ON MODULAR RIGIDITY FOR GL_n

by

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Abstract. — Let k be a global field and $\mathbb{A}_k$ be its ring of adèles. Let ℓ be a prime number and fix a field isomorphism from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$. Let Π_1, Π_2 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$ for some integer $n \geqslant 1$. In this paper, we study the following question: assuming that there is a finite set S of places of k containing all Archimedean places and all finite places above ℓ such that, for all $v \notin S$, the local components $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified and their Satake parameters are congruent mod ℓ, are the local components $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ integral, and do their reductions mod ℓ share an irreducible factor for all non-Archimedean places w not dividing ℓ? We show that, under certain conditions on Π_1, Π_2, the answer is yes. We also give a simple proof when k is a function field.

Keywords and Phrases: Automorphic forms, Congruences mod ℓ, Satake parameters, Whittaker models, Automorphic representations

1. Introduction

1.1. Let k be a number field and $\mathbb{A}_k$ be its ring of adèles. Let Π_1 and Π_2 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$ for some integer $n \geqslant 1$. The rigidity (or strong multiplicity 1) theorem asserts that, if there is a finite set S of places of k containing all Archimedean places such that, for all $v \notin S$, the local components $\Pi_{1,v}$ and $\Pi_{2,v}$ are unramified and have the same Satake parameter, then Π_1 and Π_2 are isomorphic ([**17, 3, 10, 11**]). A similar result holds over function fields.

1.2. Now fix a field isomorphism ι from $\mathbb{C}$ to an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of ℓ-adic numbers for some prime number ℓ, and consider the collections of irreducible smooth $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(k_v)$ defined by

$$\pi_{i,v} = \Pi_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad i \in \{1,2\}, \tag{1.1}$$

where the tensor product is taken with respect to ι, v runs over all finite places of k and k_v is the completion of k at v. As Π_1 and Π_2 are cuspidal, these representations are generic (see §2.2).

Suppose that there exists a finite set S of places of k containing all Archimedean places and all finite places above ℓ such that, for all $v \notin S$, the following are satisfied:

(1) the representations $\pi_{1,v}$ and $\pi_{2,v}$ are unramified representations of $\mathrm{GL}_n(k_v)$,

(2) the Satake parameters $\sigma_{1,v}$ and $\sigma_{2,v}$ of these unramified representations, considered as conjugacy classes of semisimple elements of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, have their characteristic polynomials $P_{1,v}(X)$ and $P_{2,v}(X)$ in $\overline{\mathbb{Z}}_\ell[X]$, where $\overline{\mathbb{Z}}_\ell$ is the ring of integers of $\overline{\mathbb{Q}}_\ell$,

(3) the reductions of $P_{1,v}(X)$ and $P_{2,v}(X)$ in $\overline{\mathbb{F}}_\ell[X]$ are equal, $\overline{\mathbb{F}}_\ell$ being the residue field of $\overline{\mathbb{Z}}_\ell$.

Assumption 2 is equivalent to saying that the unramified representations $\pi_{1,v}$ and $\pi_{2,v}$ are integral, that is, their $\overline{\mathbb{Q}}_\ell$-vectors spaces contain $\mathrm{GL}_n(k_v)$-stable $\overline{\mathbb{Z}}_\ell$-lattices (see §2.4). One can then consider their reductions mod ℓ, denoted $\mathbf{r}_\ell(\pi_{1,v})$ and $\mathbf{r}_\ell(\pi_{2,v})$, which are finite length, semisimple smooth $\overline{\mathbb{F}}_\ell$-representations of $\mathrm{GL}_n(k_v)$ (see Section 2 for a precise definition of reduction mod ℓ). Assumption 3 is then equivalent to saying that the representations $\mathbf{r}_\ell(\pi_{1,v})$ and $\mathbf{r}_\ell(\pi_{2,v})$ are equal (see Remarks 4.2 and 4.3).

Now let w be a finite place of k not dividing ℓ. Our first question is

Question 1.1. — *Are the irreducible representations $\pi_{1,w}$ and $\pi_{2,w}$ integral?*

Assume that this is the case. One can then form $\mathbf{r}_\ell(\pi_{1,w})$ and $\mathbf{r}_\ell(\pi_{2,w})$. These representations may not be equal (see Remark 4.4 for an example), but one may address the following question.

Question 1.2. — *Do $\mathbf{r}_\ell(\pi_{1,w})$ and $\mathbf{r}_\ell(\pi_{2,w})$ have an irreducible component in common?*

If k is a totally real (respectively, CM) number field, and if Π_1, Π_2 are algebraic regular, selfdual (respectively, conjugate-selfdual) cuspidal automorphic representations, then [**16**] Theorem 8.2 says that the answers to Questions 1.1 and 1.2 are yes. More precisely:

– the representations $\pi_{1,w}$ and $\pi_{2,w}$ are integral for all finite places w of k not dividing ℓ,
– their reductions mod ℓ have a unique generic irreducible component in common,
– this unique common generic irreducible component occurs with multiplicity 1.

Such a result, which can be thought of as a modular rigidity theorem, has been used in [**16**] in order to study the behavior of local transfer for cuspidal $\overline{\mathbb{Q}}_\ell$-representations of quasi-split classical groups with respect to congruences mod ℓ.

More generally, thanks to the results of [**6, 20, 23**], one can make the argument of the proof of [**16**] Theorem 8.2 work with no duality assumption on Π_1 and Π_2: if k is a totally real or CM number field, and if Π_1, Π_2 are algebraic regular, cuspidal automorphic representations, the answers to Questions 1.1 and 1.2 are still yes; more precisely, the three properties above still hold. (See §4.2 below for a detailed argument, which relies on the existence of a correspondence from algebraic regular cuspidal automorphic representation to Galois representations with local-global compatibility at all finite places not dividing ℓ.)

It is natural to ask whether the 'totally real or CM' assumption on k, or the 'algebraic regular' assumption on the representations Π_1 and Π_2, or the cuspidality assumption, can be removed. We will not investigate these questions in the present article.

It is also natural to seek an elementary, purely automorphic proof of such a modular rigidity theorem, avoiding the use of Galois representations and local-global compatibility theorems. We will study this question in the case of function fields, which is easier since there are no Archimedean places.

1.3. We now assume that k is a function field of characteristic p, with ring of adèles $\mathbb{A}_k$. Recall that we have fixed a field isomorphism ι from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ for some prime number ℓ which we assume to be different from p. In this article, we prove the following theorem (see Theorem 4.5).

Theorem 1.3. — *Let Π_1 and Π_2 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$. Associated with them, there are the representations $\pi_{i,v}$ defined by (1.1). Suppose that there exists a finite set S of places of k such that, for all $v \notin S$, one has:*

(1) *the representations $\pi_{1,v}$ and $\pi_{2,v}$ are unramified,*

(2) *the characteristic polynomials of their Satake parameters are in $\overline{\mathbb{Z}}_\ell[X]$ and have the same reduction in $\overline{\mathbb{F}}_\ell[X]$.*

Let w be a place of k. Then

– *the representations $\pi_{1,w}$ and $\pi_{2,w}$ are integral,*
– *their reductions mod ℓ share a generic irreducible component,*
– *and such a generic component is unique and occurs with multiplicity 1 in both reductions.*

This theorem can be easily deduced from L. Lafforgue's global Langlands correspondence [**13**] (see Remark 4.9). Our purpose is to give a simple proof of Theorem 1.3 which does not rely on the Langlands correspondence for function fields. Our argument, inspired from Piatetski-Shapiro's proof of the classical rigidity theorem [**17, 3**] is described below. We currently do not know how to extend our argument to number fields.

1.4. Before explaining the proof of Theorem 1.3, we introduce our main local ingredients. Let F be a non-Archimedean locally compact field of residue characteristic p (and characteristic 0 or p). Fix a non-trivial smooth $\overline{\mathbb{Q}}_\ell$-character ϑ of F.

