of irreducible representations whose coefficients are non-negative integers. In fact, it is characterized such that

$$\mathrm{Jac}_P(\pi) = \rho^{\times k} \otimes D_\rho^{(k)}(\pi) + \sum_i \tau_i \boxtimes \pi_i,$$

where $P^\circ = M^\circ N_P$ is such that $M^\circ = \mathrm{GL}_{dk}(E) \times G_0^\circ$, and $\tau_i \boxtimes \pi_i \in \mathrm{Irr}(M)$ is such that $\tau_i \not\cong \rho^{\times k}$.

Lemma C.1.2. *Suppose that ρ is not conjugate-self-dual. Then for any $\pi \in \mathrm{Irr}(G)$, its highest derivative $D_\rho^{\max}(\pi)$ is also irreducible. Moreover, the map $\mathrm{Irr}(G) \ni \pi \mapsto D_\rho^{\max}(\pi)$ is injective in the following sense: for $\pi, \pi' \in \mathrm{Irr}(G)$, if $D_\rho^{\max}(\pi) = D_\rho^{(k)}(\pi) = D_\rho^{(k)}(\pi')$, then $\pi \cong \pi'$.*

Proof. Write $D_\rho^{\max}(\pi) = D_\rho^{(k)}(\pi)$. One can take an irreducible summand π_0 of $D_\rho^{\max}(\pi)$ such that

$$\mathrm{Jac}_P(\pi) \twoheadrightarrow \underbrace{\rho \boxtimes \cdots \boxtimes \rho}_{k} \boxtimes \pi_0$$

in $\mathrm{Rep}(M)$, where $P^\circ = M^\circ N_P$ is the appropriate standard parabolic subgroup of G°. By Frobenius reciprocity, we have

$$\pi \hookrightarrow \rho^{\times k} \rtimes \pi_0.$$

In particular, we have

$$\pi_0 \leq D_\rho^{(k)}(\pi) \leq D_\rho^{(k)}\left(\rho^{\times k} \rtimes \pi_0\right)$$

in $\mathcal{R}(G_0)$ for some classical group G_0. (Here, for $A, B \in \mathcal{R}(G_0)$, we write $A \leq B$ if $B - A$ is a non-negative combination of irreducible representations.) Since π_0 is ρ-reduced and since ρ is not conjugate-self-dual, by applying Tadić's formula ([Tad1, Theorems 5.4, 6.5], [Ban, Theorem 7.3]) to $\rho^{\times k} \rtimes \pi_0$, we have

$$D_\rho^{(k)}\left(\rho^{\times k} \rtimes \pi_0\right) = \pi_0,$$

and hence we have $D_\rho^{(k)}(\pi) = \pi_0$.

Suppose that $\pi' \in \mathrm{Irr}(G)$ satisfies $D_\rho^{(k)}(\pi') = \pi_0$ and $\pi' \not\cong \pi$. Then π' is also an irreducible subrepresentation of $\rho^{\times k} \rtimes \pi_0$. However, in the Grothendieck group, since $\pi' \leq (\rho^{\times k} \rtimes \pi_0) - \pi$, we have

$$\pi_0 = D_\rho^{(k)}(\pi') \leq D_\rho^{(k)}\left((\rho^{\times k} \rtimes \pi_0) - \pi\right) = 0.$$

This is a contradiction. □

Remark C.1.3. Tadić's formula holds even for $\mathrm{O}_{2n}(F)$ (see [X1, (5.5)] or [Ban, Theorem 7.3]). However, as in [X1, page 463], for $\mathrm{SO}_{2n}(F)$, this formula needs to be modified. This is one of the reasons why we do not work with $\mathrm{SO}_{2n}(F)$ but with $\mathrm{O}_{2n}(F)$.

Similarly, let $P = MN_P$ be a θ-stable standard parabolic subgroup of $\mathrm{GL}_N(E)$ such that $M \cong \mathrm{GL}_d(E) \times \mathrm{GL}_{N_-}(E) \times \mathrm{GL}_d(E)$. For $\widetilde{\pi} \in \mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$, if we write

$$\mathrm{Jac}_{\widetilde{P}}(\widetilde{\pi}) = \sum_{i \in I} \tau_i \otimes \widetilde{\sigma}_i \otimes {}^c\tau_i^\vee + \sum_{j \in J}(\pi_j + \pi_j \circ \theta)$$

with $\tau_i \in \mathrm{Irr}(\mathrm{GL}_d(E))$, $\widetilde{\sigma}_i \in \mathrm{Irr}(\widetilde{\mathrm{GL}}_{N_-}(E))$ and $\pi_j \in \mathrm{Irr}(M)$ such that $\pi_j \not\cong \pi_j \circ \theta$, we define

$$\widetilde{D}_\rho(\widetilde{\pi}) = \sum_{\substack{i \in I \\ \tau_i \cong \rho}} \widetilde{\sigma}_i \in \mathcal{R}(\widetilde{\mathrm{GL}}_{N_-}(E)).$$

Moreover, for $k \geq 0$, we set

$$\widetilde{D}_\rho^{(k)}(\widetilde{\pi}) = \frac{1}{k!} \underbrace{\widetilde{D}_\rho \circ \cdots \circ \widetilde{D}_\rho}_{k}(\widetilde{\pi}).$$

Note that a priori, $\widetilde{D}_\rho^{(k)}(\widetilde{\pi})$ is in $\mathcal{R}(\widetilde{\mathrm{GL}}_{N_0}(E)) \otimes_{\mathbb{Z}} \mathbb{Q}$ for some $N_0 \geq 0$.

C.2. **Compatibility of Jacquet functors with twisted endoscopy.** We denote by $\mathbb{C}[\Pi(G)]$ (resp. $\mathbb{C}^\theta[\Pi(N)]$) the space of invariant (resp. twisted invariant) distributions on G (resp. $\mathrm{GL}_N(E)$) which are finite in the sense that they are finite linear combinations of irreducible characters. Then the representation theoretic character provides an isomorphism

$$\mathbb{C}[\Pi(G)] \cong \mathcal{R}(G) \otimes_{\mathbb{Z}} \mathbb{C}.$$

For any $\pi \in \mathcal{R}(G)$, we denote its character by $\Theta_\pi \in \mathbb{C}[\Pi(G)]$.

By the above isomorphism, one can transfer our definition of Jacquet functors to the spaces $\mathbb{C}[\Pi(G)]$ and $\mathbb{C}^\theta[\Pi(N)]$. Now we define N from G as in Section 1.5. We fix an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(E)$, and $x \in \mathbb{R}$. We consider Levi subgroups

$$\mathrm{GL}_d(E) \times \mathrm{GL}_{N_-}(E) \times \mathrm{GL}_d(E)$$

of $\mathrm{GL}_N(E)$ and

$$M^\circ = \mathrm{GL}_d(E) \times G_-^\circ$$

of G°, respectively. Then by [X1, (6.6)], we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[\Pi(G)] & \xrightarrow{(G:G^\circ)^{-1}\mathrm{Trans}^\theta_{G^\circ}} & \mathbb{C}^\theta[\Pi(N)] \\ {\scriptstyle D_{\rho|\cdot|^x}}\downarrow & & \downarrow{\scriptstyle \widetilde{D}_{\rho|\cdot|^x}} \\ \mathbb{C}[\Pi(G_-)] & \xrightarrow{(G_-:G_-^\circ)^{-1}\mathrm{Trans}^\theta_{G_-^\circ}} & \mathbb{C}^\theta[\Pi(N_-)] \end{array}$$

where the above horizontal map is the composition of the averaged restriction map $\Theta \mapsto (G : G^\circ)^{-1}\Theta|_{C_c^\infty(G^\circ)}$ and the twisted endoscopic transfer map $\mathrm{Trans}^\theta_{G^\circ} \colon \mathbb{C}[\Pi(G^\circ)] \to \mathbb{C}^\theta[\Pi(N)]$. The bottom horizontal map is defined similarly. This commutativity was proven for general quasi-split connected reductive groups in [X1, Appendix C], and can be extended to the case of O_{2n} easily from the case of SO_{2n} (see Remark 1.6.2 (3)). Applying the above commutative diagram repeatedly, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\Pi(G)] & \xrightarrow{(G:G^\circ)^{-1}\mathrm{Trans}^\theta_{G^\circ}} & \mathbb{C}^\theta[\Pi(N)] \\ {\scriptstyle D^{(k)}_{\rho|\cdot|^x}}\downarrow & & \downarrow{\scriptstyle \widetilde{D}^{(k)}_{\rho|\cdot|^x}} \\ \mathbb{C}[\Pi(G_0)] & \xrightarrow{(G_0:G_0^\circ)^{-1}\mathrm{Trans}^\theta_{G_0^\circ}} & \mathbb{C}^\theta[\Pi(N_0)] \end{array}$$

for suitable G_0 and N_0.

Let ϕ be a tempered L-parameter for G, and let Π_ϕ be the L-packet associated to ϕ. Then we have an invariant distribution

$$\sum_{\pi \in \Pi_\phi} \Theta_\pi \in \mathbb{C}[\Pi(G)],$$

whose image under $(G : G^\circ)^{-1}\mathrm{Trans}^\theta_{G^\circ}$ is the twisted character $\Theta_{\widetilde{\pi}_\phi} \in \mathbb{C}^\theta[\Pi(N)]$ by (**ECR1**).

From now on, we suppose that

- ρ is conjugate-self-dual;
- $2x$ is a positive integer;
- ϕ contains $\rho \boxtimes S_{2x+1}$ with multiplicity $k \geq 0$.

Then

$$\widetilde{D}^{(k)}_{\rho|\cdot|^x}(\widetilde{\pi}_\phi)\Big|_{\mathrm{GL}_{N-2dk}(E)} = C \cdot \pi_{\phi_0},$$

for some $C \in \mathbb{Q}$, where

$$\phi_0 = \phi - (\rho \boxtimes S_{2x+1})^{\oplus k} \oplus (\rho \boxtimes S_{2x-1})^{\oplus k}.$$

In particular, $\widetilde{D}^{(k+1)}_{\rho|\cdot|^x}(\widetilde{\pi}_\phi) = 0$ and one can define $\epsilon(\phi) \in \mathbb{Q}$ such that

$$\widetilde{D}^{(k)}_{\rho|\cdot|^x}(\widetilde{\pi}_\phi) \overset{\theta}{=} \epsilon(\phi) \cdot \widetilde{\pi}_{\phi_0}.$$

Here, we recall from Section 4.1 that for $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \mathcal{R}(\widetilde{\mathrm{GL}}_N(E))$, we write $\widetilde{\pi}_1 \overset{\theta}{=} \widetilde{\pi}_2$ if $\Theta_{\widetilde{\pi}_1}(\widetilde{f}) = \Theta_{\widetilde{\pi}_2}(\widetilde{f})$ for any $\widetilde{f} \in C_c^\infty(\mathrm{GL}_N(E) \rtimes \theta)$. With the above commutative diagram, it implies the following lemma.

Lemma C.2.1. *We have $D^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$ for any $\pi \in \Pi_\phi$. Moreover, $\epsilon(\phi) \in \{0, 1\}$ and*

$$D^{(k)}_{\rho|\cdot|^x}\left(\sum_{\pi \in \Pi_\phi} \pi\right) = \epsilon(\phi) \sum_{\pi_0 \in \Pi_{\phi_0}} \pi_0.$$

Proof. By (**ECR1**), we have

$$\begin{aligned}
\frac{1}{(G_0 : G_0^\circ)} \mathrm{Trans}^\theta_{G_0^\circ} \circ D^{(k)}_{\rho|\cdot|^x}\left(\sum_{\pi \in \Pi_\phi} \Theta_\pi\right) &= \widetilde{D}^{(k)}_{\rho|\cdot|^x} \circ \mathrm{Trans}^\theta_{G^\circ}\left(\frac{1}{(G : G^\circ)} \sum_{\pi \in \Pi_\phi} \Theta_\pi\right) \\
&= \widetilde{D}^{(k)}_{\rho|\cdot|^x}(\Theta_{\widetilde{\pi}_\phi}) \overset{\theta}{=} \epsilon(\phi) \cdot \Theta_{\widetilde{\pi}_{\phi_0}} \\
&= \frac{\epsilon(\phi)}{(G_0 : G_0^\circ)} \mathrm{Trans}^\theta_{G_0^\circ}\left(\sum_{\pi_0 \in \Pi_{\phi_0}} \Theta_{\pi_0}\right).
\end{aligned}$$

Hence

$$\sum_{\pi \in \Pi_\phi} D^{(k)}_{\rho|\cdot|^x}(\Theta_\pi) = \epsilon(\phi) \sum_{\pi_0 \in \Pi_{\phi_0}} \Theta_{\pi_0}.$$

A similar argument shows that $\sum_{\pi \in \Pi_\phi} D^{(k+1)}_{\rho|\cdot|^x}(\Theta_\pi) = 0$, which implies that $D^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$ for any $\pi \in \Pi_\phi$. Hence by Lemma C.1.2, $\sum_{\pi \in \Pi_\phi} D^{(k)}_{\rho|\cdot|^x}(\pi)$ is a multiplicity-free sum of irreducible representations (possibly zero). This implies that $\epsilon(\phi) \in \{0, 1\}$, and we obtain the desired equation for $D^{(k)}_{\rho|\cdot|^x}(\Pi_\phi)$. □

C.3. **Computation of highest derivatives.** Recall that we fix an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(E)$, and $x > 0$ such that $2x \in \mathbb{Z}$. We compute $D^{(k)}_{\rho|\cdot|^x}(\pi)$ for each $\pi \in \Pi_\phi$.

First, we suppose that $\rho \boxtimes S_{2x+1}$ is conjugate-self-dual of the opposite type to ϕ. Then the multiplicity k of $\rho \boxtimes S_{2x+1}$ is even. Moreover, if we set $\phi' = \phi - (\rho \boxtimes S_{2x+1})^k$, then $A_\phi = A_{\phi'}$ and

$$\pi = \underbrace{\Delta([-x,x]_\rho) \times \cdots \times \Delta([-x,x]_\rho)}_{k/2} \rtimes \pi',$$

where $\pi' \in \Pi_{\phi'}$ is such that $\langle \cdot, \pi' \rangle_{\phi'} = \langle \cdot, \pi \rangle_\phi$. See [Ar2, Theorem 1.5.1] and [Mok, Theorem 2.5.1]. Hence

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \underbrace{\Delta([-(x-1),x-1]_\rho) \times \cdots \times \Delta([-(x-1),x-1]_\rho)}_{k/2} \rtimes \pi'.$$

This is equal to $\pi_0 \in \Pi_{\phi_0}$ with $\langle \cdot, \pi_0 \rangle_{\phi_0} = \langle \cdot, \pi \rangle_\phi$ via the canonical identification $A_{\phi_0} = A_\phi$, where ϕ_0 is the same as in the previous subsection.

In the rest of this subsection, we assume that $\rho \boxtimes S_{2x+1}$ is conjugate-self-dual of the same type as ϕ. In this case, to compute $D^{(k)}_{\rho|\cdot|^x}(\pi)$ for each $\pi \in \Pi_\phi$, we use (**ECR2**). Let $s \in A_\phi$. Suppose first that $s \in A_\phi^+$ or $G = \mathrm{U}_n$. Let I_1 be an index set such that $s = \sum_{i \in I_1} e(\rho_i, a_i, 1)$, and set

$$\phi_1 = \bigoplus_{i \in I_1} \rho_i \boxtimes S_{a_i}, \quad \phi_2 = \phi - \phi_1.$$

For $i = 1, 2$, fix a conjugate-self-dual character η_i of $E^\times$ such that $\phi_i \otimes \eta_i \in \Phi_{\mathrm{temp}}(G_i)$ for some classical group G_i. Then $H^\circ = G_1^\circ \times G_2^\circ$ is an elliptic endoscopic group of G°, and (η_1, η_2) gives an L-homomorphism

$$\xi \colon {}^L(G_1^\circ \times G_2^\circ) \to {}^L G^\circ.$$

Let

$$\begin{aligned} M^\circ &= \mathrm{GL}_d(E) \times G_-^0, \\ M_1^\circ &= \mathrm{GL}_d(E) \times G_{1,-}^0, \\ M_2^\circ &= \mathrm{GL}_d(E) \times G_{2,-}^0 \end{aligned}$$

be Levi subgroups of G°, G_1° and G_2°, respectively, and write

$$\begin{aligned} H &= G_1 \times G_2, \\ H_{1,-} &= G_{1,-} \times G_2, \\ H_{2,-} &= G_1 \times G_{2,-}. \end{aligned}$$

Then by [X1, (6.2)–(6.4)], we have a commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}[\Pi(H)] & \xrightarrow{\alpha(G,H)\mathrm{Trans}_{H^\circ}^{G^\circ}} & \mathbb{C}[\Pi(G^\circ)] \\
{\scriptstyle D_{\rho\eta_1|\cdot|^x}\oplus D_{\rho\eta_2|\cdot|^x}}\Big\downarrow & & \Big\downarrow{\scriptstyle D_{\rho|\cdot|^x}} \\
\mathbb{C}[\Pi(H_{1,-})]\oplus\mathbb{C}[(\Pi(H_{2,-}))] & \xrightarrow{\sum_{i=1}^{2}\alpha(G_-:H_{i,-})\mathrm{Trans}_{H_{i,-}^\circ}^{G_-^\circ}} & \mathbb{C}[\Pi(G_-^\circ)],
\end{array}$$

where

$$\alpha(G,H)=\frac{(G:G^\circ)}{(H:H^\circ)}.$$

For the proof, see [X1, Appendix C].

Similarly, when $G=\mathrm{O}_{2n}(F)$ and $s\in A_\phi\setminus A_\phi^+$, assuming that $G_-^\circ\neq\{1\}$, by [X1, (6.7)], we have a commutative diagram:

$$\begin{array}{ccc}
\mathbb{C}[\Pi(H)] & \xrightarrow{\alpha(G,H)\mathrm{Trans}_{H^\circ}^{G\setminus G^\circ}} & \mathbb{C}[\Pi(G\setminus G^\circ)] \\
{\scriptstyle D_{\rho\eta_1|\cdot|^x}\oplus D_{\rho\eta_2|\cdot|^x}}\Big\downarrow & & \Big\downarrow{\scriptstyle D_{\rho|\cdot|^x}} \\
\mathbb{C}[\Pi(H_{1,-})]\oplus\mathbb{C}[(\Pi(H_{2,-}))] & \xrightarrow{\sum_{i=1}^{2}\alpha(G_-,H_{i,-})\mathrm{Trans}_{H_{i,-}^\circ}^{G_-\setminus G_-^\circ}} & \mathbb{C}[\Pi(G_-\setminus G_-^\circ)],
\end{array}$$

where $\mathbb{C}[\Pi(G\setminus G^\circ)]$ is the space of finite linear combinations of $\mathrm{O}_{2n}(F)$-twisted irreducible characters of $\mathrm{SO}_{2n}(F)$. More precisely, $\mathrm{Trans}_{H^\circ}^{G\setminus G^\circ}$ stands for the twisted endoscopic transfer map

$$\mathrm{Trans}_{H^\circ}^{G\setminus G^\circ}\left(\prod_{i=1}^{2}\frac{1}{(G_i:G_i^\circ)}\sum_{\pi_i\in\Pi_{\phi_i\otimes\eta_i}}\Theta_{\pi_i}\right)=\frac{1}{(G:G^\circ)}\sum_{\pi\in\Pi_\phi}\langle s,\pi\rangle_\phi\Theta_\pi|_{C_c^\infty(G\setminus G^\circ)}.$$

We denote the multiplicity of $\rho\boxtimes S_{2x+1}$ in ϕ_i by k_i, and set

$$\phi_{i,0}=\phi_i-(\rho\boxtimes S_{2x+1})^{\oplus k_i}\oplus(\rho\boxtimes S_{2x-1})^{\oplus k_i}.$$

Let $G_{i,0}$ be the classical group such that $\phi_{i,0}\otimes\eta_i\in\Phi_{\mathrm{temp}}(G_{i,0})$.

Proposition C.3.1. *Let s_0 be the image of s under the surjection $A_\phi\twoheadrightarrow A_{\phi_0}$ defined by*

$$e(\rho',d,1)\mapsto\begin{cases}e(\rho,2x-1,1) & \text{if } \rho'\cong\rho,\ d=2x+1>2,\\ 0 & \text{if } \rho'\cong\rho,\ d=2x+1=2,\\ e(\rho',d,1) & \text{otherwise.}\end{cases}$$

Then

$$D_{\rho|\cdot|^x}^{(k)}\left(\sum_{\pi\in\Pi_\phi}\langle s,\pi\rangle_\phi\Theta_\pi\right)=\epsilon(\phi_1\otimes\eta_1)\epsilon(\phi_2\otimes\eta_2)\sum_{\pi_0\in\Pi_{\phi_0}}\langle s_0,\pi_0\rangle_{\phi_0}\Theta_{\pi_0}.$$

Proof. Note that if $\phi_0 = 0$, then $2x+1 = 2$ and $\phi = (\rho \boxtimes S_2)^{\oplus k}$. In this case, $A_\phi^+ = A_\phi$. In other words, if $G = \mathrm{O}_{2n}(F)$ and $s \in A_\phi \setminus A_\phi^+$, then we have $G_-^\circ \neq \{1\}$ automatically, and we can use the last diagram above.

If $s \in A_\phi^+$ or $G = \mathrm{U}_n$, then by applying the first diagram repeatedly together with (**ECR2**) and Lemma C.2.1, we have

$$\begin{aligned}
&D^{(k)}_{\rho|\cdot|^x}\left(\sum_{\pi\in\Pi_\phi}\langle s,\pi\rangle_\phi\Theta_\pi\right)\\
&= D^{(k)}_{\rho|\cdot|^x}\circ\alpha(G:H)\mathrm{Trans}^{G^\circ}_{G_1^\circ\times G_2^\circ}\left(\prod_{i=1}^{2}\sum_{\pi_i\in\Pi_{\phi_i\otimes\eta_i}}\Theta_{\pi_i}\right)\\
&= \alpha(G_0, G_{1,0}\times G_{2,0})\mathrm{Trans}^{G_0^\circ}_{G_{1,0}^\circ\times G_{2,0}^\circ}\left(\prod_{i=1}^{2}D^{(k_i)}_{\rho\eta_i|\cdot|^x}\left(\sum_{\pi_i\in\Pi_{\phi_i\otimes\eta_i}}\Theta_{\pi_i}\right)\right)\\
&= \alpha(G_0, G_{1,0}\times G_{2,0})\mathrm{Trans}^{G_0^\circ}_{G_{1,0}^\circ\times G_{2,0}^\circ}\left(\prod_{i=1}^{2}\epsilon(\phi_i\otimes\eta_i)\sum_{\pi_{i,0}\in\Pi_{\phi_{i,0}\otimes\eta_i}}\Theta_{\pi_{i,0}}\right)\\
&= \epsilon(\phi_1\otimes\eta_1)\epsilon(\phi_2\otimes\eta_2)\sum_{\pi_0\in\Pi_{\phi_0}}\langle s_0,\pi_0\rangle_{\phi_0}\Theta_{\pi_0}.
\end{aligned}$$

This completes the proof unless $G = \mathrm{O}_{2n}(F)$ since A_ϕ is generated by A_ϕ^+ and z_ϕ when $G = \mathrm{SO}_{2n+1}(F)$ or $G = \mathrm{Sp}_{2n}(F)$ (whereas when $G = \mathrm{U}_n$, there is no restriction). On the other hand, using the second diagram involving $\mathrm{Trans}^{G\setminus G^\circ}_{H^\circ}$, the same argument works when $G = \mathrm{O}_{2n}(F)$ and $s \in A_\phi \setminus A_\phi^+$. □

As a first consequence, we obtain the following important result.

Corollary C.3.2. *We have* $\epsilon(\phi) = 1$.

Proof. Recall that $\epsilon(\phi) \in \{0,1\}$ satisfies $\widetilde{D}^{(k)}_{\rho|\cdot|^x}(\widetilde{\pi}_\phi) \overset{\theta}{=} \epsilon(\phi)\cdot\widetilde{\pi}_{\phi_0}$ and

$$\sum_{\pi\in\Pi_\phi} D^{(k)}_{\rho|\cdot|^x}(\Theta_\pi) = \epsilon(\phi)\sum_{\pi_0\in\Pi_{\phi_0}}\Theta_{\pi_0}.$$

We show the claim by induction on k.

The case $k=0$ is trivial since $\widetilde{D}^{(0)}_{\rho|\cdot|^x}(\widetilde{\pi}_\phi) = \widetilde{\pi}_\phi$ by definition. Similarly, if $k=1$, then $\widetilde{D}^{(1)}_{\rho|\cdot|^x}(\widetilde{\pi}_\phi)|_{\mathrm{GL}_{N-2d}(E)} = \pi_{\phi_0}$ so that $\epsilon(\phi) \in \{\pm 1\}$. Since $\epsilon(\phi) \in \{0,1\}$, we must have $\epsilon(\phi) = 1$.

Suppose now that $k > 1$. We apply Proposition C.3.1 to $s = e(\rho, 2x+1, 1)$. Then by induction hypothesis, we have $\epsilon(\phi_i\otimes\eta_i) = 1$ for $i = 1,2$ since $k_1 = 1$ and $k_2 = k-1$.

Hence

$$D^{(k)}_{\rho|\cdot|^x}\left(\sum_{\pi\in\Pi_\phi}\langle s,\pi\rangle_\phi\Theta_\pi\right)\neq 0.$$

This implies that $\epsilon(\phi)\neq 0$ and hence $\epsilon(\phi)=1$. □

In particular, by Lemma C.2.1 and Corollary C.3.2, we have

$$D^{(k)}_{\rho|\cdot|^x}\left(\sum_{\pi\in\Pi_\phi}\pi\right)=\sum_{\pi_0\in\Pi_{\phi_0}}\pi_0.$$

Now we compute $D^{(k)}_{\rho|\cdot|^x}(\pi)$ for $\pi\in\Pi_\phi$.

Theorem C.3.3. *Let ϕ be a tempered L-parameter for G. Suppose that ϕ contains $\rho\boxtimes S_{2x+1}$ with multiplicity $k>0$. Let $\pi\in\Pi_\phi$. Then $D^{(k)}_{\rho|\cdot|^x}(\pi)=0$ if and only if one of the following holds:*

- *$x\geq 1$, $\phi\supset\rho\boxtimes S_{2x-1}$ and $\langle e(\rho,2x+1,1),\pi\rangle_\phi\neq\langle e(\rho,2x-1,1),\pi\rangle_\phi$;*
- *$x=\frac{1}{2}$ and $\langle e(\rho,2,1),\pi\rangle_\phi=-1$.*

Moreover, if $D^{(k)}_{\rho|\cdot|^x}(\pi)\neq 0$, then $\pi_0=D^{(k)}_{\rho|\cdot|^x}(\pi)\in\Pi_{\phi_0}$ and it is characterized such that $\langle\cdot,\pi\rangle_\phi$ is the composition of $\langle\cdot,\pi_0\rangle_{\phi_0}$ with the surjection $A_\phi\twoheadrightarrow A_{\phi_0}$ in Proposition C.3.1.

Proof. First, suppose that $x\geq 1$ and $\phi\supset\rho\boxtimes S_{2x-1}$. We apply Proposition C.3.1 to $s=e(\rho,2x+1,1)+e(\rho,2x-1,1)$. Then $s_0\in A^0_{\phi_0}$ so that

$$D^{(k)}_{\rho|\cdot|^x}\left(\sum_{\pi\in\Pi_\phi}\langle s,\pi\rangle_\phi\Theta_\pi\right)$$

is a sum of irreducible characters with non-negative coefficients. Hence, if $\langle s,\pi\rangle_\phi=\langle e(\rho,2x+1,1),\pi\rangle_\phi\cdot\langle e(\rho,2x-1,1),\pi\rangle_\phi=-1$, then $D^{(k)}_{\rho|\cdot|^x}(\pi)=0$.

Similarly, when $x=\frac{1}{2}$, by applying Proposition C.3.1 for $s=e(\rho,2,1)$, we see that if $\langle e(\rho,2,1),\pi\rangle_\phi=-1$, then $D^{(k)}_{\rho|\cdot|^x}(\pi)=0$.

Note that via the surjection $A_\phi\twoheadrightarrow A_{\phi_0}$, we have $z_\phi\mapsto z_{\phi_0}$, and the image of A^0_ϕ is included in $A^0_{\phi_0}$. Hence this map induces a surjection $\mathcal{A}_\phi\twoheadrightarrow\mathcal{A}_{\phi_0}$. Then π does not satisfy the above two assumptions if and only if the character $\langle\cdot,\pi\rangle_\phi\colon\mathcal{A}_\phi\to\{\pm1\}$ factors through the surjection $\mathcal{A}_\phi\twoheadrightarrow\mathcal{A}_{\phi_0}$. In this case, we denote the character of $\mathcal{A}_{\phi_0}$ associated to π by η_π, i.e.,

$$\langle\cdot,\pi\rangle_\phi\colon\mathcal{A}_\phi\twoheadrightarrow\mathcal{A}_{\phi_0}\xrightarrow{\eta_\pi}\{\pm1\}.$$

For $s_0\in A_{\phi_0}$, choose a lift $s\in A_\phi$ of it. By applying Proposition C.3.1 for s together with Corollary C.3.2, we have

$$\sum_{\pi\in\Pi_\phi}\langle s,\pi\rangle_\phi D^{(k)}_{\rho|\cdot|^x}(\pi)=\sum_{\pi_0\in\Pi_{\phi_0}}\langle s_0,\pi_0\rangle_{\phi_0}\pi_0.$$

This equation can be written as

$$\sum_{\substack{\pi\in\Pi_\phi \\ D^{(k)}_{\rho|\cdot|^x}(\pi)\neq 0}} \eta_\pi(s_0) D^{(k)}_{\rho|\cdot|^x}(\pi) = \sum_{\pi_0\in\Pi_{\phi_0}} \langle s_0, \pi_0\rangle_{\phi_0}\pi_0$$

for $s_0 \in \mathcal{A}_{\phi_0}$. This implies that $D^{(k)}_{\rho|\cdot|^x}(\pi) = \pi_0$ if and only if $\eta_\pi = \langle\cdot,\pi_0\rangle_{\phi_0}$. This completes the proof. □

When $D^{(k)}_{\rho|\cdot|^x}(\pi) = 0$, the highest derivative of π is given as follows.

Theorem C.3.4. *Let ϕ be a tempered L-parameter for G. Suppose that ϕ contains $\rho\boxtimes S_{2x+1}$ with multiplicity $k>0$. Let $\pi\in\Pi_\phi$ and assume that $D^{(k)}_{\rho|\cdot|^x}(\pi) = 0$.*

(1) *If k is odd, then $\pi_1 = D^{(k-1)}_{\rho|\cdot|^x}(\pi)$ is nonzero, tempered and belongs to Π_{ϕ_1} with*

$$\phi_1 = \phi - (\rho\boxtimes S_{2x+1})^{\oplus k-1} \oplus (\rho\boxtimes S_{2x-1})^{\oplus k-1}.$$

Moreover, there is a canonical isomorphism $\mathcal{A}_{\phi_1}\cong\mathcal{A}_\phi$, and $\langle\cdot,\pi_1\rangle_{\phi_1} = \langle\cdot,\pi\rangle_\phi$ via this isomorphism.