Proposition 1.4. — *Let π_1 and π_2 be integral generic irreducible $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(F)$. Suppose that there are functions W_1 and W_2 in the Whittaker models of π_1 and π_2 with respect to ϑ satisfying the following conditions:*

(1) *W_1 and W_2 are $\overline{\mathbb{Z}}_\ell$-valued and $W_1(1) = W_2(1) = 1$,*

(2) *the reductions of $W_1(g)$ and $W_2(g)$ in $\overline{\mathbb{F}}_\ell$ are equal for all $g \in G$.*

Then $\mathbf{r}_\ell(\pi_1)$ and $\mathbf{r}_\ell(\pi_2)$ share a generic irreducible component, such a generic irreducible component is unique and it occurs with multiplicity 1.

Let P be the mirabolic subgroup of GL_n, made of all matrices with last row $(0\ \dots\ 0\ 1)$, and N be its unipotent radical. We have the following remarkable integrality criterion (see Proposition 2.5), which follows from [**8**] and [**14**].

Proposition 1.5. — *Let π be a generic irreducible $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_n(F)$. The following assertions are equivalent.*

(1) *The representation π is integral.*

(2) *Given any function in the Whittaker model of π with respect to ϑ whose restriction to $P(F)$ is compactly supported mod $N(F)$, this function is $\overline{\mathbb{Z}}_\ell$-valued on $\mathrm{GL}_n(F)$ if and only if it is $\overline{\mathbb{Z}}_\ell$-valued on $P(F)$.*

1.5. We now introduce our main global ingredients. Fix a continuous unitary complex character ψ of $\mathbb{A}_k$, and consider a cuspidal automorphic form φ on $\mathrm{GL}_n(\mathbb{A}_k)$. Associated with it by (3.3), with respect to the choice of ψ, there is a Whittaker function W on $\mathrm{GL}_n(\mathbb{A}_k)$. Note that ψ is valued in the group μ_p of complex pth roots of 1. Let Z be the centre of GL_n.

Given any sub-$\mathbb{Z}[\mu_p]$-module I of $\mathbb{C}$, we prove in Section 3 that:

– if W takes values in I on $P(\mathbb{A}_k)$, then φ takes values in I on $P(\mathbb{A}_k)$ (Theorem 3.1),

– if φ takes values in I on $Z(\mathbb{A}_k)P(\mathbb{A}_k)$, then W takes values in I on $\mathrm{GL}_n(\mathbb{A}_k)$ (Theorem 3.7).

1.6. We now consider two cuspidal automorphic representations Π_1, Π_2 of $\mathrm{GL}_n(\mathbb{A}_k)$ as in Theorem 1.3. Let A be the local ring $\iota^{-1}(\overline{\mathbb{Z}}_\ell)$ and $\mathfrak{m}$ be its maximal ideal. Fix a place $w \in S$.

We first observe that the central characters of Π_1 and Π_2 are A-valued and congruent mod $\mathfrak{m}$ (see Proposition 4.8), thanks to the information we have at all places $v \notin S$.

For each place v of k, let ψ_v be the local component of ψ at v. It is a smooth character of k_v, which we may assume, for all $v \notin S$, to be trivial on the ring of integers of k_v but not on the inverse of its maximal ideal.

Let $W_{i,v}$ be any function in the Whittaker model of $\Pi_{i,v}$ with respect to ψ_v such that:

(1) if $v \notin S$, then $W_{i,v}$ is $\mathrm{GL}_n(\mathcal{O}_v)$-invariant (see §2.5),

(2) if $v \in S$, then $W_{1,v}$ and $W_{2,v}$ coincide on $P(k_v)$, and their restriction to $P(k_v)$ is compactly supported mod $N(k_v)$ and A-valued (see §2.2),

(3) and $W_{1,v}(1) = W_{2,v}(1) = 1$ for all $v \neq w$.

The tensor product of the $W_{i,v}$ is a function W_i in the Whittaker model of Π_i with respect to ψ. First, it follows from the Shintani formula (see [**21, 3**] and Proposition 2.4) that, for all $v \notin S$:

– the functions $W_{1,v}$ and $W_{2,v}$ are A-valued,

– the difference $W_{1,v} - W_{2,v}$ is $\mathfrak{m}$-valued.

The functions W_1 and W_2 are thus A-valued and $W_1 - W_2$ is $\mathfrak{m}$-valued on $Z(\mathbb{A}_k)P(\mathbb{A}_k)$.

Since A and $\mathfrak{m}$ are sub-$\mathbb{Z}[\mu_p]$-modules of $\mathbb{C}$, we may apply the result of Paragraph 1.5, from which we deduce that the functions W_1 and W_2 are A-valued and $W_1 - W_2$ is $\mathfrak{m}$-valued on the whole of $\mathrm{GL}_n(\mathbb{A}_k)$. Consequently, thanks to Assumption (3) above, we get:

($\star$) the functions $W_{1,w}$ and $W_{2,w}$ are A-valued,

($\star\star$) the difference $W_{1,w} - W_{2,w}$ is $\mathfrak{m}$-valued.

Starting with any function in the Whittaker model of $\pi_{i,w}$ with respect to ψ_w whose restriction to P_w is compactly supported mod N_w and $\overline{\mathbb{Z}}_\ell$-valued, we thus proved that this function is $\overline{\mathbb{Z}}_\ell$-valued (see ($\star$) above). Applying Proposition 1.5, we deduce that $\pi_{1,w}$ and $\pi_{2,w}$ are integral, proving the first assertion of Theorem 1.3.

Now assume that $W_{1,w}$ and $W_{2,w}$ satisfy the additional condition $W_{1,w}(1) = W_{2,w}(1) = 1$. The remaining two assertions of Theorem 1.3 then follow from ($\star$) and ($\star\star$) by Proposition 1.4.

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2. Local considerations

In this section, F denotes a locally compact non-Archimedean field of residue characteristic p and $n \geqslant 1$ is a positive integer. We write $\mathcal{O}_F$ for the ring of integers of F and $\mathfrak{p}_F$ for its maximal ideal. We also write G for the locally profinite group $\mathrm{GL}_n(F)$.

Let ℓ be a prime number different from p. We write $\overline{\mathbb{Q}}_\ell$ for an algebraic closure of the field of ℓ-adic integers, $\overline{\mathbb{Z}}_\ell$ for its ring of integers and $\overline{\mathbb{F}}_\ell$ for its residue field.

Let $\psi : F \to \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial smooth character. It defines a non-degenerate character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of N, the subgroup of upper triangular unipotent matrices of G, still denoted ψ. Note that this character takes values in $\overline{\mathbb{Z}}_\ell$, and even more precisely in the group of roots of unity in $\overline{\mathbb{Q}}_\ell$ whose order is a power of p.

Let P denote the mirabolic subgroup of G, made of all matrices whose last row is $(0 \dots 0\ 1)$.

The representations we will consider will be smooth representations of locally profinite groups with coefficients in $\mathbb{Z}[1/p]$-algebras.

2.1. A $\overline{\mathbb{Q}}_\ell$-representation of finite length π of G is said to be integral if its vector space V contains a G-stable $\overline{\mathbb{Z}}_\ell$-lattice. (A G-stable $\overline{\mathbb{Z}}_\ell$-lattice is a G-stable free $\overline{\mathbb{Z}}_\ell$-module generated by a basis of V or, equivalently, an admissible $\overline{\mathbb{Z}}_\ell[G]$-module containing a basis of V.)

If this is the case, and if L is such a G-stable $\overline{\mathbb{Z}}_\ell$-lattice, the representation of G on $L \otimes \overline{\mathbb{F}}_\ell$ is smooth and has finite length, and its semisimplification does not depend on the choice of L (see [**26**] Theorem 1). This semisimplified $\overline{\mathbb{F}}_\ell$-representation is called the reduction mod ℓ of π and is denoted $\mathbf{r}_\ell(\pi)$.