(2) *If k is even, then $D^{(k-1)}_{\rho|\cdot|^x}(\pi)$ is nonzero but not tempered. It is the Langlands quotient of the standard module*

$$\Delta([-(x-1),x]_\rho)\rtimes\pi_2,$$

where $\pi_2\in\Pi_{\phi_2}$ with

$$\phi_2 = \phi - (\rho\boxtimes S_{2x+1})^{\oplus k} \oplus (\rho\boxtimes S_{2x-1})^{\oplus k-2}.$$

Moreover, there is a canonical inclusion $\mathcal{A}_{\phi_2}\hookrightarrow\mathcal{A}_\phi$, and $\langle\cdot,\pi_2\rangle_{\phi_2} = \langle\cdot,\pi\rangle_\phi|_{\mathcal{A}_{\phi_2}}$.

Proof. First, we consider the case $k=2$. Then

$$\pi\hookrightarrow\Delta([-x,x]_\rho)\rtimes\pi_2\hookrightarrow\rho|\cdot|^x\times\Delta([-x,x-1]_\rho)\rtimes\pi_2.$$

By Frobenius reciprocity, we have $D^{(1)}_{\rho|\cdot|^x}(\pi)\neq 0$. Since $D^{(2)}_{\rho|\cdot|^x}(\pi) = 0$, by Lemma C.1.2, we know that $D^{(1)}_{\rho|\cdot|^x}(\pi)$ is irreducible. Moreover, we have an equivariant map

$$D^{(1)}_{\rho|\cdot|^x}(\pi)\to\Delta([-x,x-1]_\rho)\rtimes\pi_2.$$

Since $D^{(1)}_{\rho|\cdot|^x}(\pi)$ is irreducible, this map must be injective. Using the MVW involution, we see that $D^{(1)}_{\rho|\cdot|^x}(\pi)$ is the unique irreducible quotient of $\Delta([-(x-1),x]_\rho)\rtimes\pi_2$. See e.g., [AG2, Section 2.7].

Note that when $k=1$, there is nothing to prove. Therefore, the remaining case is where $k\geq 3$. Write $k = 2l+k'$ with $k'\in\{1,2\}$. Consider $\pi'\in\Pi_{\phi'}$ with

$$\phi' = \phi - (\rho\boxtimes S_{2x+1})^{\oplus 2l}$$

so that $\mathcal{A}_{\phi'} \cong \mathcal{A}_\phi$, and $\langle \cdot, \pi' \rangle_{\phi'} = \langle \cdot, \pi \rangle_\phi|_{\mathcal{A}_{\phi'}}$. Then by [Ar2, Theorem 1.5.1] and [Mok, Theorem 2.5.1], the parabolically induced representation

$$\underbrace{\Delta([-x,x]_\rho) \times \cdots \times \Delta([-x,x]_\rho)}_{l} \rtimes \pi'$$

is irreducible and is equal to π. By Tadić's formula ([Tad1, Theorems 5.4, 6.5], [Ban, Theorem 7.3]), we see that

$$\begin{aligned} D^{(k-1)}_{\rho|\cdot|^x}(\pi) &= D^{(k-1)}_{\rho|\cdot|^x}(\Delta([-x,x]_\rho) \times \cdots \times \Delta([-x,x]_\rho) \rtimes \pi') \\ &= \underbrace{\Delta([-(x-1),x-1]_\rho) \times \cdots \times \Delta([-(x-1),x-1]_\rho)}_{l} \rtimes D^{(k'-1)}_{\rho|\cdot|^x}(\pi'). \end{aligned}$$

This shows the assertions. □

Finally, we give a characterization of (almost) supercuspidal representations. (See also [Mœ, Theorem 2.5.1].)

Corollary C.3.5. *Let ϕ be a tempered L-parameter for G, and $\pi \in \Pi_\phi$. Then the following are equivalent:*

(1) *π is $\rho|\cdot|^x$-reduced for any irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(E)$ and nonzero real number $x \neq 0$;*
(2) *the following conditions hold:*
   - *If we denote the multiplicity of $\rho \boxtimes S_d$ in ϕ by $m_\phi(\rho,d)$, then $m_\phi(\rho,d) \leq 1$ whenever $d \geq 2$;*
   - *if $\rho \boxtimes S_d \subset \phi$ with $d > 2$, then $\rho \boxtimes S_{d-2} \subset \phi$ and*
   $$\langle e(\rho,d,1), \pi \rangle_\phi = -\langle e(\rho,d-2,1), \pi \rangle_\phi;$$
   - *if $\rho \boxtimes S_2 \subset \phi$, then*
   $$\langle e(\rho,2,1), \pi \rangle_\phi = -1.$$

Moreover, π is supercuspidal if and only if the above conditions hold and ϕ is a discrete parameter.

Proof. By Casselman's criterion (see [Kon2, Lemma 2.4]), we know that any tempered representation π is $\rho|\cdot|^x$-reduced for any $x < 0$. Then the first equivalence follows from Theorems C.3.3 and C.3.4.

If π is supercuspidal, then π is discrete series so that ϕ is discrete. Conversely, if π satisfies the conditions in (2) but not cuspidal, then one can find an irreducible cuspidal unitary representation ρ of $\mathrm{GL}_d(E)$ such that $D_\rho(\pi) \neq 0$. In particular, $\pi \hookrightarrow \rho \rtimes \pi_0$ for some irreducible representation π_0. Then Casselman's criterion shows that π_0 is also tempered. Hence ϕ contains $\rho \oplus {}^c\rho^\vee$ so that ϕ is not discrete. □

Remark C.3.6. Fix an irreducible unitary cuspidal representation ρ of $\mathrm{GL}_d(E)$. In general, for $\pi \in \mathrm{Irr}(G)$, the highest ρ-derivative $D^{\max}_\rho(\pi)$ is not necessarily irreducible, and it is difficult to describe it completely. However, when π is tempered, it is easy.

More strongly, the next claim follows from [Ar2, Proposition 2.4.3] and [Mok, Proposition 3.4.4].

Let ϕ be the tempered L-parameter of $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G)$. Then there exists $\pi' \in \mathrm{Irr}(G')$ such that

$$\pi \hookrightarrow \rho \rtimes \pi'$$

if and only if ϕ contains $\rho \oplus {}^c\rho^\vee$. In this case, π' is uniquely determined by the conditions $\pi' \in \Pi_{\phi'}$ with $\phi = \phi' \oplus \rho \oplus {}^c\rho^\vee$, and $\langle \cdot, \pi' \rangle_{\phi'} = \langle \cdot, \pi \rangle_\phi|_{\mathcal{A}_{\phi'}}$. In particular, if we write $\phi = \phi_0 \oplus (\rho \oplus {}^c\rho^\vee)^{\oplus k}$ such that $\phi_0 \not\supset \rho \oplus {}^c\rho^\vee$, and if $\pi_0 \in \Pi_{\phi_0}$ is such that $\langle \cdot, \pi_0 \rangle_{\phi_0} = \langle \cdot, \pi \rangle_\phi|_{\mathcal{A}_{\phi_0}}$, then $\pi \hookrightarrow \rho^{\times k} \rtimes \pi_0$ and

$$D_\rho^{\max}(\pi) = D_\rho^{(k)}(\pi) = m \cdot \pi_0$$

with $m = 2^k$ or $m = 1$ according to $\rho \cong {}^c\rho^\vee$ or not.

Appendix D. The tempered L-packet conjecture and Arthur's Lemma 2.5.5

In this appendix, we prove two results whose statements appear unrelated, but whose proofs follow from the same argument.

The first statement is that Arthur's construction of tempered L-packets satisfies the strong form of Shahidi's tempered packet conjecture (Corollary D.1.3). This conjecture was initially formulated as [Sha7, Conjecture 9.4] and stipulated that every tempered L-packet should contain a generic representation. It was later strengthened to include the statement that, for an arbitrarily fixed Whittaker datum $\mathfrak{w}$, every tempered L-packet should contain exactly one $\mathfrak{w}$-generic member. For classical groups, this was proven by Konno [Kon1] and Varma [V2] assuming the twisted endoscopic transfer to GL_N, which is the basis of the constructions of [Ar2] and [Mok]. A further strengthening of the conjecture would require that the $\mathfrak{w}$-generic member is matched with the trivial character of the centralizer component group $\mathcal{S}_\phi$. It is this version that we formulate here (Conjecture D.1.3) and prove for classical groups, thus providing a mild strengthening of the result of Konno and Varma for classical groups. We hasten to note that our proof does not replace those of Konno and Varma, but rather it uses them crucially. In fact, we prove a result (Theorem D.1.2) that is valid for general reductive groups and infers the validity of this conjecture from its validity for endoscopic groups. This result is a strengthening of [Sha7, Proposition 9.6] and the proof follows from a similar outline, with a few additional arguments, and a key input from the work of Kottwitz [Kot]. When combined with the results of Konno and Varma, this relative result delivers Conjecture D.1.3 for classical groups.

The second statement (Theorem D.2.1) is of a more technical nature. It infers the local intertwining relation from an a priori weaker statement. It was formulated for symplectic and orthogonal groups as [Ar2, Lemma 2.5.5], whose proof was deferred to [A27]. We formulate and prove it here for arbitrary connected reductive groups, subject to assuming basic expected properties of tempered L-packets that are known in the setting of classical groups.

Arthur suggested in [Ar2] that Theorem D.2.1 can be shown by applying results of Konno [Kon1] concerning the behavior of local character expansions under the (twisted) endoscopic transfer. We largely follows his suggestion in the proofs below. However, it is also clear that the results of Konno needed to be refined and made more precise before they can be applied. Thankfully, in the intervening years, Varma has provided in [V2] such a refined result, and this is the main ingredient in our proof below. It reduces the proof to an identity between Weil indices and ε-factors, which follows at once from the work of Kottwitz [Kot], and in the case of classical groups can also be verified by hand. Thanks to [V1], we can also remove the assumption in [Ar2, Lemma 2.5.5] that the residual characteristic of F is odd.

We first employ this argument to obtain a proof of Theorem D.1.2. The flow is as in [Sha7, Proposition 9.6], and is in some sense opposite to that of [V2]. Then we employ a similar argument, in a slightly more abbreviated form, to obtain Theorem D.2.1.

Remark D.0.1. While the proofs of Theorems D.1.2 and D.2.1 use the same argument, the statements have different assumptions. The main assumption for Theorem D.1.2 is that (**ECR2**) holds with respect to any endoscopic group, and Corollary D.1.3 assumes in addition that (**ECR1**) holds. Thus, in the setting of Arthur's argument, Theorem D.1.2 and Corollary D.1.3 become available after the inductive argument has been completed for the group of interest. On the other hand, the main assumption in Theorem D.2.1 is that a weakened form of the local intertwining relation holds. This assumption replaces (**ECR2**). The reason is that Theorem D.2.1 will be used in the middle of the inductive proof, where (**ECR2**) is not yet available. In addition, the validity of Corollary D.1.3 is assumed for groups of lower rank. In the setting of Arthur's argument, this is part of the inductive assumption.

D.1. **Shahidi's tempered packet conjecture for quasi-split classical groups.** Let F be a local field of characteristic zero and let G be a quasi-split connected reductive F-group. Fix a Whittaker datum $\mathfrak{w}$ for G. Let ϕ be a tempered L-parameter for G. We assume that the corresponding L-packet Π_ϕ and its pairing with $\mathcal{S}_\phi$ have been constructed, giving an injective map from Π_ϕ to the set of irreducible representations of $\mathcal{S}_\phi$; we use the notation $\langle s, \pi\rangle_\phi$ for the trace at s of the representation of $\mathcal{S}_\phi$ associated to π. Note that the pairing $\langle \cdot, \pi\rangle_\phi$ depends on $\mathfrak{w}$. Then Shahidi's strong tempered packet conjecture is the following statement, a strengthening of [Sha7, Conjecture 9.4].

Conjecture D.1.1. *Fix a Whittaker datum $\mathfrak{w}$ for G. The L-packet Π_ϕ contains exactly one $\mathfrak{w}$-generic member $\pi_{\mathfrak{w}}$, and it satisfies $\langle \cdot, \pi_{\mathfrak{w}}\rangle_\phi = \mathbf{1}$.*

This conjecture is known for archimedean base fields due to the work of Shelstad in [She1], see also [AK]. We may therefore concentrate on a non-archimedean base field F.

Theorem D.1.2. *Assume $|\Pi_\phi| > 1$ and that, for any factorization ϕ' of ϕ through a proper endoscopic group, the character identity* (**ECR2**) *holds and Conjecture D.1.1 holds for $\Pi_{\phi'}$. Then Conjecture D.1.1 holds for Π_ϕ.*

We will show Theorem D.1.2 in Section D.3 below. For now, we extract the following result in the setting of classical groups.

Corollary D.1.3. *Let G be a quasi-split connected classical group. Assuming that the local results* (**ECR1**) *and* (**ECR2**) *of* [Ar2] *and* [Mok] *are known for the parameter ϕ and for its factorizations through endoscopic groups. Then Conjecture D.1.1 holds for Π_ϕ.*

Proof. As already remarked, the archimedean case is known by the work of Shelstad, so we focus on non-archimedean F. We induct on the size of L-packets.

Assume first that Π_ϕ is a singleton $\{\pi\}$. Then $\mathcal{S}_\phi$ is the trivial group, so trivially $\langle \cdot, \pi\rangle_\phi = \mathbf{1}$ and it remains to show that π is $\mathfrak{w}$-generic. Let π_ϕ be the representation of $\mathrm{GL}_N(E)$ with parameter ϕ, seen as a parameter for $\mathrm{GL}_N(E)$ via the standard representation of ${}^LG$. Then π_ϕ is tempered, hence generic. Using (**ECR1**), the claim follows from [Kon1] and [V2].

Assume next that Π_ϕ is not a singleton. Then $\mathcal{S}_\phi \neq \{1\}$. For any $s \in S_\phi \setminus Z(\widehat{G})^\Gamma$, the pair (s, ϕ) leads to a proper endoscopic datum G' and a parameter ϕ' for G'. The L-packet $\Pi_{\phi'}$ has strictly smaller size than Π_ϕ, so Conjecture D.1.1 holds for $\Pi_{\phi'}$ by induction hypothesis. Thus Conjecture D.1.1 holds for Π_ϕ by Theorem D.1.2. □

D.2. **A weakening of the local intertwining relation.** Let F be a non-archimedean local field of characteristic zero and let G be a quasi-split connected reductive F-group equipped with a Whittaker datum $\mathfrak{w}$. Let $P = MN$ be a proper parabolic subgroup and let ϕ_M be a tempered L-parameter for M. We assume the existence of an associated L-packet Π_{ϕ_M}. Let ϕ be the parameter for G obtained by composing ϕ_M with the natural inclusion ${}^LM \to {}^LG$. Recall that $N_\phi = N(A_{\widehat{M}}, \widehat{G}) \cap S_\phi$ and $\mathfrak{N}_\phi = \pi_0(N_\phi/Z(\widehat{G})^\Gamma)$. We now recall some material from Section 1.10 in this more general setting. For $f \in C_c^\infty(G(F))$ and $u \in \mathfrak{N}_\phi$, define the distribution

$$f \mapsto f_G(\phi, u) = \sum_{\substack{\pi_M \in \Pi_{\phi_M} \\ w_u \pi_M \cong \pi_M}} \mathrm{tr}(\langle u, \widetilde{\pi}_M\rangle R_P(w_u, \widetilde{\pi}_M, \phi_M) I_P(\pi_M, f)),$$

where w_u is the Weyl element given by u which preserves M, and we are using a representation $\widetilde{\pi}_M$ of the disconnected group

$$M(F) \rtimes \langle w_u \rangle$$

that extends π_M and the associated pairing $\langle u, \widetilde{\pi}_M\rangle$, noting that the product

$$\langle u, \widetilde{\pi}_M\rangle R_P(w_u, \widetilde{\pi}_M, \phi_M)$$

is independent of the choice $\widetilde{\pi}_M$ extending π_M.

Let $s \in \mathcal{S}_\phi$ be the image of u. From the pair (s, ϕ), we obtain an endoscopic datum (G', s, η) and a parameter ϕ' for G'. We define a second distribution

$$f \mapsto f'_G(\phi, s) = \mathrm{Trans}(S\Theta_{\phi'})(f),$$

where

$$S\Theta_{\phi'} = \sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \Theta_{\pi'}$$

is the stable character of ϕ', and $f' \in C_c^\infty(G'(F))$ is a $\Delta[\mathfrak{w}]$-transfer of f, i.e. the two functions f and f' have matching orbital integrals with respect to the transfer factor $\Delta[\mathfrak{w}]$ normalized with respect to the Whittaker datum $\mathfrak{w}$.

We now state the assumptions under which our result holds. The main assumption is that there exists a constant $e(s,u)$ such that

$$f'_G(\phi, s) = e(s,u) f_G(\phi, u).$$

The supplementary assumptions concern expected properties of L-packets, as follows. We assume that Conjecture D.1.1 holds for the parameter ϕ_M as well as for the endoscopic factorizations ϕ' of ϕ. In the setting of classical groups, this follows from Corollary D.1.3. Thus, we know that there is a unique $\mathfrak{w}_M$-generic member $\pi_{M,\mathfrak{w}} \in \Pi_{\phi_M}$ and it satisfies $\langle \cdot, \pi_{M,\mathfrak{w}} \rangle_{\phi_M} = \mathbf{1}$. Let Π_ϕ be the set consisting of the irreducible constituents of the representations $I_P(\pi_M)$, as π_M runs over Π_{ϕ_M}. According to the heredity property ([Rod], [CS, Corollary 1.7]), there is a unique $\mathfrak{w}$-generic member $\pi_{\mathfrak{w}}$ of Π_ϕ and it lies in $I_P(\pi_{M,\mathfrak{w}})$. Then the action of the operator $\langle u, \widetilde{\pi}_{M,\mathfrak{w}} \rangle R_P(w_u, \widetilde{\pi}_{M,\mathfrak{w}}, \phi_M)$ on $I_P(\pi_{M,\mathfrak{w}})$ must preserve $\pi_{\mathfrak{w}}$, and hence acts on it by a scalar. We assume that this scalar is 1. For classical groups, this follows from Theorem 1.8.1 (1) together with Lemma 6.3.1.

Theorem D.2.1. *Under the above assumptions,*

$$e(s,u) = 1.$$

In other words, equation (**A-LIR**) *in Section 1.10 holds.*

We will prove Theorem D.2.1 in Section D.4 below.

Remark D.2.2. Since we are also interested in the case of the orthogonal group $G = \mathrm{O}_{2n}$, which is disconnected, we will make the following slight modification in this situation. We will take $f \in C_c^\infty(G^\circ(F))$, and $u \in \mathfrak{N}_\phi$ will also be taken with respect to G°. The distribution $f_G(\phi, u)$ and the stable character $S\Theta_{\phi'}$ will be defined as

$$\frac{1}{(G : G^\circ)} \sum_{\substack{\pi_M \in \Pi_{\phi_M} \\ w_u \pi_M \cong \pi_M}} \mathrm{tr}(\langle u, \widetilde{\pi}_M \rangle R_P(w_u, \widetilde{\pi}_M, \phi_M) I_P(\pi_M, f))$$

and

$$S\Theta_{\phi'} = \frac{1}{(G' : G'^\circ)} \sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \Theta_{\pi'},$$

respectively. Note that, when G is connected, these formulas recover the previous formulas. Then the modifications for the case of $G = \mathrm{O}_{2n}$ follows by applying the following argument to $G^\circ = \mathrm{SO}_{2n}$. See Remark 1.6.2. In the following proof, we will continue working with a connected reductive group G.

D.3. **Proof of Theorem D.1.2.** For $\pi \in \Pi_\phi$, let

$$c(\pi, \mathfrak{w}) = \begin{cases} 1 & \text{if } \pi \text{ is } \mathfrak{w}\text{-generic}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that it is enough to show the following statement: For any $s \in \mathcal{S}_\phi$ with $s \neq 1$, we have

$$(\dagger) \qquad \sum_{\pi \in \Pi_\phi} c(\pi, \mathfrak{w}) \langle s, \pi \rangle_\phi = 1.$$

Indeed, assume that we have shown this statement. Let

$$X = \{\pi \in \Pi_\phi \,|\, c(\pi, \mathfrak{w}) = 1\} \subset \Pi_\phi$$

and consider the conjugation-invariant function f on $\mathcal{S}_\phi$ defined by

$$f(s) = \sum_{\pi \in \Pi_\phi} c(\pi, \mathfrak{w}) \langle s, \pi \rangle_\phi.$$

Equation (†) and the assumption $|\mathcal{S}_\phi| > 1$ implies $f \neq 0$, hence $X \neq \emptyset$, and $f(1)$ is a natural number greater than 0 (if $\mathcal{S}_\phi$ is abelian, then $f(1) = |X|$). The scalar product of f with the trivial character of $\mathcal{S}_\phi$ is non-trivial, hence X contains the representation π_1 with $\langle \cdot, \pi_1 \rangle_\phi = \mathbf{1}$. Thus, π_1 is $\mathfrak{w}$-generic. Let $X^1 = X \setminus \{\pi_1\}$ and let $f^1 = f - \langle \cdot, \pi_1 \rangle_\phi$. Then $f^1(s) = 0$ for all $s \neq 1$. On the one hand, since f is a multiplicity free sum of irreducible characters of $\mathcal{S}_\phi$, the construction of f^1 implies that the scalar product of f^1 and the trivial character of $\mathcal{S}_\phi$ is equal to zero, while on the other hand, this scalar product is equal to $f^1(1)$ by evaluation. Thus $f^1(1) = 0$ and we conclude $f^1 = 0$, hence $X^1 = \emptyset$. Therefore $X = \{\pi_1\}$.

This reduces the proof of Theorem D.1.2 to the proof of Equation (†). To prove it, let $1 \neq s \in \mathcal{S}_\phi$ and let G' and ϕ' be the corresponding endoscopic datum and factored parameter, respectively. Consider the distributions

$$f \mapsto f'_G(\phi, s) = \mathrm{Trans}(S\Theta_{\phi'})(f), \quad S\Theta_{\phi'} = \sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \Theta_{\pi'}$$

and

$$f_G(\phi, s) = \sum_{\pi \in \Pi_\phi} \langle s, \pi \rangle_\phi \Theta_\pi(f),$$

both linear forms on $C_c^\infty(G(F))$. According to (**ECR2**) we have

$$f_G(\phi, s) = f'_G(\phi, s).$$

For each $\pi \in \Pi_\phi$, we have the Harish-Chandra local character expansion

$$\Theta_\pi(f) = \sum_O c(\pi, O) \widehat{\mu}_O(f \circ \exp)$$

for all $f \in C^\infty(G(F))$ with support close to the identity, where O runs over the set of nilpotent orbits in $\mathfrak{g}(F) = \mathrm{Lie}(G)(F)$. To form the Fourier transform $\widehat{\mu}_O$, we have chosen arbitrarily a non-degenerate $G(F)$-invariant symmetric bilinear form β on $\mathfrak{g}(F)$

and a non-trivial unitary character $\psi_F \colon F \to \mathbb{C}^\times$. Note that the measures on $G(F)$ (used to define Θ_π) and on O (used to define μ_O) also depend on the choice of β, as in [MW1, Section I.8].

Putting these together we obtain

$$f_G(\phi, s) = \sum_{\pi \in \Pi_\phi} \langle s, \pi \rangle_\phi \sum_O c(\pi, O) \widehat{\mu}_O(f \circ \exp).$$

In the same way, we obtain

$$f'_G(\phi, s) = \sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \sum_{O'} c(\pi', O') \widehat{\mu}_{O'}(f' \circ \exp),$$

where now O' runs over the set of nilpotent orbits in $\mathfrak{g}'(F)$. We have used another non-degenerate $G'(F)$-invariant symmetric bilinear form β' on $\mathfrak{g}'(F)$, at the moment unrelated to β.

Thus (**ECR2**) implies that, in a neighborhood of the identity,

$$\mathrm{Trans}\left(\sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \sum_{O'} c(\pi', O') \widehat{\mu}_{O'} \right) = \sum_{\pi \in \Pi_\phi} \langle s, \pi \rangle_\phi \sum_O c(\pi, O) \widehat{\mu}_O,$$

and using the homogeneity of nilpotent orbital integrals, we obtain from this

$$\mathrm{Trans}\left(\sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \sum_{O' : \text{regular}} c(\pi', O') \widehat{\mu}_{O'} \right) = \sum_{\pi \in \Pi_\phi} \langle s, \pi \rangle_\phi \sum_{O : \text{regular}} c(\pi, O) \widehat{\mu}_O,$$

see [Sha7, pp. 325–326].

We now use a result of Mœglin–Waldspurger [MW1] and its extension to the dyadic case by Varma [V1]. There exists a certain regular nilpotent orbit $O_{\psi_F, \beta, \mathfrak{w}}$, depending on the Whittaker datum $\mathfrak{w}$, the form β, and the character ψ_F, such that

$$c(\pi, O_{\psi_F, \beta, \mathfrak{w}}) = \begin{cases} 1 & \text{if } \pi \text{ is } \mathfrak{w}\text{-generic}, \\ 0 & \text{otherwise}. \end{cases}$$

We will specify the orbit $O_{\psi_F, \beta, \mathfrak{w}}$ more precisely below. For now, we note that we can apply this to both sides of the above identity. On the left-hand side, we use the uniqueness of $\mathfrak{w}'$-generic constituent in $\Pi_{\phi'}$ for each Whittaker datum $\mathfrak{w}'$ on G', the fact that $\langle 1, \pi' \rangle_{\phi'} = 1$ for such a constituent (which is the assumption that Conjecture D.1.1 holds for G' and an application of [Kal1, Theorem 4.3]), and the fact that each regular nilpotent orbit O' is equal to $O_{\psi_F, \beta', \mathfrak{w}'}$ for some choice of $\mathfrak{w}'$, to conclude that

$$\sum_{\pi' \in \Pi_{\phi'}} \langle 1, \pi' \rangle_{\phi'} \sum_{O' : \text{regular}} c(\pi', O') \widehat{\mu}_{O'} = \sum_{O' : \text{regular}} \widehat{\mu}_{O'}.$$

Therefore, our identity becomes

$$\mathrm{Trans}\left(\sum_{O':\mathrm{regular}} \widehat{\mu}_{O'}\right) = \sum_{\pi\in\Pi_\phi} \langle s,\pi\rangle_\phi \sum_{O:\mathrm{regular}} c(\pi,O)\widehat{\mu}_O.$$

According to [LSh, Corollary 5.5.B], we have

$$\mathrm{Trans}\left(\sum_{O':\mathrm{regular}} \mu_{O'}\right) = \sum_{O:\mathrm{regular}} \Delta[\mathfrak{w}](O)\mu_O,$$

where the transfer factor $\Delta[\mathfrak{w}](O)$ is defined at the end of [LSh, (5.1)], in which it was denoted by $\Delta(u)$, with u a regular unipotent element in $G(F)$. We are using here the bijection between regular unipotent orbits in $G(F)$ and regular nilpotent orbits in $\mathfrak{g}(F)$.

Since the endoscopic transfer commutes with the Fourier transform up to an explicit scalar by [W1, p. 91, Conjecture 1] (which is a theorem due to [W2], [W3], [CHL], [N]), we obtain

$$\mathrm{Trans}\left(\sum_{O':\mathrm{regular}} \widehat{\mu}_{O'}\right) = \frac{\gamma(\mathfrak{g},\beta,\psi_F)}{\gamma(\mathfrak{g}',\beta',\psi_F)} \sum_{O:\mathrm{regular}} \Delta[\mathfrak{w}](O)\widehat{\mu}_O,$$

where $\gamma(\mathfrak{g},\beta,\psi_F)$ and $\gamma(\mathfrak{g}',\beta',\psi_F)$ are the corresponding Weil indices. This identity requires the forms β and β' to be synchronized, as explained in [W1, Section VIII.6], and as we now recall. Extending scalars from F to $\overline{F}$ identifies the space of non-degenerate symmetric bilinear forms on $\mathfrak{g}(F)$ that are invariant under $G(F)$ with the space of non-degenerate symmetric bilinear forms on $\mathfrak{g}(\overline{F})$ that are invariant under $G(\overline{F})$ and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. If $T \subset G$ is an arbitrary maximal torus, then the restriction to its Lie algebra $\mathfrak{t}(\overline{F})$ identifies the latter space with the space of non-degenerate symmetric bilinear forms on $\mathfrak{t}(\overline{F})$ that are invariant under the absolute Weyl group and Γ. Note that, if $T' \subset G$ is a second maximal torus, then the spaces for $\mathfrak{t}$ and $\mathfrak{t}'$ are canonically identified, namely by $\mathrm{Ad}(g)$ for any $g \in G(\overline{F})$ that conjugates T to T'. In this way, taking a maximal torus of G' and transferring it to G, we can transfer β to $\mathfrak{g}'$, and we take β' to be that transfer.

With this proviso, our identity becomes

$$\frac{\gamma(\mathfrak{g},\beta,\psi_F)}{\gamma(\mathfrak{g}',\beta',\psi_F)} \sum_{O:\mathrm{regular}} \Delta[\mathfrak{w}](O)\widehat{\mu}_O = \sum_{\pi\in\Pi_\phi} \langle s,\pi\rangle_\phi \sum_{O:\mathrm{regular}} c(\pi,O)\widehat{\mu}_O.$$

By [HC, Theorem 5.11], we can separate terms. Comparing coefficients for the orbit $O_{\psi_F,\beta,\mathfrak{w}}$ and applying the results of Mœglin–Waldspurger and Varma recalled above, this identity turns

$$\frac{\gamma(\mathfrak{g},\beta,\psi_F)}{\gamma(\mathfrak{g}',\beta',\psi_F)} \Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}) = \sum_{\pi\in\Pi_\phi} c(\pi,\mathfrak{w})\langle s,\pi\rangle_\phi.$$

Our goal is to show that the left-hand side of this expression equals 1. As a first step, we will rewrite this left-hand side in a way that does not involve the form β and the

Whittaker datum $\mathfrak{w}$. For this, let $S' \subset G'$ be a maximal torus and let $S \subset G$ be its transfer. Decompose $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, where $\mathfrak{r}$ is the direct sum of the root spaces $\mathfrak{g}_\alpha$ for all absolute roots α of S in G. As discussed in [Kot, Section 1.3.3], there is a canonical quadratic form Q on $\mathfrak{r}$ defined as the orthogonal sum of quadratic forms $Q_{\pm\alpha}$ on each $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ given by

$$[Y_\alpha, Y_{-\alpha}] = Q_{\pm\alpha}(Y_\alpha + Y_{-\alpha}) \cdot H_\alpha,$$

where H_α is the coroot for α. We can thus form the Weil index $\gamma(\mathfrak{r}, Q, \psi_F)$. Note however that this form does not always extend to $\mathfrak{g}$, see [Kot, Section I.5].

We have the analogous decomposition $\mathfrak{g}' = \mathfrak{s}' \oplus \mathfrak{r}'$ and the analogous quadratic form Q' on $\mathfrak{r}'$, hence also the Weil index $\gamma(\mathfrak{r}', Q', \psi_F)$. Finally, we have the maximally split maximal tori (equivalently, minimal Levi subgroups) $T \subset G$ and $T' \subset G'$.