An irreducible $\overline{\mathbb{Q}}_\ell$-representation π which embeds in the parabolic induction of some cuspidal irreducible representation ρ of some Levi subgroup M of G is integral if and only if the central character of ρ is $\overline{\mathbb{Z}}_\ell$-valued (see [**25**] II.4.12, II.4.14 and [**4**] Proposition 6.7).

2.2. In this paragraph, π is a generic irreducible $\overline{\mathbb{Q}}_\ell$-representation of G, that is, its vector space V carries a non-zero $\overline{\mathbb{Q}}_\ell$-linear form Λ such that $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$ for all $u \in N$, $v \in V$. Let

$$\mathcal{W}(\pi,\psi) \subseteq \mathrm{Ind}_N^G(\psi) \qquad (2.1)$$

denote its Whittaker model with respect to ψ, where Ind_N^G denotes smooth induction from N to G. Let $\mathcal{K}(\pi,\psi)$ denote the Kirillov model of π, that is, the space of smooth $\overline{\mathbb{Q}}_\ell$-valued functions on P which extend to a function in $\mathcal{W}(\pi,\psi)$. By Kirillov's theory, restriction from G to P induces a P-equivariant isomorphism from $\mathcal{W}(\pi,\psi)$ to $\mathcal{K}(\pi,\psi)$, and one has the containments

$$\mathrm{ind}_N^P(\psi) \subseteq \mathcal{K}(\pi,\psi) \subseteq \mathrm{Ind}_N^P(\psi)$$

where ind_N^P denotes compact induction from N to P (see [**1**]).

2.3. In this paragraph, π_1 and π_2 are integral generic irreducible $\overline{\mathbb{Q}}_\ell$-representations of G. Let $\mathfrak{m}_\ell$ denote the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

Proposition 2.1. — *Suppose there are Whittaker functions $W_1 \in \mathcal{W}(\pi_1,\psi)$ and $W_2 \in \mathcal{W}(\pi_2,\psi)$ with values in $\overline{\mathbb{Z}}_\ell$ such that:*

(1) *$W_1(1) = W_2(1) = 1$,*
(2) *$W_1(g)$ and $W_2(g)$ are congruent mod $\mathfrak{m}_\ell$ for all $g \in G$.*

Then the reductions mod ℓ of π_1 and π_2 share a generic irreducible component, such a generic irreducible component is unique and it occurs with multiplicity 1.

Proof. — We will need the following result.

Lemma 2.2. — *Let π be an integral generic irreducible $\overline{\mathbb{Q}}_\ell$-representation of G. Then its reduction mod ℓ contains a unique irreducible generic factor, occuring with multiplicity* 1.

Proof. — The existence of an irreducible generic factor follows from the fact that any non-zero linear form in $\mathrm{Hom}_N(\pi,\psi)$ is non-zero on any G-stable $\overline{\mathbb{Z}}_\ell$-lattice of π. Its uniqueness follows for instance from [**15**] Proposition 8.4 applied to the representation parabolically induced from the cuspidal support of π. □

Let $i \in \{1,2\}$. By [**26**] Theorem 2, the $\overline{\mathbb{Z}}_\ell$-module L_i made of all $\overline{\mathbb{Z}}_\ell$-valued Whittaker functions in $\mathcal{W}(\pi_i,\psi)$ is a G-stable $\overline{\mathbb{Z}}_\ell$-lattice. Let Λ_i be the $\overline{\mathbb{Z}}_\ell[G]$-module generated by W_i in $\mathcal{W}(\pi_i,\psi)$. It contains a $\overline{\mathbb{Q}}_\ell$-basis of $\mathcal{W}(\pi_i,\psi)$ since π_i is irreducible, and it is contained in L_i. It follows that it is a G-stable $\overline{\mathbb{Z}}_\ell$-lattice in $\mathcal{W}(\pi_i,\psi)$. Let M_i denote the submodule of L_i made of all $\mathfrak{m}_\ell$-valued functions. The containment of $\mathfrak{m}_\ell\Lambda_i$ in $\Lambda_i \cap M_i$ implies that we have morphisms:

$$\Lambda_i \otimes \overline{\mathbb{F}}_\ell \to \Lambda_i/(\Lambda_i \cap M_i) \simeq (\Lambda_i + M_i)/M_i \subseteq L_i/M_i \to \mathrm{Ind}_N^G(\vartheta \otimes \overline{\mathbb{F}}_\ell)$$

where the left hand side morphism α_i is surjective, the right hand side morphism β_i is injective, and $\vartheta \otimes \overline{\mathbb{F}}_\ell$ denotes the $\overline{\mathbb{F}}_\ell$-character of N obtained by reducing ϑ mod $\mathfrak{m}_\ell$.

As W_1 and W_2 are congruent mod $\mathfrak{m}_\ell$ on G and take 1 to 1, the intersection:

$$\beta_1((\Lambda_1 + M_1)/M_1) \cap \beta_2((\Lambda_2 + M_2)/M_2) \tag{2.2}$$

is non-zero in $\mathrm{Ind}_N^G(\vartheta \otimes \overline{\mathbb{F}}_\ell)$ for it contains the function $\beta_1(W_1 \bmod M_1) = \beta_2(W_2 \bmod M_2)$ and the latter is non-zero. The socle of (2.2), denoted Σ, is made of generic irreducible $\overline{\mathbb{F}}_\ell$-representations appearing in both the reductions mod ℓ of π_1 and π_2. By Lemma 2.2, the reduction mod ℓ of π_i contains a unique irreducible generic factor ρ_i. The socle Σ is thus irreducible, reduced to ρ_i. It follows that ρ_1 and ρ_2 are isomorphic. □

2.4. Let K be the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. In this paragraph, π is an unramified irreducible $\overline{\mathbb{Q}}_\ell$-representation of G, that is, π has a non-zero K-fixed vector. It defines a conjugacy class of semisimple elements in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, called its Satake parameter. The characteristic polynomial of this conjugacy class is denoted $\chi(\pi)$. This is a polynomial of degree n in $\overline{\mathbb{Q}}_\ell[X]$.

Lemma 2.3. — *The unramified representation π is integral if and only if the polynomial $\chi(\pi)$ has all its coefficients in $\overline{\mathbb{Z}}_\ell$.*

Proof. — Since $\overline{\mathbb{Z}}_\ell$ is integrally closed, $\chi(\pi)$ has all its coefficients in $\overline{\mathbb{Z}}_\ell$ if and only if its roots are in $\overline{\mathbb{Z}}_\ell$, that is, if and only if π is parabolically induced from an integral unramified character of the diagonal torus of $\mathrm{GL}_n(F)$. The lemma then follows from Paragraph 2.1. □

2.5. Now assume that π is a generic unramified irreducible $\overline{\mathbb{Q}}_\ell$-representation of G, and that ψ is trivial on $\mathcal{O}_F$ but not on $\mathfrak{p}_F^{-1}$. Its Whittaker model $\mathcal{W}(\pi,\psi)$ contains a unique Whittaker function W_π such that:

(1) one has $W_\pi(gk) = W_\pi(g)$ for all $g \in G$ and $k \in K$,
(2) and $W_\pi(1) = 1$.

Let us recall the Shintani–Casselman–Shalika formula [**21, 3**], which gives the values of W_π at diagonal elements in terms of the Satake parameter of π.

Fix a representative $(\mu_1,\dots,\mu_n) \in \overline{\mathbb{Q}}_\ell^{\times n}$ of the Satake parameter of π. If we write

$$\chi(\pi) = X^n + c_1(\pi)X^{n-1} + \cdots + c_n(\pi) \in \overline{\mathbb{Q}}_\ell[X],$$

then

$$(-1)^r c_r(\pi) = \sum_{1\leqslant i_1<\cdots<i_r\leqslant n} \mu_{i_1}\dots\mu_{i_r}$$

for all $r \in \{1,\dots,n\}$. Let q be the cardinality of the residue field of F. Fix a uniformizer $\varpi \in F$ and let $\mathit{\Delta}$ be the subgroup of G made of all diagonal matrices whose eigenvalues are integral powers of ϖ. The Iwasawa decomposition $G = N\mathit{\Delta}K$ shows that W_π is entirely determined by its restriction to $\mathit{\Delta}$. Given $a \in \mathbb{Z}^n$, write ϖ^a for the diagonal matrix whose ith eigenvalue is ϖ^{a_i}.