Lemma D.3.1. *The identity*

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}) = \frac{\gamma(\mathfrak{r}, Q, \psi_F)}{\gamma(\mathfrak{r}', Q', \psi_F)} \varepsilon(1/2, X^*(T)_{\mathbb{C}} - X^*(T')_{\mathbb{C}}, \psi_F)$$

holds for any choices of character ψ_F, Whittaker datum $\mathfrak{w}$, and non-degenerate $G(F)$-invariant symmetric bilinear form β, where β' is the transfer of β to $\mathfrak{g}'(F)$ according to [W1, Section VIII.6]. *In particular, the left-hand side is independent of the choices of $\mathfrak{w}$ and β. Furthermore, both sides are independent of the choice of ψ_F.*

Proof. Following the definition of $\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}})$ given at the end of [LSh, (5.1)], we have

$$\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}) = \frac{\Delta[\mathfrak{w}](\overline{\gamma}_{G'}, \overline{\gamma}_G)}{\Delta[\boldsymbol{spl}'_\infty](\overline{\gamma}_{G'}, \overline{\gamma}_G)} \langle \mathrm{inv}_{\boldsymbol{spl}'}(O_{\psi_F,\beta,\mathfrak{w}}), s \rangle,$$

where $\boldsymbol{spl}'$ is an arbitrarily chosen splitting of G and $\boldsymbol{spl}'_\infty$ is its opposite splitting; the left-hand side is independent of this choice, as well as of the choice of related elements $\overline{\gamma}_{G'}$ and $\overline{\gamma}_G$.

We will choose $\boldsymbol{spl}'$ in such a way that $\mathrm{inv}_{\boldsymbol{spl}'}(O_{\psi_F,\beta,\mathfrak{w}}) = 1$. To see what this entails, we review the definition of $O_{\psi_F,\beta,\mathfrak{w}}$ given in [MW1] and [V1]. Choose a splitting $\boldsymbol{spl} = (B, T, \{X_\alpha\})$ that, together with ψ_F, induces $\mathfrak{w}$ in the sense of [KoSh1, Section 5.3]. Let U be the unipotent radical of B and let $\overline{U}$ be the unipotent radical of the T-opposite Borel subgroup to B. We write $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ for the Lie algebras of U and $\overline{U}$, respectively. Then $O_{\psi_F,\beta,\mathfrak{w}}$ is the $G(F)$-orbit of a regular nilpotent element $Y_{\psi_F,\beta,\mathfrak{w}} \in \overline{\mathfrak{u}}(F)$ whose property is that the character

$$X \mapsto \psi_F(\beta(X, Y_{\psi_F,\beta,\mathfrak{w}}))$$

of $\mathfrak{u}(F)$ equals the composition of the exponential map $\exp\colon \mathfrak{u}(F) \to U(F)$ and the generic character $U(F) \to \mathbb{C}^\times$ that makes up the Whittaker datum $\mathfrak{w}$. One can check that $Y_{\psi_F,\beta,\mathfrak{w}}$ is given by

$$Y_{\psi_F,\beta,\mathfrak{w}} = \sum_\alpha \beta(X_\alpha, X_{-\alpha})^{-1} X_{-\alpha},$$

where $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ is determined by $[X_\alpha, X_{-\alpha}] = H_\alpha$. Thus, if we take $\boldsymbol{spl}'$ to be $(\overline{B}, T, \{\beta(X_\alpha, X_{-\alpha})^{-1}X_{-\alpha}\})$, then $\mathrm{inv}_{\boldsymbol{spl}'}(O_{\psi_F,\beta,\mathfrak{w}}) = 1$. Hence we have

$$\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}) = \frac{\Delta[\mathfrak{w}](\overline{\gamma}_{G'}, \overline{\gamma}_G)}{\Delta[\boldsymbol{spl}'_\infty](\overline{\gamma}_{G'}, \overline{\gamma}_G)},$$

for this particular choice of $\boldsymbol{spl}'$. On the other hand, by the definition of the Whittaker normalization [KoSh1, Section 5.3], [KoSh2],

$$\Delta[\mathfrak{w}](\overline{\gamma}_{G'}, \overline{\gamma}_G) = \varepsilon(1/2, X^*(T)_\mathbb{C} - X^*(T')_\mathbb{C}, \psi_F) \cdot \Delta[\boldsymbol{spl}](\overline{\gamma}_{G'}, \overline{\gamma}_G).$$

Thus we need to compute the ratio between $\Delta[\boldsymbol{spl}](\overline{\gamma}_{G'}, \overline{\gamma}_G)$ and $\Delta[\boldsymbol{spl}'_\infty](\overline{\gamma}_{G'}, \overline{\gamma}_G)$. Both of these being transfer factors, this ratio does not depend on the elements $\overline{\gamma}_{G'}$ and $\overline{\gamma}_G$, as long as they are related (otherwise both factors are zero). So we choose arbitrarily a pair of related strongly regular semi-simple elements $\overline{\gamma}_{G'} \in S'(F)$ and $\overline{\gamma}_G \in S(F)$. Note that this choice determines an admissible isomorphism $S' \xrightarrow{\sim} S$, namely the unique one that maps $\overline{\gamma}_{G'}$ to $\overline{\gamma}_G$, and hence determines an inclusion of absolute root systems $R(S', G') \to R(S, G)$.

The only term of the transfer factor that depends on the splitting is the term Δ_I, where the splitting enters the definition of the splitting invariant. We have the relation $\boldsymbol{spl}'_\infty = b \cdot \boldsymbol{spl}$, where $b_\alpha = \beta(X_\alpha, X_{-\alpha})$ for all $\alpha \in R(T, G)$. Then [Kal1, Lemma 5.1] shows that rescaling the splitting has the same effect as rescaling the a-data, which also enters the definition of the splitting invariant. On the other hand, [LSh, Lemma 3.2.C] shows how the transfer factor changes when the a-data is rescaled. With this, we obtain

$$\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}) = \varepsilon(1/2, X^*(T)_\mathbb{C} - X^*(T')_\mathbb{C}, \psi_F) \prod_{\alpha \in R(S,G/G')_{\mathrm{sym}}/\Gamma} \kappa_\alpha(b_\alpha),$$

where $R(S, G/G')$ is a short-hand notation for $R(S, G) \setminus R(S', G')$, the subscript "sym" indicates those roots that are *symmetric*, i.e. those α with $-\alpha \in \Gamma \cdot \alpha$, and κ_α is the sign character of the quadratic extension $F_\alpha/F_{\pm\alpha}$.

Using that the decompositions $\mathfrak{g}' = \mathfrak{s}' \oplus \mathfrak{r}'$ and $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ are orthogonal for β' and β, respectively, we see that the Weil indices decompose accordingly as products, and since $\beta|_\mathfrak{s} = \beta'|_{\mathfrak{s}'}$ by the synchronization of β and β', the left-hand side of the equation in the statement of the lemma becomes

$$\frac{\gamma(\mathfrak{r}, \beta, \psi_F)}{\gamma(\mathfrak{r}', \beta', \psi_F)} \varepsilon(1/2, X^*(T)_\mathbb{C} - X^*(T')_\mathbb{C}, \psi_F) \prod_{\alpha \in R(S,G/G')_{\mathrm{sym}}/\Gamma} \kappa_\alpha(b_\alpha).$$

According to [Kal2, Lemma 4.8] and the fact that $Q(X_\alpha + X_{-\alpha}) = 1$, we have

$$\frac{\gamma(\mathfrak{r}, \beta, \psi_F)}{\gamma(\mathfrak{r}, Q, \psi_F)} = \prod_{\alpha \in R(S,G)_{\mathrm{sym}}/\Gamma} \kappa_\alpha(b_\alpha).$$

The analogous identity holds with (G, S) replaced by (G', S'). This proves the identity claimed in the lemma. The independence of the left-hand side of the choices of β and $\mathfrak{w}$ follows from that identity. To see the independence of both sides of the choice of

ψ_F, note that any other choice of character is of the form $(a\psi_F)(x) = \psi_F(ax)$ for some $a \in F^\times$, but one sees directly from the definitions that replacing ψ_F and β by $a\psi_F$ and $a^{-1} \cdot \beta$ does not change the Weil indices or the orbit $O_{\psi_F,\beta,\mathfrak{w}}$. □

To complete the proof of Theorem D.1.2, we need to show that the right-hand side of the identity of Lemma D.3.1 equals to one. This is accomplished by the following lemma.

Lemma D.3.2. *Let $S' \subset G'$ be any maximal torus and let $S \subset G$ be its transfer. Decompose the Lie algebras $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ and $\mathfrak{g}' = \mathfrak{s}' \oplus \mathfrak{r}'$. Let Q (resp. Q') be the canonical quadratic form on $\mathfrak{r}$ (resp. $\mathfrak{r}'$) described before the statement of Lemma D.3.1. Let $T' \subset G'$ and $T \subset G$ be minimal Levi subgroups. Then*

$$\frac{\gamma(\mathfrak{r}, Q, \psi_F)}{\gamma(\mathfrak{r}', Q', \psi_F)} \varepsilon(1/2, X^*(T)_{\mathbb{C}} - X^*(T')_{\mathbb{C}}, \psi_F) = 1.$$

Proof. From [Kot, Theorem 1.1], we have

$$\gamma(\mathfrak{r}, Q, \psi_F) = \varepsilon(1/2, X^*(S)_{\mathbb{C}} - X^*(T)_{\mathbb{C}}, \psi_F).$$

Applying this result once to G and the torus S, and once to G' and the torus S', and using that $X^*(S) \cong X^*(S')$ as Γ-modules, we obtain the desired result. □

We obtain Equation (†), and hence complete the proof of Theorem D.1.2.

Remark D.3.3. In the case of classical groups, one can also compute the left-hand side in Lemma D.3.2 explicitly and check that it equals 1. See Section D.5 below.

D.4. **Proof of Theorem D.2.1.** The argument is very close to that of the proof of Theorem D.1.2. Since Theorem D.2.1 is an important part of the argument of [Ar2], we will still present all steps of the proof, but we will be more brief with the justifications, which are the same as in the previous proof.

We consider the same distribution $f'_G(\phi, s)$ as in that proof, but replace the distribution $f_G(\phi, s)$ by the distribution $f_G(\phi, u)$ from the statement of Theorem D.2.1. This distribution can be expanded as

$$f_G(\phi, u) = \sum_{\pi \in \Pi_\phi} c(\pi)\Theta_\pi(f),$$

where $c(\pi)$ is defined as follows. Let $\pi \otimes M_\pi$ be the maximal π-isotypic constituent of $I_P(\pi_M)$. By Schur's lemma $\langle u, \widetilde{\pi}_M\rangle R_P(w_u, \widetilde{\pi}_M, \phi_M)$ induces an operator on the finite-dimensional $\mathbb{C}$-vector space M_π and $c(\pi)$ is the trace of that operator. Our assumptions imply that $M_{\pi_\mathfrak{w}}$ is 1-dimensional and $c(\pi_\mathfrak{w}) = 1$.

Instead of (**ECR2**) in Theorem D.1.2, we now use the assumed identity

$$f'_G(\phi, s) = e(s, u) f_G(\phi, u).$$

Expanding both sides using the Harish-Chandra local character expansion and using the homogeneity of nilpotent orbital integrals, we arrive at the identity

$$\mathrm{Trans}\left(\sum_{\pi'\in\Pi_{\phi'}}\langle 1,\pi'\rangle_{\phi'}\sum_{O':\text{regular}}c(\pi',O')\widehat{\mu}_{O'}\right)=e(s,u)\sum_{\pi\in\Pi_\phi}c(\pi)\sum_{O:\text{regular}}c(\pi,O)\widehat{\mu}_O.$$

Here we have again fixed a non-trivial character $\psi_F\colon F\to\mathbb{C}^\times$ and a non-degenerate $G(F)$-invariant symmetric bilinear form β on $\mathfrak{g}(F)$, which we have transferred to a form β' on $\mathfrak{g}'(F)$, and have used these to form the Fourier transforms. We consider again the nilpotent $G(F)$-orbit $O_{\psi_F,\beta,\mathfrak{w}}$ in $\mathfrak{g}(F)$, which according to the results of Mœglin–Waldspurger and Varma has the property

$$c(\pi,O_{\psi_F,\beta,\mathfrak{w}})=\begin{cases}1 & \text{if } \pi=\pi_{\mathfrak{w}},\\ 0 & \text{otherwise.}\end{cases}$$

Using the assumed validity of Conjecture D.1.1 for G', which gives the uniqueness of $\mathfrak{w}'$-generic constituent in $\Pi_{\phi'}$ for each Whittaker datum $\mathfrak{w}'$ on G' and the fact that $\langle 1,\pi'\rangle_{\phi'}=1$ for such a constituent, and using further the fact that each regular nilpotent orbit O' is equal to $O_{\psi_F,\beta',\mathfrak{w}'}$ for some choice of $\mathfrak{w}'$, we conclude that

$$\sum_{\pi'\in\Pi_{\phi'}}\langle 1,\pi'\rangle_{\phi'}\sum_{O':\text{regular}}c(\pi',O')\widehat{\mu}_{O'}=\sum_{O':\text{regular}}\widehat{\mu}_{O'}.$$

Therefore, our identity becomes

$$\mathrm{Trans}\left(\sum_{O':\text{regular}}\widehat{\mu}_{O'}\right)=e(s,u)\sum_{\pi\in\Pi_\phi}c(\pi)\sum_{O:\text{regular}}c(\pi,O)\widehat{\mu}_O.$$

Applying [LSh, Corollary 5.5.B], [W1, p. 91, Conjecture 1], [W2], [W3], [CHL], [N], we have

$$\mathrm{Trans}\left(\sum_{O':\text{regular}}\widehat{\mu}_{O'}\right)=\frac{\gamma(\mathfrak{g},\beta,\psi_F)}{\gamma(\mathfrak{g}',\beta',\psi_F)}\sum_{O:\text{regular}}\Delta[\mathfrak{w}](O)\widehat{\mu}_O.$$

We separate terms using [HC, Theorem 5.11] and comparing coefficients for the orbit $O_{\psi_F,\beta,\mathfrak{w}}$, we arrive at

$$e(s,u)=\frac{\gamma(\mathfrak{g},\beta,\psi_F)}{\gamma(\mathfrak{g}',\beta',\psi_F)}\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}).$$

By Lemmas D.3.1 and D.3.2, the right-hand side is equal to 1. This completes the proof of Theorem D.2.1.

D.5. **An explicit computation of Weil indices and ε-factors.** The proofs of Theorems D.1.2 and D.2.1 rely on the key Lemma D.3.2. We gave a proof of this lemma for general reductive groups using the work of Kottwitz [Kot]. For classical groups, this lemma can also be verified by an explicit computation, which we shall give here.

For explicit computation, the left-hand side of Lemma D.3.2 is inconvenient, because it stipulates that one has to work with a maximal torus of G that transfers to G', while the computation of the Weil index is most convenient when one uses as a maximal torus a minimal Levi subgroup of G. So we return to the left-hand side of Lemma D.3.1. As shown there, the left-hand side is independent of the choices of the character ψ_F, the Whittaker datum $\mathfrak{w}$, and the form β (recalled that β' is the transfer of β). We are thus free to choose these objects in a convenient way. We first choose a character ψ_F and an F-splitting $\boldsymbol{spl} = (B, T, \{X_\alpha\})$ and then take the Whittaker datum $\mathfrak{w}$ determined by $\boldsymbol{spl}$ and ψ_F as in [KoSh1, Section 5.3]. We have the $G(F)$-orbit $O_{\psi_F,\beta,\mathfrak{w}}$ of the regular nilpotent element

$$Y_{\psi_F,\beta,\mathfrak{w}} = \sum_\alpha \beta(X_\alpha, X_{-\alpha})^{-1} X_{-\alpha}.$$

It may be a priori different from the $G(F)$-orbit O_∞ of the regular nilpotent element

$$Y_\infty = \sum_\alpha X_{-\alpha}$$

that is "associated" to the splitting $\boldsymbol{spl}_\infty = (\overline{B}, T, \{X_{-\alpha}\})$ opposite to $\boldsymbol{spl}$. In our computations, we will show that β can be chosen such that

$$(\ddagger) \qquad O_{\psi_F,\beta,\mathfrak{w}} = O_\infty.$$

This identity implies that $\mathrm{inv}_{\boldsymbol{spl}_\infty}(O_{\psi_F,\beta,\mathfrak{w}}) = 1$. We can then apply the computation of $\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}})$ given in the proof of Lemma D.3.1 but with $\boldsymbol{spl}' = \boldsymbol{spl}_\infty$ and obtain

$$\Delta[\mathfrak{w}](O_{\psi_F,\beta,\mathfrak{w}}) = \frac{\Delta[\mathfrak{w}](\overline{\gamma}_{G'}, \overline{\gamma}_G)}{\Delta[\boldsymbol{spl}](\overline{\gamma}_{G'}, \overline{\gamma}_G)} = \varepsilon(1/2, X^*(T)_\mathbb{C} - X^*(T')_\mathbb{C}, \psi_F),$$

thereby reducing the desired identity to

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_\mathbb{C}, \psi_F)}{\varepsilon(1/2, X^*(T')_\mathbb{C}, \psi_F)} = 1$$

for those particular choices of ψ_F, β and $\boldsymbol{spl}$. We will show this identity for classical groups by explicitly computing all terms.