One has the formula:

$$W_\pi(\varpi^a) = q^{\sum_{j=1}^{n} a_j(j-(n+1)/2)} \cdot \frac{\det((\mu_j^{a_l+n-l})_{j,l})}{\prod_{j<l}(\mu_j-\mu_l)} \quad \text{if } a_1 \geqslant \ldots \geqslant a_n \tag{2.3}$$

and W_π vanishes at ϖ^a otherwise.

2.6. Formula (2.3) has the following application.

Proposition 2.4. — *Let π_1 and π_2 be integral generic unramified irreducible representations of G. Assume that the polynomials $\chi(\pi_1)$ and $\chi(\pi_2)$ have the same reduction mod $\mathfrak{m}_\ell$ in $\overline{\mathbb{F}}_\ell[X]$. Then the Whittaker functions W_{π_1} and W_{π_2} are $\overline{\mathbb{Z}}_\ell$-valued on G and one has:*

$$W_{\pi_1}(g) \equiv W_{\pi_2}(g) \bmod \mathfrak{m}_\ell \tag{2.4}$$

for all $g \in G$.

Proof. — Notice that $\chi(\pi_1)$ and $\chi(\pi_2)$ have coefficients in $\overline{\mathbb{Z}}_\ell$ by Lemma 2.3, thus the reduction of $\chi(\pi_1)$ and $\chi(\pi_2)$ mod $\mathfrak{m}_\ell$ is well-defined.

It suffices to prove that W_{π_1} and W_{π_2} are $\overline{\mathbb{Z}}_\ell$-valued on Δ and that the relation (2.4) is satisfied for all $g \in \Delta$. Fix a representative $\boldsymbol{\mu}_i = (\mu_{i,1}, \dots, \mu_{i,n}) \in \overline{\mathbb{Q}}_\ell^{\times n}$ of the Satake parameter of π_i for $i = 1, 2$. The scalars $\mu_{i,1}, \dots, \mu_{i,n}$ are the roots of $\chi(\pi_i)$. They are thus in $\overline{\mathbb{Z}}_\ell$. The fact that the polynomials $\chi(\pi_1)$ and $\chi(\pi_2)$ and congruent mod $\mathfrak{m}_\ell$ ensures that, up to reordering, we may assume that $\mu_{1,j}$ and $\mu_{2,j}$ are congruent mod $\mathfrak{m}_\ell$ for all $j = 1, \dots, n$. The lemma now follows from the Shintani formula (2.3). □

2.7. In this paragraph, π is a generic irreducible $\overline{\mathbb{Q}}_\ell$-representation of G. We have the following remarkable integrality criterion, based on [**8**] and [**14**].

Proposition 2.5. — *Let π be a generic irreducible $\overline{\mathbb{Q}}_\ell$-representation of G. The following assertions are equivalent:*

(1) *The representation π is integral.*

(2) *Given any $\overline{\mathbb{Z}}_\ell$-valued function $f \in \mathrm{ind}_N^P(\psi)$, the Whittaker function in $\mathcal{W}(\pi, \psi)$ extending f is $\overline{\mathbb{Z}}_\ell$-valued.*

Remark 2.6. — Assertion 2 can be restated as follows: given any function $W \in \mathcal{W}(\pi, \psi)$ whose restriction to P is compactly supported mod N, the function W is $\overline{\mathbb{Z}}_\ell$-valued on G if and only if it is $\overline{\mathbb{Z}}_\ell$-valued on P.

Proof. — That the first assertion implies the second one follows from [**14**] Corollary 4.3 (which gives an even stronger result: it says that a Whittaker function $W \in \mathcal{W}(\pi, \psi)$ is $\overline{\mathbb{Z}}_\ell$-valued on G if and only if it is $\overline{\mathbb{Z}}_\ell$-valued on P).

Let us prove that the second assertion implies the first one. We will use [**8**] Theorem 3.2, which is stated for Noetherian algebras over the ring $\mathbb{W}_\ell$ of Witt vectors of $\overline{\mathbb{F}}_\ell$. Let us explain how it applies to a generic $\overline{\mathbb{Q}}_\ell$-representation π satisfying Assertion 2.

Let V be the $\overline{\mathbb{Q}}_\ell$-vector space of π. By [**25**] II.4.9, there exists a finite extension E of $\mathbb{Q}_\ell^{\mathrm{ur}}$, the maximal unramified extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, such that π is defined over E, that is, V contains a G-stable E-vector space V_E such that $V = V_E \otimes_E \overline{\mathbb{Q}}_\ell$. Let π_E denote the E-representation of G on V_E. If ψ_E denotes the character ψ considered as being valued in E, then π_E is generic with respect to ψ_E. Let K be the completion of E. Then $\pi_K = \pi_E \otimes_E K$ is generic with respect to the character $\psi_K = \psi_E \otimes_E K$. Since the complete discrete valuation ring $\mathbb{W}_\ell$ is isomorphic to the completion of the ring of integers of $\mathbb{Q}_\ell^{\mathrm{ur}}$, the ring $\mathcal{O}$ of integers of K is a Noetherian $\mathbb{W}_\ell$-algebra. Let us show that π_K satisfies the analogue of Assertion 2 for the ring $\mathcal{O}$.

Lemma 2.7. — *Given any $\mathcal{O}$-valued function $f \in \mathrm{ind}_N^P(\psi_K)$, there exists an $\mathcal{O}$-valued function in $\mathcal{W}(\pi_K, \psi_K)$ extending f.*

Proof. — This f can be written $a_1 f_1 + \dots + a_r f_r$ with $a_1, \dots, a_r \in \mathcal{O}$, and where the functions $f_1, \dots, f_r \in \mathrm{ind}_N^G(\psi)$ are $\overline{\mathbb{Z}}_\ell$-valued. By assumption on π, the function $W_i \in \mathcal{W}(\pi, \psi)$ extending f_i is $\overline{\mathbb{Z}}_\ell$-valued. Thus $a_1 W_1 + \dots + a_r W_r$ is in $\mathcal{W}(\pi_K, \psi_K)$, it extends f and it is $\mathcal{O}$-valued. □

Let us collect some results from [**7**] about the category $\mathbf{Rep}_{\mathbb{W}_\ell}(G)$ of all smooth $\mathbb{W}_\ell$-representations of G. This category decomposes into a product of blocks indexed by inertial classes Ω of supercuspidal $\overline{\mathbb{F}}_\ell$-representations of G. Associated with each block, there is its centre $\mathfrak{z}_\Omega$, which is a finitely generated commutative $\mathbb{W}_\ell$-algebra, and its *universal co-Whittaker* module $\mathcal{W}_\Omega$, which is an admissible $\mathfrak{z}_\Omega[G]$-module.

The representation π_K is absolutely irreducible and generic. It is thus a co-Whittaker $K[G]$-module in the sense of [**8**] Definition 2.1. Also, by Schur's lemma, if Ω is the inertial class associated with it, the action of the centre $\mathfrak{z}_\Omega$ on π_K defines a morphism of $\mathbb{W}_\ell$-algebras $\chi : \mathfrak{z}_\Omega \to K$. By [**7**] Theorem 6.3, the representation π_K is a quotient of $\mathcal{W}_\Omega \otimes_{\mathfrak{z}_\Omega} K$.

We now apply [**8**] Theorem 3.2 to π_K (with $A = K$ and $A' = \mathcal{O}$), which says that, thanks to Lemma 2.7, χ is valued in $\mathcal{O}$, which makes $\mathcal{O}$ into a $\mathfrak{z}_\Omega$-algebra. By [**7**] Lemma 6.4 (or more precisely its proof), the image L of $\mathcal{W}_\Omega \otimes_{\mathfrak{z}_\Omega} \mathcal{O}$ in π_K is an $\mathcal{O}$-torsion free co-Whittaker $\mathcal{O}[G]$-module such that $L \otimes_{\mathcal{O}} K = \pi_K$. By [**8**] Definition 2.1, this $\mathcal{O}[G]$-module L is admissible. The representation π_K is thus $\mathcal{O}$-integral.

Fix a parabolic subgroup Q of G with Levi subgroup M, and a cuspidal irreducible representation ρ of M such that π embeds in the parabolic induction $\boldsymbol{i}_Q^G(\rho)$. We may and will choose E so that ρ is also defined over E: we thus have an E-representation ρ_E such that $\rho_E \otimes_E \overline{\mathbb{Q}}_\ell = \rho$ and π_E embeds in $\boldsymbol{i}_Q^G(\rho_E)$. Thus π_K embeds in $\boldsymbol{i}_Q^G(\rho_K)$, where $\rho_K = \rho_E \otimes_E K$ is cuspidal and absolutely irreducible. Note that the central character ω of ρ_K takes values in E.

Since L is admissible, [**4**] Proposition 6.7 implies that the Jacquet module $\boldsymbol{r}_Q^G(L)$ is admissible, thus ρ_K is $\mathcal{O}$-integral. Its central character ω thus takes values in $\mathcal{O}_E$, thus the central character of ρ takes values in $\overline{\mathbb{Z}}_\ell$. It follows (see Paragraph 2.1) that π is integral, as expected. □

3. Global considerations

3.1. Let k be a global field, that is, either a finite extension of $\mathbb{Q}$ or the field of rational functions over a smooth irreducible projective curve X defined over a finite field of cardinality q. Let $\mathbb{A}$ be the ring of adèles of k.

Given an integer $n \geqslant 2$, let $N = N_n$ be the subgroup of upper triangular unipotent matrices of GL_n and $P = P_n$ be its mirabolic subgroup, made of all matrices whose last row is $(0 \dots 0\ 1)$. More generally, for $m \in \{0, \dots, n\}$, let $N_{m,n-m}$ denote the unipotent radical of the parabolic subgroup of GL_n generated by upper triangular matrices and the Levi subgroup $\mathrm{GL}_m \times \mathrm{GL}_{n-m}$.

Let $\psi : \mathbb{A} \to \mathbb{C}^\times$ be a non-trivial continuous character trivial on k. It defines in the usual way a non-degenerate character of $N(k)\backslash N(\mathbb{A})$, namely

$$u \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n})$$

for all $u \in N(\mathbb{A})$, which we still denote by ψ.

For any place v of k, let k_v denote the completion of k at v. If v is finite, we write $\mathcal{O}_v$ for the ring of integers of k_v and $\mathfrak{p}_v$ for its maximal ideal. The character ψ decomposes as

$$\psi = \bigotimes_v \psi_v \tag{3.1}$$

where ψ_v is a non-trivial continuous character of k_v, trivial on $\mathcal{O}_v$ but not on $\mathfrak{p}_v^{-1}$ for almost all finite v.

3.2. Let us fix a Haar measure du on $N(k)\backslash N(\mathbb{A})$. Given a cuspidal irreducible automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$, the linear form

$$\varphi \mapsto \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1}\varphi(u)\ \mathrm{d}u \tag{3.2}$$

on Π is known to be well-defined and non-zero (see [**3**] Theorem 1.1 or (3.4) below).

Associated to $\varphi \in \Pi$, there is a Whittaker function W_φ defined by:

$$W_\varphi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1}\varphi(ug)\ \mathrm{d}u \tag{3.3}$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$. The map $\varphi \mapsto W_\varphi$ is a morphism from Π to its Whittaker model $\mathcal{W}(\Pi,\psi)$.

If we choose for du the Haar measure giving measure 1 to the compact group $N(k)\backslash N(\mathbb{A})$, one also has a converse expansion:

$$\varphi(g) = \sum_{\gamma\in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right) \tag{3.4}$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$, with absolute and uniform convergence on compact subsets (see for instance [**5**] Theorem 13.5.4 or [**3**] Theorem 1.1).

3.3. From now on, assume that k is a function field of characteristic p. There is thus no Archimedean place. Let I be any sub-$\mathbb{Z}[\mu_p]$-module of $\mathbb{C}$, where μ_p denotes the subgroup of pth roots of unity in $\mathbb{C}$. Note that ψ takes values in μ_p.

Theorem 3.1. — *Let $\phi : P_n(\mathbb{A}) \to \mathbb{C}$ be a smooth function such that:*

(1) *one has $\phi(\gamma g) = \phi(g)$ for all $\gamma \in P_n(k)$ and all $g \in P_n(\mathbb{A})$,*

(2) *the function ϕ is cuspidal in the sense that*

$$\int_{N_{m,n-m}(k)\backslash N_{m,n-m}(\mathbb{A})} \phi(ug)\ \mathrm{d}u = 0$$

for all $g \in P_n(\mathbb{A})$ and all $m \in \{1,\dots,n-1\}$,

(3) *one has*

$$\int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1}\phi(ug)\ \mathrm{d}u \in I$$

for all $g \in P_n(\mathbb{A})$.

Then $\phi(g) \in I$ for all $g \in P_n(\mathbb{A})$.

We will prove this theorem by induction on $n \geqslant 2$. Given a function ϕ as in Theorem 3.1, it is useful to define a function W_ϕ on $P_n(\mathbb{A})$ by setting

$$W_\phi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1}\phi(ug)\ \mathrm{d}u$$

for all $g \in P_n(\mathbb{A})$. We will often use the fact that, by Assumption 3, it takes values in I.

3.4. We first treat the case where $n = 2$. We will need the following lemma.

Lemma 3.2. — *For any smooth functions $f, g \in \mathcal{C}^\infty(\mathbb{A}/k, \mathbb{C})$, we have*

$$\int_{\mathbb{A}/k} f(x)g(x)\ \mathrm{d}x = \sum_{\gamma\in k} \int_{\mathbb{A}/k} \psi^{-1}(\gamma x)f(x)\ \mathrm{d}x \cdot \int_{\mathbb{A}/k} \psi(\gamma x)g(x)\ \mathrm{d}x.$$

Proof. — Start with the Fourier expansion formula

$$f(y) = \sum_{\gamma\in k} \psi(\gamma y) \cdot \int_{\mathbb{A}/k} \psi^{-1}(\gamma x)f(x)\ \mathrm{d}x$$

for $y \in \mathbb{A}/k$. Then multiply by $g(y)$ and integrate over $\mathbb{A}/k$. □

Let $f \in \mathcal{C}^\infty(\mathbb{A}/k, \mathbb{C})$. For any $g \in P_2(\mathbb{A})$, we thus get

$$\int_{\mathbb{A}/k} \phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) f(u)\ \mathrm{d}u = \sum_{\gamma\in k} \Phi(\gamma, g)F(\gamma) \tag{3.5}$$

where

$$\Phi(\gamma, g) = \int_{\mathbb{A}/k} \psi^{-1}(\gamma u)\phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right)\ \mathrm{d}u \quad \text{and} \quad F(\gamma) = \int_{\mathbb{A}/k} \psi(\gamma u)f(u)\ \mathrm{d}u.$$

Therefore, we have $\Phi(0, g) = 0$ by cuspidality of ϕ and, if $\gamma \neq 0$, we have

$$\begin{aligned}
\Phi(\gamma, g) &= \int_{\mathbb{A}/k} \psi^{-1}(u)\phi\left(\begin{pmatrix}1 & \gamma^{-1}u\\ 0 & 1\end{pmatrix} g\right)\ \mathrm{d}u\\
&= \int_{\mathbb{A}/k} \psi^{-1}(u)\phi\left(\begin{pmatrix}\gamma & u\\ 0 & 1\end{pmatrix} g\right)\ \mathrm{d}u\\
&= W_\phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)
\end{aligned}$$

where the first equality follows from the fact that the module of γ is 1 by the product formula, and the second one follows from the fact that ϕ is $P_2(k)$-invariant.