First, we compute the relevant Weil indices. To recall the definition and properties of Weil indices, we introduce some notation. Let V be a finite dimensional vector space over F equipped with a non-degenerate symmetric bilinear form β. Following [W1, Section VIII.1], we put

$$\gamma(V, \beta, \psi_F) = \frac{I}{|I|} \quad \text{with} \quad I = \int_L \psi_F\left(\frac{\beta(x,x)}{2}\right) dx,$$

where L is a sufficiently large lattice of V and dx is a Haar measure on V. Note that $\gamma(V, \beta, \psi_F)$ does not depend on the choice of L and dx. If $V = F$ and $\beta(x, y) = 2axy$ for some $a \in F^\times$, then we have $\gamma(V, \beta, \psi_F) = \gamma(a\psi_F)$ with the convention in [Rao, Appendix]. Here $a\psi_F$ is the non-trivial additive character of F given by $a\psi_F(x) = \psi_F(ax)$. Note that $\gamma(\psi_F)\gamma(-\psi_F) = 1$ and

$$\frac{\gamma(\psi_F)}{\gamma(a\psi_F)} = \varepsilon(1/2, \eta_a, \psi_F)$$

(see [Kah], [Sz]). Here η_a is the (possibly trivial) quadratic character of $F^\times$ given by $\eta_a(x) = (a, x)_F$, where $(\cdot, \cdot)_F$ is the quadratic Hilbert symbol of F, so that η_a is associated to $F(\sqrt{a})/F$ by the local class field theory. One can prove that if (V, β) is a direct sum of (V_0, β_0) and the hyperbolic plane $\mathbb{H}$, then $\gamma(V, \beta, \psi_F) = \gamma(V_0, \beta_0, \psi_F)$.

Recall that U is the unipotent radical of the Borel subgroup $B = TU$, and $\mathfrak{u}$ is its Lie algebra. Since G is quasi-split, $\mathfrak{u} \oplus \overline{\mathfrak{u}}$ is an orthogonal direct sum of hyperbolic planes and hence

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\mathfrak{t}, \beta, \psi_F).$$

Now we take the splitting ***spl*** given in Section A.3 and compute $\gamma(\mathfrak{g}, \beta, \psi_F)$ explicitly.

The case of symplectic groups: Suppose that $G = \mathrm{Sp}_{2n}(F)$, so that

$$\mathfrak{t} = \{\mathrm{diag}(X_1, \dots, X_n, -X_n, \dots, -X_1) \mid X_1, \dots, X_n \in F\}.$$

Take the non-degenerate G-invariant symmetric bilinear form β on $\mathfrak{g}$ given by $\beta(X, Y) = \mathrm{tr}(XY)$. Noting that $X_{-\alpha_i} = E_{i+1,i} - E_{2n+1-i,2n-i}$ for $1 \leq i \leq n-1$ and $X_{-\alpha_n} = E_{n+1,n}$, we have

$$Y_{\psi_F,\beta,\mathfrak{w}} = 2^{-1}(X_{-\alpha_1} + \cdots + X_{-\alpha_{n-1}}) + X_{-\alpha_n} = \mathrm{Ad}(t_0)\left(\sum_{i=1}^{n} X_{-\alpha_i}\right) \in O_\infty,$$

where

$$t_0 = \mathrm{diag}(2^{n-1}, 2^{n-2}, \dots, 1, 1, \dots, 2^{-n+2}, 2^{-n+1}) \in T.$$

Hence β satisfies the condition (‡). Since

$$\beta(X, Y) = 2(X_1Y_1 + \cdots + X_nY_n)$$

for $X, Y \in \mathfrak{t}$, we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\psi_F)^n.$$

The case of odd special orthogonal groups: Suppose that $G = \mathrm{SO}_{2n+1}(F)$, so that

$$\mathfrak{t} = \{\mathrm{diag}(X_1, \dots, X_n, 0, -X_n, \dots, -X_1) \mid X_1, \dots, X_n \in F\}.$$

Take the non-degenerate G-invariant symmetric bilinear form β on $\mathfrak{g}$ given by $\beta(X, Y) = \mathrm{tr}(XY)$. Noting that $X_{-\alpha_i} = E_{i+1,i} - E_{2n+2-i,2n+1-i}$ for $1 \leq i \leq n-1$ and $X_{-\alpha_n} = 2(E_{n+1,n} - E_{n+2,n+1})$, we have

$$Y_{\psi_F,\beta,\mathfrak{w}} = 2^{-1}(X_{-\alpha_1} + \cdots + X_{-\alpha_{n-1}}) + 2^{-2}X_{-\alpha_n} = \mathrm{Ad}(t_0)\left(\sum_{i=1}^{n} X_{-\alpha_i}\right) \in O_\infty,$$

where

$$t_0 = \mathrm{diag}(2^{n+1}, 2^n, \dots, 2^2, 1, 2^{-2}, \dots, 2^{-n}, 2^{-n-1}) \in T.$$

Hence β satisfies the condition (‡). Since

$$\beta(X, Y) = 2(X_1Y_1 + \cdots + X_nY_n)$$

for $X, Y \in \mathfrak{t}$, we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\psi_F)^n.$$

The case of even special orthogonal groups: Suppose that $G = \mathrm{SO}_{2n}^{\eta}(F)$, so that $\mathfrak{t}$ is equal to

$$\begin{cases} \{\mathrm{diag}(X_1, \dots, X_n, -X_n, \dots, -X_1) \,|\, X_1, \dots, X_n \in F\} & \text{if } \eta = \mathbf{1}, \\ \{\mathrm{diag}(X_1, \dots, X_n, -X_n, \dots, -X_1) \,|\, X_1, \dots, X_{n-1} \in F,\ X_n \in K_0\} & \text{otherwise,} \end{cases}$$

where K is the quadratic extension of F associated to η by the local class field theory and K_0 is the set of trace zero elements in K. Take the non-degenerate G-invariant symmetric bilinear form β on $\mathfrak{g}$ given by $\beta(X, Y) = \mathrm{tr}(XY)$. Noting that $X_{-\alpha_i} = E_{i+1,i} - E_{2n+1-i,2n-i}$ for $1 \le i \le n-1$ and $X_{-\alpha_n} = E_{n+1,n-1} - E_{n+2,n}$, we have

$$Y_{\psi_F,\beta,\mathfrak{w}} = 2^{-1}(X_{-\alpha_1} + \cdots + X_{-\alpha_n}) = \mathrm{Ad}(t_0)\left(\sum_{i=1}^{n} X_{-\alpha_i}\right) \in O_\infty,$$

where

$$t_0 = \mathrm{diag}(2^{n-1}, 2^{n-2}, \dots, 1, 1, \dots, 2^{-n+2}, 2^{-n+1}) \in T.$$

Hence β satisfies the condition (‡). Since

$$\beta(X, Y) = \begin{cases} 2(X_1Y_1 + \cdots + X_nY_n) & \text{if } \eta = \mathbf{1}, \\ 2(X_1Y_1 + \cdots + X_{n-1}Y_{n-1}) + \mathrm{tr}_{K/F}(X_nY_n) & \text{otherwise} \end{cases}$$

for $X, Y \in \mathfrak{t}$, we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\psi_F)^{n-1}\gamma(a\psi_F),$$

where we write $\eta = \eta_a$ with $a \in F^\times$ so that $K = F(\sqrt{a})$. Note that $K_0 = F\sqrt{a}$.

The case of unitary groups: Suppose that $G = \mathrm{U}_n$, so that

$$\mathfrak{t} = \begin{cases} \{\mathrm{diag}(X_1, \dots, X_r, -\overline{X_r}, \dots, -\overline{X_1}) \,|\, X_1, \dots, X_r \in E\}, \\ \{\mathrm{diag}(X_1, \dots, X_{r+1}, -\overline{X_r}, \dots, -\overline{X_1}) \,|\, X_1, \dots, X_r \in E,\ X_{r+1} \in E_0\} \end{cases}$$

according to $n = 2r$ or $n = 2r+1$, where E_0 is the set of trace zero elements in E. Write $E = F(\sqrt{a})$ with $a \in F^\times$ and put $\eta = \eta_a$. Take the non-degenerate G-invariant symmetric bilinear form β on $\mathfrak{g}$ given by $\beta(X, Y) = \frac{1}{2}\mathrm{tr}_{E/F}(\mathrm{tr}(XY))$. Noting that $X_{-\alpha_i} = E_{i+1,i}$ for $1 \le i \le n-1$ and that

$$u_1 X_{\alpha_1} + \cdots + u_{n-1} X_{\alpha_{n-1}} \in \mathfrak{u} \iff \overline{u_i} = u_{n-i} \quad (1 \le i \le n-1)$$

for $u_1, \dots, u_{n-1} \in E$, we have

$$Y_{\psi_F,\beta,\mathfrak{w}} = \sum_{i=1}^{n-1} X_{-\alpha_i} \in O_\infty.$$

Hence β satisfies the condition (‡). Since

$$\beta(X,Y) = \begin{cases} \mathrm{tr}_{E/F}(X_1Y_1 + \cdots + X_rY_r) & \text{if } n = 2r, \\ \mathrm{tr}_{E/F}(X_1Y_1 + \cdots + X_rY_r + (1/2)X_{r+1}Y_{r+1}) & \text{if } n = 2r+1 \end{cases}$$

for $X, Y \in \mathfrak{t}$, we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \begin{cases} \gamma(\psi_F)^r \gamma(a\psi_F)^r & \text{if } n = 2r, \\ \gamma(\psi_F)^r \gamma(a\psi_F)^r \gamma((a/2)\psi_F) & \text{if } n = 2r+1. \end{cases}$$

The computation of the Weil indices for the various classical groups is now complete. We turn to the proof of the equation

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F)}{\varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F)} = 1.$$

Recall that G' is of the form

$$G' = \begin{cases} \mathrm{Sp}_{2n_1}(F) \times \mathrm{SO}^{\eta}_{2n_2}(F) & \text{if } G = \mathrm{Sp}_{2n}(F), \\ \mathrm{SO}_{2n_1+1}(F) \times \mathrm{SO}_{2n_2+1}(F) & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \mathrm{SO}^{\eta_1}_{2n_1}(F) \times \mathrm{SO}^{\eta_2}_{2n_2}(F) & \text{if } G = \mathrm{SO}^{\eta}_{2n}(F), \\ \mathrm{U}_{n_1} \times \mathrm{U}_{n_2} & \text{if } G = \mathrm{U}_n \end{cases}$$

with $n_1 + n_2 = n$ and $\eta_1\eta_2 = \eta$ (see e.g., [W5, Section 1.8]). Recall also that β' is the transfer of β to $\mathfrak{g}'$ according to [W1, Section VIII.6]. Explicitly, if we write $G' = G_1 \times G_2$, then we have

$$\beta' = \beta_1 \oplus \beta_2,$$

where β_i is the bilinear form on $\mathfrak{g}_i$ as above.

The case of symplectic groups: Suppose that $G = \mathrm{Sp}_{2n}(F)$ and $G' = \mathrm{Sp}_{2n_1}(F) \times \mathrm{O}^{\eta}_{2n_2}(F)$ with $\eta = \eta_a$. Then we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\psi_F)^n, \quad \gamma(\mathfrak{g}', \beta', \psi_F) = \gamma(\psi_F)^{n-1}\gamma(a\psi_F)$$

and

$$\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F) = 1, \quad \varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, \eta_a, \psi_F).$$

Hence

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F)}{\varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F)} = \frac{\gamma(\psi_F)}{\gamma(a\psi_F)} \cdot \frac{1}{\varepsilon(1/2, \eta_a, \psi_F)} = 1.$$

The case of odd special orthogonal groups: Suppose that $G = \mathrm{SO}_{2n+1}(F)$ and $G' = \mathrm{SO}_{2n_1+1}(F) \times \mathrm{SO}_{2n_2+1}(F)$. Then we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\mathfrak{g}', \beta', \psi_F) = \gamma(\psi_F)^n$$

and

$$\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F) = 1.$$

Hence

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F)}{\varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F)} = 1.$$

The case of even special orthogonal groups: Suppose that $G = \mathrm{O}_{2n}^{\eta}(F)$ and $G' = \mathrm{O}_{2n_1}^{\eta_1}(F) \times \mathrm{O}_{2n_2}^{\eta_2}(F)$ with $\eta = \eta_a$ and $\eta_i = \eta_{a_i}$. Then we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\psi_F)^{n-1}\gamma(a\psi_F), \quad \gamma(\mathfrak{g}', \beta', \psi_F) = \gamma(\psi_F)^{n-2}\gamma(a_1\psi_F)\gamma(a_2\psi_F)$$

and

$$\begin{aligned} \varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F) &= \varepsilon(1/2, \eta, \psi_F), \\ \varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F) &= \varepsilon(1/2, \eta_1, \psi_F)\varepsilon(1/2, \eta_2, \psi_F). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F)}{\varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F)} \\ &= \frac{\gamma(\psi_F)\gamma(a\psi_F)}{\gamma(a_1\psi_F)\gamma(a_2\psi_F)} \cdot \frac{\varepsilon(1/2, \eta, \psi_F)}{\varepsilon(1/2, \eta_1, \psi_F)\varepsilon(1/2, \eta_2, \psi_F)} = 1. \end{aligned}$$

The case of odd unitary groups: Suppose that $G = \mathrm{U}_n$ and $G' = \mathrm{U}_{n_1} \times \mathrm{U}_{n_2}$ with $n = 2r + 1$. Then we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\mathfrak{g}', \beta', \psi_F) = \gamma(\psi_F)^r\gamma(a\psi_F)^r\gamma((a/2)\psi_F)$$

and

$$\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, \eta, \psi_F)^{r+1}.$$

Hence

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F)}{\varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F)} = 1.$$

The case of even unitary groups: Suppose that $G = \mathrm{U}_n$ and $G' = \mathrm{U}_{n_1} \times \mathrm{U}_{n_2}$ with $n = 2r$. If n_1 and n_2 are even, then we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\mathfrak{g}', \beta', \psi_F) = \gamma(\psi_F)^r\gamma(a\psi_F)^r$$

and

$$\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, \eta, \psi_F)^r.$$

Hence

$$\frac{\gamma(\mathfrak{g}, \beta, \psi_F)}{\gamma(\mathfrak{g}', \beta', \psi_F)} \cdot \frac{\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F)}{\varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F)} = 1.$$

If n_1 and n_2 are odd, then we have

$$\gamma(\mathfrak{g}, \beta, \psi_F) = \gamma(\psi_F)^r\gamma(a\psi_F)^r, \quad \gamma(\mathfrak{g}', \beta', \psi_F) = \gamma(\psi_F)^{r-1}\gamma(a\psi_F)^{r-1}\gamma((a/2)\psi_F)^2$$

and

$$\varepsilon(1/2, X^*(T)_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, \eta, \psi_F)^r, \quad \varepsilon(1/2, X^*(T')_{\mathbb{C}}, \psi_F) = \varepsilon(1/2, \eta, \psi_F)^{r+1}.$$

Hence

$$\begin{aligned}\frac{\gamma(\mathfrak{g},\beta,\psi_F)}{\gamma(\mathfrak{g}',\beta',\psi_F)}\cdot\frac{\varepsilon(1/2,X^*(T)_{\mathbb{C}},\psi_F)}{\varepsilon(1/2,X^*(T')_{\mathbb{C}},\psi_F)} &= \frac{\gamma(\psi_F)\gamma(a\psi_F)}{\gamma((a/2)\psi_F)^2}\cdot\frac{1}{\varepsilon(1/2,\eta,\psi_F)}\\ &= \frac{\varepsilon(1/2,\eta_{a/2},\psi_F)^2}{\varepsilon(1/2,\eta,\psi_F)^2} = \frac{\eta_{a/2}(-1)}{\eta(-1)}\\ &= \frac{(a/2,-1)_F}{(a,-1)_F} = (2,-1)_F = 1.\end{aligned}$$

Here, we use the fact that $(x, 1-x)_F = 1$ for $x \neq 0, 1$.

This completes the proof of Lemma D.3.2 for classical groups by explicit computation.

Appendix E. Endoscopic character relations for the archimedean case

In this appendix, we will argue that [Ar2, Theorem 2.2.1(a)] holds for $F = \mathbb{R}$ and tempered parameters $\psi = \phi$, and [Ar2, Theorem 2.2.4] holds for $F = \mathbb{R}$ and discrete parameters $\psi = \phi$. The validity of these theorems is assumed at various places in [Ar2] with the remark that they will follow from a forthcoming work of Shelstad and Mezo. In the meantime, the work of Shelstad [She2] has appeared, which proves the transfer of functions in twisted endoscopy for $F = \mathbb{R}$, and the works of Mezo [Mez1, Mez2], which prove a weaker version of the desired theorems: the character identities are shown to hold up to a scalar.

In this appendix, we will use different sources. The forthcoming work [KM] treats a general class of disconnected real reductive groups and L-parameters which are discrete for the identity component; this will suffice for the purposes of [Ar2, Theorem 2.2.4], because the general tempered case can be reduced to the discrete case by a global argument. However, this is insufficient for the purposes of [Ar2, Theorem 2.2.1(a)], because the group GL_N has no discrete parameters when $F = \mathbb{R}$ and $N > 2$. Here we will build on [AMR] and [Cl]. This will again involve a global reduction argument, but now to the case that ϕ is discrete for the endoscopic group. It will further involve a local argument that allows for a variation of the L-homomorphism.

E.1. **Theorem 2.2.4 for archimedean discrete parameters.** We briefly review the results of [KM]. One considers rigid inner forms of groups of the form $G^+ = G \rtimes A$, where G is a quasi-split connected reductive $\mathbb{R}$-group and A is a finite group of automorphisms of G that preserve a pinning. (These groups were denoted by $\widetilde{G}$ in loc. cit., but here we are using the notation G^+ of Arthur.) In the case at hand $G = \mathrm{SO}_{2n}$, $A = \mathbb{Z}/2\mathbb{Z}$ acting by the unique pinned outer automorphism, and we are interested in the trivial rigid inner form, i.e. the group $G^+ = G \rtimes A$ itself.

It is shown in loc. cit. that to each discrete parameter $\phi\colon W_{\mathbb{R}} \to {}^LG$ one can associate an L-packet $\Pi_\phi(G^+)$ of irreducible discrete series representations of $G^+(\mathbb{R})$ and an injection of this L-packet into $\mathrm{Irr}(S_\phi^+)$, where S_ϕ^+ is the centralizer of ϕ in $\widehat{G} \rtimes A = \mathrm{SO}_{2n}(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$, where again A acts by preserving a pinning of $\widehat{G}$. Note that we are using the superscript $+$ here again in the sense of Arthur, and not in the sense of

[Kal3]. This injection is normalized using a Whittaker datum $\mathfrak{w}$ that is A-admissible, i.e. it is a $G(\mathbb{R})$-conjugacy class of pairs (B, ψ) that contains an A-stable such pair; such Whittaker data always exist.

When the $\widehat{G}$-conjugacy class of ϕ is not stable under A, then $S_\phi^+ = S_\phi$ is the centralizer of ϕ in $\widehat{G}$. The setting of [Ar2, Theorem 2.2.4] is the opposite, namely the $\widehat{G}$-conjugacy class of ϕ is stable under A, in which case S_ϕ is a normal subgroup of index 2 in S_ϕ^+, with quotient $A = \mathbb{Z}/2\mathbb{Z}$.

For any $s \in S_\phi^+$ one can associate an endoscopic datum $(G', \mathcal{G}', s, \eta)$, which is twisted when $s \notin S_\phi$. The following character identity is proved in loc. cit.:

(ECR)
$$S\Theta_{\phi'}(f') = \sum_{\pi \in \Pi_\phi(G)} \langle s, \widetilde{\pi} \rangle \Theta_{\widetilde{\pi}}(f),$$

where the notation is as follows:

(1) f is a smooth compactly supported function on $G(\mathbb{R}) = \mathrm{SO}_{2n}(\mathbb{R})$, which we interpret as a function on $G^+(\mathbb{R})$ that is supported on the *non-identity* coset, by means of the injection $G(\mathbb{R}) \to G^+(\mathbb{R})$ sending g to $g\theta$, with $\theta \in A = \mathbb{Z}/2\mathbb{Z}$ the non-identity element.
(2) f' is the transfer of f to the twisted endoscopic group G' in terms of the Kottwitz–Shelstad transfer factor normalized by the fixed A-admissible Whittaker datum $\mathfrak{w}$.
(3) ϕ' is the factorization of the parameter ϕ through $\eta\colon {}^LG' \to {}^LG$.
(4) $S\Theta$ is the stable character of the L-packet $\Pi_{\phi'}(G')$ of G'.
(5) $\widetilde{\pi}$ is an arbitrary extension to $G^+(\mathbb{R})$ of the representation π. We will argue below that such an extension always exists in the special case we are considering.
(6) $\Theta_{\widetilde{\pi}}$ is the distribution character of $\widetilde{\pi}$. It is shown in loc. cit. that the product $\langle s, \widetilde{\pi} \rangle \Theta_{\widetilde{\pi}}(f)$ does not depend on the choice of extension $\widetilde{\pi}$ of π.

Identity (**ECR**) is the desired identity (2.2.17) in [Ar2, Theorem 2.2.4], albeit in different notation. This proves part (a) of that theorem for discrete parameters over $\mathbb{R}$.

Part (b) is the assertion that we made in (5) above that every π has an extension $\widetilde{\pi}$. It is shown in loc. cit., in the general setting of disconnected groups discussed there, that if $\pi \in \Pi_\phi(G)$ corresponds to $\rho \in \mathrm{Irr}(S_\phi)$, then the extensions of π to $G^+(\mathbb{R})$ are in natural bijection with the extensions of ρ to S_ϕ^+. Thus, it is enough to show that in the particular case of $G^+ = \mathrm{SO}_{2n} \rtimes \mathbb{Z}/2\mathbb{Z}$ any $\rho \in \mathrm{Irr}(S_\phi)$ has an extension to S_ϕ^+.

To see this we note that $\widehat{G} \rtimes \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathrm{O}_{2n}(\mathbb{C})$. Therefore, the centralizer S_ϕ^+ is isomorphic to

$$\prod_i \mathrm{O}(V_i) \times \prod_i \mathrm{Sp}(W_i)$$

and S_ϕ is the subgroup of index 2 on which the product of the determinants of the individual factors is trivial. The representation ρ kills the identity component of S_ϕ, which is also the identity component of S_ϕ^+. We are therefore looking for an extension of ρ from $\pi_0(S_\phi)$ to $\pi_0(S_\phi^+)$. But the latter group is visibly abelian, in fact a 2-group, so such an extension always exists.

E.2. **Theorem 2.2.1(a) for archimedean tempered parameters.** In the remainder of this appendix we will show that the twisted character identities in [Ar2, Theorem 2.2.1 (a)] and their unitary group analogue over $\mathbb{R}$ for tempered representations are valid. Note that [KM] treats general disconnected real groups, but only those tempered parameters that are discrete for the identity component, and this is not sufficient for our purposes. Mezo has treated in [Mez2] general twisted real groups and general tempered parameters, but the desired identity is proved there only up to a scalar. We need to know that this scalar factor is equal to 1. This is done in [AMR, Appendix A] (resp. in [Cl]) for $G = \mathrm{Sp}_{2n}$ and $G = \mathrm{SO}_n$ (resp. for $G = \mathrm{U}_n$), but only for special L-embeddings ${}^LG \to \mathrm{GL}_N$ and only for L-parameters that are discrete for G. We will reduce the general case to that core case.

E.3. **Reductions.** For convenience we adopt the notation of [Ar2] and make references to there. (These are closely followed in [Mok] with minor differences.)

We can first reduce the identities to the case that G is a simple twisted endoscopic group. Indeed, if $G = G_S \times G_O$ is not simple, then by [Ar2, Proposition 6.6.1], we may assume $\phi = \phi_S \times \phi_O \in \widetilde{\Phi}_2(G)$ is square-integrable. (If G possesses no square-integrable parameters, then there is nothing to prove.) There exist stable linear forms

$$f^S(\phi_S),\ f^O(\phi_O), \quad f^S \in \widetilde{\mathcal{H}}(G_S),\ f^O \in \widetilde{\mathcal{H}}(G_O)$$

either by the induction hypothesis or by well-known results in real endoscopy. (These are the stable characters associated with the discrete series L-packets for ϕ_S and ϕ_O.) We define a stable linear form $f(\phi)$ by the formula

$$f(\phi) = f^S(\phi_S) f^O(\phi_O), \quad f = f^S \times f^O.$$

Mezo [Mez2] has shown that the twisted character $\tilde{f}_N(\phi)$ (defined in [Ar2, (2.2.1)]) on $\tilde{f} \in \widetilde{\mathcal{H}}(N)$ satisfies the following twisted character identity up to a scalar $c(\phi) \in \mathbb{C}^\times$, where $\tilde{f}^G$ denotes a transfer of $\tilde{f}$:

$$\tilde{f}_N(\phi) = c(\phi) \tilde{f}^G(\phi).$$

Lemma E.3.1. *In the above setting, $c(\phi) = 1$.*

Remark E.3.2. This lemma was assumed in the proof of [Ar2, Lemma 6.6.3] and [Mok, Proposition 7.7.1] when the authors write “assumed as part of the theory of twisted endoscopy” or “by the results of Mezo and Shelstad”. Unfortunately the current state of twisted endoscopy for real groups is not enough to imply the lemma as a special case. So we give a global proof in much the same way as Arthur treats p-adic places.

Proof of Lemma E.3.1. Let $d > 0$ be a sufficiently large integer; this number controls the number of archimedean places. (Arthur wants d to be large in the globalization of [Ar2, Sections 6.2–6.3]. For our purpose $d = 2$ is enough because we do not need [Ar2, Proposition 6.3.1 (iii)]. We do need $d > 1$ to be able to appeal to the simple trace formula.)

By [Ar2, Proposition 6.3.1] (disregarding condition (iii) there) we have the following globalization for each of $i \in \{0,1\}$:

- a totally real field $\dot{F}_i$ such that $[\dot{F}_i : \mathbb{Q}] = d + i$,
- a real place u_i of $\dot{F}_i$,
- a twisted endoscopic datum $\dot{G}_i \in \dot{\widetilde{\mathcal{E}}}_{\mathrm{ell}}(N)$ over $\dot{F}_i$,
- a parameter $\dot{\phi}_i \in \widetilde{\Phi}_2^{\mathrm{sim}}(\dot{G}_i)$,

such that $(\dot{F}_i, \dot{G}_i, \dot{\phi}_i)$ specializes to (F, G, ϕ) at the place u_i, and such that

- $\dot{G}_i$ possesses discrete series at all $v \in S_{i,\infty}^{u_i}$ (in addition to $v = u_i$),
- $\dot{\phi}_{i,v}$ is square-integrable and in relative general position at all $v \in S_{i,\infty}^{u_i}$,

where $S_{i,\infty}$ stands for the set of real places of $\dot{F}_i$, and we put $S_{i,\infty}^{u_i} = S_{i,\infty} \backslash \{u_i\}$. (The degree $[\dot{F}_i : \mathbb{Q}]$ is not prescribed in *loc. cit.* but the existence argument there works for a fixed choice of $\dot{F}_i$ and u_i.) In fact the real groups

$$\dot{G}_{i,v} \quad (i \in \{0,1\},\, v \in S_{i,\infty})$$

are all isomorphic to G (canonically up to inner automorphism) as they are quasi-split real forms accommodating discrete series. We fix such isomorphisms.

Now the point is that, in addition to the above, we can arrange that the real components of $\dot{\phi}_0$ and $\dot{\phi}_1$ away from u_0, u_1 are all equal, i.e.,

$$(\flat) \qquad \dot{\phi}_{i,v} = \phi_{\mathrm{gen}} \quad (i \in \{0,1\},\, v \in S_{i,\infty}^{u_i})$$

for some $\phi_{\mathrm{gen}} \in \widetilde{\Phi}_2(G)$ in general position via $\dot{G}_{i,v} \cong G$. This is possible since Arthur can prescribe the parameters at $v \in S_{i,\infty}^{u_i}$ quite flexibly. More precisely, for $i = 0, 1$ and $v \in S_{i,\infty}^{u_i}$, his argument fixes a regular infinitesimal character $\mu_{i,v}$ (of a discrete series representation) and shows that there exists a global parameter $\dot{\phi}_i$ such that $\dot{\phi}_{i,v}$ has infinitesimal character $n\mu_{i,v}$ and trivial central character, as long as $n \in \mathbb{Z}_{>0}$ is sufficiently large. Thus we can achieve ($\flat$) by starting from the same $\mu_{i,v}$ for all i, v as in ($\flat$) and then choosing n large. (We use the same n for all i, v.)

The rest of the argument proceeds as in the proof of [Ar2, Lemma 6.6.3], with u a real place rather than a finite place. Namely we apply [Ar2, Lemma 5.4.2] (applicable since [Ar2, Assumption 5.4.1 (b)] therein is satisfied at all archimedean places) to deduce that

$$\dot{\tilde{f}}_N(\dot{\phi}_i) = \dot{\tilde{f}}^G(\dot{\phi}_i), \quad \dot{\tilde{f}} = \prod_v \dot{\tilde{f}}_v \in \widetilde{\mathcal{H}}(N)_{\dot{F}_i}.$$

Here the subscript $\dot{F}_i$ is there to remind us that the Hecke algebra is for the twisted general linear group over $\dot{F}_i$. Condition (ii) of [Ar2, Proposition 6.3.1] allows us to cancel out the terms at all finite places from both sides, as in the proof of [Ar2, Lemma

6.6.3]. Hence

$$\prod_{v\in S_{i,\infty}} \dot{\tilde{f}}_{N,v}(\dot{\phi}_{i,v}) = \prod_{v\in S_{i,\infty}} \dot{\tilde{f}}_v^G(\dot{\phi}_{i,v}).$$

In light of the equation $\tilde{f}_N(\phi) = c(\phi)\tilde{f}^G(\phi)$, this implies that

$$c(\phi)c(\phi_{\text{gen}})^{d-1+i} = 1.$$

Since this holds for both $i \in \{0,1\}$, it follows that $c(\phi) = c(\phi_{\text{gen}}) = 1$, as desired. □

Now we may assume that G is simple. We further can reduce to the case that the parameter ϕ is square-integrable. Indeed, if ϕ is tempered but not square-integrable, [Ar2, Theorem 2.2.1] follows as explained at the beginning of Section 6.6 in loc. cit.