Now let $U = U(\phi, g)$ be a compact open subgroup of $\mathbb{A}/k$ such that

$$\phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) = \phi(g) \quad \text{for all } u \in U.$$

Let f be the characteristic function of U. Thus

$$F(\gamma) = \int_U \psi(\gamma u)\ \mathrm{d}u.$$

On the one hand, we have

$$\int_{\mathbb{A}/k} \phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) f(u)\ \mathrm{d}u = \phi(g)\cdot |U|$$

(where $|U|$ is the volume of U with respect to $\mathrm{d}u$). On the other hand, we have

$$\int_{\mathbb{A}/k} \phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) f(u)\ \mathrm{d}u = \sum_{\gamma\in k^\times} W_\phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)\cdot \int_U \psi(\gamma u)\ \mathrm{d}u,$$

which gives the identity

$$\phi(g) = \sum_{\gamma\in k^\times} W_\phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)\cdot \int_U \psi(\gamma u)\ \frac{\mathrm{d}u}{|U|}.$$

As $W_\phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right) \in I$, for all $\gamma\in k^\times$, and

$$\int_U \psi(\gamma u)\ \frac{\mathrm{d}u}{|U|} = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial on } \gamma U,\\ 1 & \text{otherwise,}\end{cases}$$

it only remains to prove that the sum over γ is finite, that is, there are only finitely many $\gamma\in k^\times$ such that $\gamma U\subseteq \mathrm{Ker}(\psi)$. Assume that

$$U = \prod_{v\in S} \mathfrak{p}_v^{m_v} \times \prod_{v\notin S} \mathcal{O}_v$$

for some finite set S of places of k and some integers $m_v\in\mathbb{Z}$. Recall that we have the decomposition (3.1) of ψ. By taking a bigger S if necessary, we may (and will) assume that the character ψ_v is trivial on $\mathcal{O}_v$ but not on $\mathfrak{p}_v^{-1}$ for all $v\notin S$. Thus $\gamma U\subseteq\mathrm{Ker}(\psi)$ if and only if $\gamma\mathcal{O}_v\subseteq\mathrm{Ker}(\psi_v)$ for all $v\notin S$ and $\gamma\mathfrak{p}_v^{m_v}\subseteq\mathrm{Ker}(\psi_v)$ for all $v\in S$. Equivalently, this means that γ belongs to the space of $f\in k$, considered as rational functions on the curve X defining the field k, such that

- f has no pole at $v\notin S$,
- f has a pole of order $\geqslant -m_v$ at $v\in S$.

The expected finiteness result now follows from the fact that these f form a finite dimensional vector space over $\mathbb{F}_q$ (see for instance [**22**] Proposition 1.4.9).

3.5. We now assume that $n\geqslant 3$, and that Theorem 3.1 has been proved for $P_{n-1}(\mathbb{A})$.

We fix an arbitrary $g\in P_n(\mathbb{A})$ and define a function $\phi'=\phi'_g$ on $P_{n-1}(\mathbb{A})$ by setting

$$\phi'(h) = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\cdot u)\phi\left(\begin{pmatrix}h & u\\ 0 & 1\end{pmatrix} g\right)\ \mathrm{d}u$$

for all $h\in P_{n-1}(\mathbb{A})$, where $\eta=(0,\dots,0,1)\in k^{n-1}$ and $\alpha\cdot u = a_1u_1+\dots\alpha_{n-1}u_{n-1}\in(\mathbb{A}/k)^{n-1}$ for any $\alpha\in k^{n-1}$ and $u\in(\mathbb{A}/k)^{n-1}$. It has the following properties.

Lemma 3.3. — *The function ϕ' is cuspidal on $P_{n-1}(\mathbb{A})$, that is, one has*

$$\int_{N_{m,n-1-m}(k)\backslash N_{m,n-1-m}(\mathbb{A})} \phi'(vh)\ \mathrm{d}v = 0$$

for all $h \in P_{n-1}(\mathbb{A})$ and all $m \in \{1, \ldots, n-2\}$.

Proof. — Let us fix an $h \in P_{n-1}(\mathbb{A})$ and an $m \in \{1, \ldots, n-2\}$. Then

$$\int_{N_{m,n-1-m}(k)\backslash N_{m,n-1-m}(\mathbb{A})} \phi'(vh)\ \mathrm{d}v = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u)\Omega_{g,h}(u)\ \mathrm{d}u \tag{3.6}$$

where

$$\Omega_{g,h}(u) = \int_{N_{m,n-1-m}(k)\backslash N_{m,n-1-m}(\mathbb{A})} \phi\left(\begin{pmatrix} vh & u \\ 0 & 1 \end{pmatrix} g\right)\ \mathrm{d}v$$

and the right hand side of (3.6) is equal to

$$\int_{(\mathbb{A}/k)^{n-1-m}} \psi^{-1}(\eta \cdot u_2) \int_{(\mathbb{A}/k)^{m}} \Omega_{g,h}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mathrm{d}u_1\ \mathrm{d}u_2 = \int_{(\mathbb{A}/k)^{n-1-m}} \psi^{-1}(\eta \cdot u_2)\Lambda_{g,h}(u_2)\ \mathrm{d}u_2$$

where

$$\Lambda_{g,h}(u_2) = \int_{N_{m,n-m}(k)\backslash N_{m,n-m}(\mathbb{A})} \phi\left(w \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right)\ \mathrm{d}w$$

and this quantity is equal to 0 thanks to the fact that ϕ is cuspidal. □

Lemma 3.4. — *One has*

$$\phi'(\alpha h) = \phi'(h)$$

for all $\alpha \in P_{n-1}(k)$ and $h \in P_{n-1}(\mathbb{A})$.

Proof. — Let us fix an $\alpha \in P_{n-1}(k)$. Thanks to the fact that ϕ is $P_n(k)$-invariant, one has

$$\begin{aligned} \phi'(\alpha h) &= \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta \cdot u)\phi\left(\begin{pmatrix} \alpha h & u \\ 0 & 1 \end{pmatrix} g\right)\ \mathrm{d}u \\ &= \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\alpha \cdot u)\phi\left(\begin{pmatrix} h & u \\ 0 & 1 \end{pmatrix} g\right)\ \mathrm{d}u. \end{aligned}$$

Since $\alpha \in P_{n-1}(k)$, we get $\eta\alpha = \eta$, thus $\phi'(\alpha h) = \phi'(h)$. □

Lemma 3.5. — *One has*

$$\int_{N_{n-1}(k)\backslash N_{n-1}(\mathbb{A})} \psi^{-1}(v)\phi'(vh)\ \mathrm{d}v \in I$$

for all $h \in P_{n-1}(\mathbb{A})$.

Proof. — It suffices to notice that

$$\int_{N_{n-1}(k)\backslash N_{n-1}(\mathbb{A})} \psi^{-1}(v)\phi'(vh)\ \mathrm{d}v = \int_{N_n(k)\backslash N_n(\mathbb{A})} \psi^{-1}(w)\phi\left(w\begin{pmatrix}h & 0\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}w = W_\phi\left(\begin{pmatrix}h & 0\\ 0 & 1\end{pmatrix}g\right)$$

which takes values in I for all $g \in P_n(\mathbb{A})$ and $h \in P_{n-1}(\mathbb{A})$. □

Applying now the inductive hypothesis to the function $\phi' = \phi'_g$, we deduce that

$$\phi'_g(h) \in I, \quad \text{for all } g \in P_n(\mathbb{A}) \text{ and all } h \in P_{n-1}(\mathbb{A}). \tag{3.7}$$

We can do even better.