Now consider the core case that G is a simple endoscopic group of twisted GL_N and ϕ is a square-integrable parameter. If the endoscopic datum is standard (i.e. the L-embedding ${}^LG \to \mathrm{GL}_N$ is standard), the result is covered by [AMR, Appendix A] for $\mathrm{Sp}_{2n}, \mathrm{SO}_n$ and [Cl] for U_n. This reduces the problem to Proposition E.4.1 below, which may be of independent interest, so we will prove it for any local field.

E.4. **Set-up and statement of the result.** Let F be a local field, G a quasi-split connected reductive F-group, $(T, B, \{X_\alpha\})$ a pinning, θ an F-automorphism of G preserving the pinning, and $\psi_F\colon F \to \mathbb{C}^\times$ a non-trivial character. Let $(H, \mathcal{H}, 1, \xi)$ be the principal endoscopic datum for (G, θ). We assume that there exists an L-isomorphism ${}^LH \to \mathcal{H}$ and write ${}^L\xi\colon {}^LH \to {}^LG$ for the composition of this isomorphism with ξ. The pinning and the character ψ_F lead to a Whittaker datum, which we use to normalize the transfer factor.

Let $\phi\colon L_F \to {}^LH$ be a tempered parameter. The expected twisted character identity is then as follows:

$$\sum_{\substack{\pi\in\Pi_{{}^L\xi\circ\phi}(G)\\ \pi\circ\theta\cong\pi}} \mathrm{tr}(\widetilde{\pi}(\widetilde{f})) = \sum_{\sigma\in\Pi_\phi(H)} \mathrm{tr}(\sigma(\widetilde{f}^H)),$$

where $\widetilde{\pi}$ is the Whittaker normalized extension of π to $G(F) \rtimes \langle\theta\rangle$, $\widetilde{f} \in C_c^\infty(G(F) \rtimes \theta)$, and $\widetilde{f}^H \in C_c^\infty(H(F))$ is its transfer.

This identity is stated in the language of distributions, but can also be restated in the language of functions as

$$\sum_{\substack{\pi\in\Pi_{{}^L\xi\circ\phi}(G)\\ \pi\circ\theta\cong\pi}} \mathrm{tr}(\widetilde{\pi}(\widetilde{\delta})) = \sum_{\gamma} \Delta[{}^L\xi](\gamma, \widetilde{\delta}) \sum_{\sigma\in\Pi_\phi(H)} \mathrm{tr}(\sigma(\gamma)),$$

where $\widetilde{\delta} \in G(F) \rtimes \theta$ is regular semi-simple, γ runs over the set of regular semi-simple elements of $H(F)$ up to stable conjugacy, and $\Delta[{}^L\xi](\gamma, \widetilde{\delta})$ is the transfer factor relative to the L-embedding ${}^L\xi$ and normalized by the Whittaker datum.

Proposition E.4.1. *Assume that the twisted character identity holds for one choice of embedding ${}^L\xi\colon {}^LH \to {}^LG$. Then it holds for any other choice of ${}^L\xi$ with the same restriction $\widehat{H}$.*

Remark E.4.2. One can contemplate various generalizations of this statement. For example, the proof given below easily generalizes to arbitrary endoscopic groups in the ordinary setting, i.e. when $\theta = 1$. It also generalizes to inner forms of (G, θ). What is not so clear to us is how to generalize it to arbitrary endoscopic groups when θ is non-trivial, but we do not need this case. Also, we will only give the proof when the derived subgroup is simply connected.

E.5. **Proof.** To simplify matters we will give the proof under the assumption that the derived subgroup G_{der} is simply connected.

First, we study the possible variations of ${}^L\xi$. Since ${}^L\xi|_{\widehat{H}}$ has been fixed, we use it to identify $\widehat{H}$ with a subgroup of $\widehat{G}$ to save notation. We have $\widehat{H} = (\widehat{G}^\theta)^\circ$.

Lemma E.5.1. *We have* $Z_{\widehat{G}}(\widehat{H}) = Z(\widehat{G})$.

Proof. Let $\widehat{S} \subset \widehat{H}$ be a maximal torus. Then since $\widehat{T} = Z_{\widehat{G}}(\widehat{S})$ is a θ-stable maximal torus of $\widehat{G}$, it is contained in a θ-stable maximal Borel subgroup $\widehat{B}$, and $\widehat{S} = (\widehat{T}^\theta)^\circ$. The root system $R(\widehat{S}, \widehat{H})$ is the set of indivisible roots in the relative root system $R(\widehat{S}, \widehat{G})$. The latter is the set of restrictions to $\widehat{S}$ of the absolute root system $R(\widehat{T}, \widehat{G})$. No such restriction vanishes, and the restriction map $R(\widehat{T}, \widehat{G}) \to R(\widehat{S}, \widehat{G})$ is surjective with fibers being the θ-orbits.

An element of $Z_{\widehat{G}}(\widehat{H})$ centralizes $\widehat{S}$, hence lies in $\widehat{T}$. Moreover, it acts trivially on each root space of $\widehat{S}$ in $\mathrm{Lie}(\widehat{H})$. For $\beta \in R(\widehat{S}, \widehat{H})$, the root space $\mathrm{Lie}(\widehat{H})_\beta$ is the space of θ-fixed points in $\bigoplus_{\alpha \mapsto \beta} \mathrm{Lie}(\widehat{G})_\alpha$. Therefore, $t \in \widehat{T}$ fixes this root space if and only if $\alpha(t) = 1$ for all $\alpha \mapsto \beta$. Since all $\widehat{B}$-simple roots α map to an indivisible root in $R(\widehat{S}, \widehat{H})$, we see that for an element $t \in \widehat{T}$ to centralize $\mathrm{Lie}(\widehat{H})$ it is necessary and sufficient that $\alpha(t) = 1$ for all $\widehat{B}$-simple roots in $R(\widehat{T}, \widehat{G})$, which is equivalent to $t \in Z(\widehat{G})$. $\square$

Lemma E.5.2. *If* ${}^L\xi \colon {}^LH \to {}^LG$ *is one choice of L-embedding, then any other choice is of the form* $\alpha \cdot {}^L\xi$ *for some* $\alpha \in Z^1(W_F, Z(\widehat{G}))$ *whose cohomology class is* θ*-fixed.*

Proof. The actions of $\sigma \in W_F$ on $\widehat{H}$ and $\widehat{G}$ are generally different, and we will write σ_H and σ_G for them. Then $\sigma_H = \mathrm{Ad}(g_\sigma) \rtimes \sigma_G$ for some $g_\sigma \in \widehat{G}$ that is well-defined up to multiplication on the left by $Z_{\widehat{G}}(\widehat{H})$. By above lemma we have $Z_{\widehat{G}}(\widehat{H}) = Z(\widehat{G})$.

Since, for any ${}^L\xi$ and $\sigma \in W_F$, the element ${}^L\xi(1 \rtimes \sigma) \in {}^LG$ normalizes $\widehat{H}$ and acts on it by σ_H, we see that if one ${}^L\xi$ is fixed, and other choice is of the form $\alpha \cdot {}^L\xi$ for a continuous map $\alpha \colon W_F \to Z(\widehat{G})$.

The multiplicativity of ${}^L\xi$ and $\alpha \cdot {}^L\xi$ implies that α is a 1-cocycle. By assumption there exist $x, y \in Z(\widehat{G})$ such that $\theta \circ {}^L\xi = x \cdot {}^L\xi \cdot x^{-1}$ and $\theta \circ (\alpha \cdot {}^L\xi) = y \cdot (\alpha \cdot {}^L\xi) \cdot y^{-1}$. It follows that $\alpha^{-1}(\sigma) \cdot \theta \circ \alpha(\sigma) = (xy^{-1})^{-1}\sigma(xy^{-1})$, i.e. $[\alpha] \in H^1(W_F, Z(\widehat{G}))^\theta$. $\square$

Next, we reduce Proposition E.4.1 to a property of the transfer factor. We now fix ${}^L\xi$ for which the character identity

$$\sum_{\substack{\pi\in\Pi_{{}^L\xi\circ\phi}(G)\\ \pi\circ\theta\cong\pi}} \mathrm{tr}(\widetilde{\pi}(\widetilde{\delta})) = \sum_{\gamma} \Delta[{}^L\xi](\gamma,\widetilde{\delta}) \sum_{\sigma\in\Pi_\phi(H)} \mathrm{tr}(\sigma(\gamma))$$

holds. If we replace ${}^L\xi$ by $\alpha\cdot{}^L\xi$ for some $\alpha\in Z^1(W_F,Z(\widehat{G}))$ whose class is θ-fixed, then on the left-hand side of the identity, the packet $\Pi_{{}^L\xi\circ\phi}$ changes to $\Pi_{\alpha\cdot{}^L\xi\circ\phi}$. If $\chi_\alpha\colon G(F)\to\mathbb{C}^\times$ is the character corresponding to α, then tensoring representations with χ_α provides a bijection

$$\Pi_{{}^L\xi\circ\phi}\to\Pi_{\alpha\cdot{}^L\xi\circ\phi}.$$

Moreover, the Whittaker extension of $\chi_\alpha\otimes\pi$ equals $\widetilde{\chi}_\alpha\otimes\widetilde{\pi}$, where $\widetilde{\pi}$ is the Whittaker extension of π, and $\widetilde{\chi}_\alpha$ is the extension of χ_α to $G(F)\rtimes\langle\theta\rangle$ specified by $\widetilde{\chi}_\alpha(\theta)=1$ (since the class of α is θ-fixed, the character χ_α is θ-invariant). Therefore, upon replacing ${}^L\xi$ by $\alpha\cdot{}^L\xi$, the left-hand side of the character identity multiplies by $\widetilde{\chi}_\alpha(\widetilde{\delta})$. Hence to prove Proposition E.4.1, it would be enough to show

$$\Delta[\alpha\cdot{}^L\xi](\gamma,\widetilde{\delta})=\widetilde{\chi}_\alpha(\widetilde{\delta})\cdot\Delta[{}^L\xi](\gamma,\widetilde{\delta}).$$

Finally, we shall show this property. The only piece of the transfer factor that depends on the choice of L-embedding is Δ_{III}. Let us briefly recall its construction following [KoSh1, Section 5.3]. There exists a θ-admissible maximal F-torus $T\subset G$ together with an element $g\in G_{sc}(\overline{F})$ such that $\widetilde{\delta}^*=g^{-1}\widetilde{\delta}g\in T(\overline{F})\rtimes\theta$, and if we write $\widetilde{\delta}^*=\delta^*\rtimes\theta$ then the image $\gamma^*\in T_\theta(\overline{F})$ of δ^* lies in $T_\theta(F)$. Writing $z_\sigma=g^{-1}\sigma(g)$ we have $(z_\sigma^{-1},\delta^*)\in Z^1(F,T_{sc}\xrightarrow{1-\theta}T)$.

Let $\gamma\in H(F)$ be related to $\widetilde{\delta}$ and let $S\subset H$ be the centralizer of γ. There exists a unique admissible isomorphism $\xi_{\gamma,\gamma^*}\colon S\to T_\theta$ mapping γ to γ^*. Choosing χ-data for $R(T^\theta,G)$ provides L-embeddings ${}^L\xi_{S,H}\colon{}^LS\to{}^LH$ and ${}^L\xi_{T,G}\colon{}^LT\to{}^LG$. There is a unique 1-cocycle $\eta\colon W_F\to\widehat{T}$ that makes the following diagram commute

$$\begin{array}{ccc} {}^LS & \xrightarrow{{}^L\xi_{S,H}} & {}^LH \\ {\scriptstyle \eta\cdot{}^L\xi_{\gamma,\gamma^*}}\downarrow & & \downarrow{\scriptstyle {}^L\xi} \\ {}^LT & \xrightarrow[{}^L\xi_{T,G}]{} & {}^LG. \end{array}$$

Here ${}^L\xi_{\gamma,\gamma^*}$ is the L-embedding obtained from the L-isomorphism $\widehat{S}\rtimes W_F\to(\widehat{T}^\theta)^\circ\rtimes W_F$ dual to the inverse of ξ_{γ,γ^*} and the inclusion $(\widehat{T}^\theta)^\circ\to\widehat{T}$, and we have multiplied it with the 1-cocycle η, composed with the projection ${}^LS\to W_F$ and the inclusion $\widehat{T}\to{}^LT$. Then $(\eta^{-1},1)\in Z^1(W_F,\widehat{T}\xrightarrow{1-\theta}\widehat{T}_{\mathrm{ad}})$. The factor Δ_{III} is the pairing of (z_σ^{-1},δ^*) and $(\eta^{-1},1)$.

If we replace ${}^L\xi$ by $\alpha\cdot{}^L\xi$ then η is replaced by $\alpha\cdot\eta$. We are using the natural inclusion $Z(\widehat{G})\to\widehat{T}$, which is equivariant under both Γ and θ, and induces an inclusion of complexes of Γ-modules $[Z(\widehat{G}) \to 1] \to [\widehat{T} \xrightarrow{1-\theta} \widehat{T}_{\mathrm{ad}}]$. The value of the transfer factor thus multiplies by the pairing of $(\alpha^{-1}, 1)$ with $(z_\sigma^{-1}, \delta^*)$. Now $(\alpha^{-1}, 1)$ is included from $[Z(\widehat{G}) \to 1]$. Using the functoriality of the Tate–Nakayama pairing and the fact that $Z(\widehat{G})$ is the torus dual to $D = G/G_{\mathrm{der}}$, we may compute the pairing of $(\alpha^{-1}, 1)$ and $(z_\sigma^{-1}, \delta^*)$ by mapping the latter under the map $[T_{sc} \xrightarrow{1-\theta} T] \to [1 \to D]$. Now $\delta^* \rtimes \theta = \widetilde{\delta}^* = g^{-1}\widetilde{\delta}g = (g^{-1}\widetilde{\delta}g\widetilde{\delta}^{-1}) \cdot \widetilde{\delta}$ and the term in the parentheses lies in G_{der}. Therefore we see that the images of $\delta^* \rtimes \theta$ and $\widetilde{\delta}$ in $D(\overline{F}) \rtimes \theta$ agree, and lie in $D(F) \rtimes \theta$. In particular, the image $\overline{\delta}$ of δ^* in D is an F-point. Since Δ_{III} enters the transfer factor with its reciprocal, we see that changing ${}^L\xi$ to $\alpha \cdot {}^L\xi$ multiplies the transfer factor by $\chi_\alpha(\overline{\delta})$, where χ_α is the character of $D(F)$ with parameter α. This character is θ-invariant and hence extends to a character $\widetilde{\chi}_\alpha$ of $D(F) \rtimes \langle\theta\rangle$ with $\widetilde{\chi}_\alpha(\theta) = 1$. Then $\chi_\alpha(\overline{\delta}) = \widetilde{\chi}_\alpha(\overline{\delta} \rtimes \theta) = \widetilde{\chi}_\alpha(\widetilde{\delta})$, where we have mapped $\widetilde{\delta}$ in $D(F) \rtimes \theta$ under the natural map $G(F) \rtimes \theta \to D(F) \rtimes \theta$. Since the character χ_α of $G(F)$ is simply the inflation to $G(F)$ of the character χ_α of $D(F)$, the desired identity

$$\Delta[\alpha \cdot {}^L\xi](\gamma, \widetilde{\delta}) = \widetilde{\chi}_\alpha(\widetilde{\delta}) \cdot \Delta[{}^L\xi](\gamma, \widetilde{\delta})$$

has been established.

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Department of Mathematics, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan
Email address: atobe@math.kyoto-u.ac.jp

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
Email address: matgwt@nus.edu.sg

Department of Mathematics, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan
Email address: ichino@math.kyoto-u.ac.jp

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, Mi 48109, USA
Email address: kaletha@umich.edu

University of Vienna, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Wien
Email address: alberto.minguez@univie.ac.at

Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA / Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
Email address: sug.woo.shin@berkeley.edu