Lemma 3.6. — *For all $g \in P_n(\mathbb{A})$ and all $g' \in \mathrm{GL}_{n-1}(\mathbb{A})$, we have*

$$\int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\cdot u)\phi\left(\begin{pmatrix}g' & u\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}u \in I. \tag{3.8}$$

Proof. — Indeed, we have

$$\int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\cdot u)\phi\left(\begin{pmatrix}g' & u\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}u = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\cdot u)\phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}g' & 0\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}u$$

which is equal to $\phi'_x(1)$ with

$$x = \begin{pmatrix}g' & 0\\ 0 & 1\end{pmatrix}g \in P_n(\mathbb{A}).$$

The lemma thus follows from (3.7) applied to the function ϕ'_x. □

We now extend $\phi' = \phi'_g$ to $\mathrm{GL}_{n-1}(\mathbb{A})$ by setting

$$\phi'(g') = \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\cdot u)\phi\left(\begin{pmatrix}g' & u\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}u$$

for all $g' \in \mathrm{GL}_{n-1}(\mathbb{A})$. By Lemma 3.6, it takes values in I on $\mathrm{GL}_{n-1}(\mathbb{A})$ for all $g \in P_n(\mathbb{A})$.

Now, by Fourier analysis on the compact Abelian group $(\mathbb{A}/k)^{n-1}$, we have

$$\begin{aligned}\phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}g\right) &= \sum_{\beta\in k^{n-1}} \psi(\beta\cdot u) \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\beta\cdot x)\phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}x\\ &= \sum_{\rho\in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \psi(\eta\rho\cdot u) \int_{(\mathbb{A}/k)^{n-1}} \psi^{-1}(\eta\cdot x)\phi\left(\begin{pmatrix}\rho & x\\ 0 & 1\end{pmatrix}g\right)\ \mathrm{d}x\\ &= \sum_{\rho\in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \psi(\eta\rho\cdot u)\phi'(\rho).\end{aligned}$$

Multiplying by $f(u)$ for some function $f \in \mathcal{C}^\infty((\mathbb{A}/k)^{n-1}, \mathbb{C})$ and integrating, we get

$$\int_{(\mathbb{A}/k)^{n-1}} \phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) f(u)\, \mathrm{d}u = \sum_{\rho\in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \phi'(\rho) \int_{(\mathbb{A}/k)^{n-1}} \psi(\eta\rho\cdot u) f(u)\, \mathrm{d}u.$$

Now let $U = U(\phi, g)$ be a compact open subgroup of $\mathbb{A}/k$ such that

$$\phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) = \phi(g) \quad \text{for all } u \in U^{n-1} \subseteq (\mathbb{A}/k)^{n-1}.$$

Now take for f the characteristic function of U^{n-1}. We get

$$\phi(g) = \sum_{\rho\in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \phi'(\rho) \int_{U^{n-1}} \psi(\eta\rho\cdot u)\, \frac{\mathrm{d}u}{|U|^{n-1}}. \tag{3.9}$$

For all ρ, we have

$$\int_{U^{n-1}} \psi(\eta\rho\cdot u)\, \frac{\mathrm{d}u}{|U|^{n-1}} = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial on } \eta\rho\cdot U^{n-1}, \\ 1 & \text{otherwise.}\end{cases}$$

A coset $\rho \in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)$ satisfies $\eta\rho\cdot U^{n-1} \subseteq \mathrm{Ker}(\psi)$ if and only if the vector

$$\beta = (\beta_1, \ldots, \beta_{n-1}) = \eta\rho \in k^{n-1}$$

satisfies $\beta\cdot U^{n-1} \subseteq \mathrm{Ker}(\psi)$, that is, $\beta_i U \subseteq \mathrm{Ker}(\psi)$ for all i. But it follows from the case where $n = 2$ that there are finitely many $\beta_i \in k$ such that $\beta_i U \subseteq \mathrm{Ker}(\psi)$. There are thus finitely many cosets ρ contributing to the sum (3.9).

Moreover, $\phi'(\rho) \in I$ for all $g \in P_n(\mathbb{A})$ and all $\rho \in P_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)$. It follows that $\phi(g) \in I$ for all $g \in P_n(\mathbb{A})$. This finishes the proof of Theorem 3.1.

3.6. As in Paragraph 3.3, k is a function field of characteristic p and I is a sub-$\mathbb{Z}[\mu_p]$-module of $\mathbb{C}$. Let $Z = Z_n$ denote the centre of GL_n. We will prove the following result.

Theorem 3.7. — *Let $\phi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ be a smooth function such that:*

(1) *one has $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathrm{GL}_n(k)$ and all $g \in \mathrm{GL}_n(\mathbb{A})$,*

(2) *the function ϕ is cuspidal in the sense that*

$$\int_{N_{m,n-m}(k)\backslash N_{m,n-m}(\mathbb{A})} \phi(ug)\, \mathrm{d}u = 0$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$ and all $m \in \{1, \ldots, n-1\}$,

(3) *one has $\phi(g) \in I$ for all $g \in Z(\mathbb{A})P(\mathbb{A})$.*

Then

$$\int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1}\phi(ug)\, \mathrm{d}u \in I$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$.

We first prove the following lemma.

Lemma 3.8. — *The image of* $\mathrm{GL}_n(k)Z(\mathbb{A})P(\mathbb{A})$ *in* $\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})$ *is dense.*

Proof. — Since the projective space $\mathbb{P}^{n-1}$ is k-isomorphic to GL_n/ZP, it is equivalent to saying that the image of the diagonal embedding of $\mathbb{P}^{n-1}(k)$ in $\mathbb{P}^{n-1}(\mathbb{A})$ is dense, that is, $\mathbb{P}^{n-1}$ satisfies the strong approximation property. We thus have to prove that, given any finite set S of places of k and, for $v \in S$, any non-empty open subset U_v of $\mathbb{P}^{n-1}(k_v)$, the intersection

$$\left(\prod_{v\in S} U_v \times \prod_{v\notin S} \mathbb{P}^{n-1}(\mathcal{O}_v)\right) \cap \mathbb{P}^{n-1}(k)$$

is non-empty. As $\mathbb{P}^{n-1}(\mathcal{O}_v) = \mathbb{P}^{n-1}(k_v)$ for all v (given any point $[x_1 : \cdots : x_n] \in \mathbb{P}^{n-1}(k_v)$, one may multiply the x_i by a suitable power of a uniformizer so that the coordinates are in $\mathcal{O}_v$), this is equivalent to proving the weak approximation property, that is, proving that there is a point $P = [x_1 : \cdots : x_n] \in \mathbb{P}^{n-1}(k)$ such that $P \in U_v$ for all $v \in S$. Restricting to the affine open subspace made of all points whose last coordinate is non-zero, we are reduced to proving the weak approximation property for an affine space. This follows from the weak approximation theorem for k (see for instance [**22**] Theorem 1.3.1). □

By Assumptions 1 and 3, the function ϕ takes values in I on $\mathrm{GL}_n(k)Z(\mathbb{A})P(\mathbb{A})$. Since ϕ is locally constant on $\mathrm{GL}_n(\mathbb{A})$, it follows from Lemma 3.8 that ϕ takes values in I on $\mathrm{GL}_n(\mathbb{A})$.

We now prove the theorem. Let us apply the map $\phi \mapsto W_\phi$ of Paragraph 3.2 defined by (3.3). This defines a function:

$$W_\phi(g) = \int_{N(k)\backslash N(\mathbb{A})} \psi(u)^{-1}\phi(ug)\ \mathrm{d}u, \quad g \in \mathrm{GL}_n(\mathbb{A}).$$

Let us prove that W_ϕ takes values in I on $\mathrm{GL}_n(\mathbb{A})$. Since $N(k)\backslash N(\mathbb{A})$ is compact, there exists a compact open subgroup C of $N(k)\backslash N(\mathbb{A})$ such that $N(k)\backslash N(\mathbb{A})$ is the union of finitely many u_iC and the function $u \mapsto \psi(u)^{-1}\phi(ug)$ is constant on these cosets. This gives us

$$W_\phi(g) = |C| \cdot \sum_{i=1}^{r} \psi^{-1}(u_i)\phi(u_ig).$$

It thus remains to prove that $|C|$ is in $\mathbb{Z}[\mu_p]$.

The measure $\mathrm{d}u$ gives measure 1 to the compact group $N(k)\backslash N(\mathbb{A})$. Thus $|C|$ is the index of C in $N(k)\backslash N(\mathbb{A})$. Let us prove that $|C|$ is a p-power. For this, it suffices to prove that $N(k)\backslash N(\mathbb{A})$ is a pro-p-group. For this, by dévissage, it suffices to prove that $\mathbb{A}/k$ is a pro-p-group.

By [**19**] Theorem 5.8, there are a finite set S of places of k and integers $m_v \geqslant 0$ for $v \in S$ such that $\mathbb{A} = k + U$ for some compact open subgroup

$$U = \prod_{v\in S} \mathfrak{p}_v^{m_v} \times \prod_{v\notin S} \mathcal{O}_v$$

thus $\mathbb{A}/k$ is a quotient of U. But U is clearly a pro-p-group, as it is a product of pro-p-groups.

4. Modular rigidity

Let k be a global field as in Paragraph 3.1. Fix a prime number ℓ different from the characteristic of k and a field isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. Let $\mathbb{A} = \mathbb{A}_k$ be its ring of adèles.

Given any place v of k, write $G_v = \mathrm{GL}_n(k_v)$ and $N_v = N(k_v)$, and $P_v = P(k_v)$ for the mirabolic subgroup of G_v.

Recall that $\mathfrak{m}_\ell$ is the maximal ideal of $\overline{\mathbb{Z}}_\ell$.

4.1. Let us state the following conjecture.

Conjecture 4.1. — *Let Π_1, Π_2 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ with central characters Ω_1, Ω_2, respectively. Let ι be a field isomorphism from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ for some prime number ℓ different from the characteristic of k. Suppose that:*

(1) *the characters $\Omega_1 \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Omega_2 \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are $\overline{\mathbb{Z}}_\ell$-valued and congruent mod $\mathfrak{m}_\ell$,*

(2) *there exists a finite set S of places of k, containing all Archimedean places and all finite places above ℓ, such that, for all $v \notin S$, one has:*

(a) *the local components $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified,*

(b) *the characteristic polynomials of their Satake parameters belong to $\overline{\mathbb{Z}}_\ell[X]$ and have the same reduction mod $\mathfrak{m}_\ell$ in $\overline{\mathbb{F}}_\ell[X]$.*

Let w be a finite place not dividing ℓ such that the representations $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral. Then the reductions mod ℓ of these representations have a common generic irreducible component, and such a generic component is unique and occurs with multiplicity 1.

Remark 4.2. — Note that the case where $w \notin S$ is easy. Indeed, if $w \notin S$, write

$$\pi_1 = \Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell, \quad \pi_2 = \Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell.$$

These representations are generic (as $\Pi_{1,w}$ and $\Pi_{2,w}$ are local components of cuspidal automorphic representations) and unramified. For each i, there is thus an unramified character ω_i of the diagonal torus T_w of G_w whose parabolic induction is isomorphic to π_i. By Lemma 2.3, these representations are integral, that is, the character ω_i takes values in $\overline{\mathbb{Z}}_\ell^\times$. By [**16**] Proposition 6.2, the fact that the characteristic polynomials of their Satake parameters are congruent implies that the reductions mod $\mathfrak{m}_\ell$ of ω_1 and ω_2 are conjugate by the normalizer of T_w in G_w. It follows that $\mathbf{r}_\ell(\pi_1)$ and $\mathbf{r}_\ell(\pi_2)$ are equal, and the conclusion follows from Lemma 2.2. We will thus concentrate on the case where $w \in S$.

Remark 4.3. — The reader should be aware that there are integral unramified irreducible $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(k_w)$ whose Satake parameters have congruent characteristic polynomials, but whose reductions mod ℓ are unequal. (For instance, this is the case for the trivial $\overline{\mathbb{Q}}_\ell$-character and any integral unramified principal series $\overline{\mathbb{Q}}_\ell$-representation whose Satake parameter has a characteristic polynomial congruent to that of the Satake parameter of the trivial $\overline{\mathbb{Q}}_\ell$-character.) However, this phenomenon does not appear for *generic* unramified representations.

Remark 4.4. — The reductions mod ℓ of $\Pi_{1,w}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ won't be equal in general for $w\in S$. Here is an example. Start with a unitary group $\mathbf{G}$ of rank 2 with respect to a totally imaginary quadratic extension l of a totally real number field k. Suppose that:

– the group $\mathbf{G}(k_v)$ is compact for all Archimedean places v,

– there is a finite place w of k above a prime number $p\neq\ell$ such that $\mathbf{G}(k_w)\simeq \mathrm{GL}_2(k_w)$ and q, the cardinality of the residue field of $\mathcal{O}_w$, has order 2 mod ℓ.

Thanks to our assumption on q, the $\overline{\mathbb{F}}_\ell$-representation induced from the trivial $\overline{\mathbb{F}}_\ell$-character of a Borel subgroup of $\mathrm{GL}_2(k_w)$ has length 3: its irreducible subquotients are the trivial character, the unramified character of order 2 and a cuspidal subquotient denoted ρ (see [**24**] Théorème 3, Corollaire 5). Let π be a cuspidal lift of ρ to $\overline{\mathbb{Q}}_\ell$, that is, π is an integral cuspidal $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_2(k_w)$ such that $\mathbf{r}_\ell(\pi)=\rho$. (The existence of such a π is granted by [**25**] III.5.10.)

Now realize π as the local component at w of some automorphic representation Π_1 of $\mathbf{G}(\mathbb{A}_k)$ which is trivial at infinity, and whose local component at another place $u\neq w$ where $\mathbf{G}$ splits is a given cuspidal representation η of $\mathrm{GL}_2(k_u)$.

We now follow [**16**] Section 3. Let K_w be the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_w)$ and $\mathbb{F}_q$ be the residue field of k_w. Let:

– κ_1 be the inflation to K_w of the cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ occurring in the parabolic induction of the trivial $\overline{\mathbb{F}}_\ell$-character of a Borel subgroup (thus the restriction of π to K_w contains κ_1),

– κ_2 be the inflation to K_w of the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_q)$.

Since the reduction mod ℓ of κ_2 contains that of κ_1, we get an automorphic representation Π_2 of $\mathbf{G}(\mathbb{A}_k)$ such that:

– the representation Π_2 is trivial at infinity,

– the representation $\Pi_{2,u}$ is isomorphic to η,

– the restriction of $\Pi_{2,w}$ to K_w contains κ_2 (thus $\Pi_{2,w}$ has non-zero Iwahori fixed vectors),

– there is a finite set of places S of k, containing all Archimedean places and u,w, such that, for all $v\notin S$, the representations $\Pi_{1,v}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ are unramified and the characteristic polynomials of their Satake parameters have coefficients in $\overline{\mathbb{Z}}_\ell$ and have the same reduction.

Using [**12**], we transfer Π_1 and Π_2 to algebraic regular, conjugate-selfdual, cuspidal automorphic representations $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ of $\mathrm{GL}_2(\mathbb{A}_l)$. Applying [**16**] Theorem 8.2, and as the local transfer at w is the identity since the group $\mathbf{G}$ splits at w, we deduce that the representations $\mathbf{r}_\ell(\pi)=\rho$ and $\mathbf{r}_\ell(\Pi_{2,w})$ share a generic irreducible component. Since $\mathbf{r}_\ell$ commutes to parabolic restriction (by [**4**] Proposition 6.7), proving that $\mathbf{r}_\ell(\Pi_{2,w})\neq\rho$ reduces to proving that $\Pi_{2,w}$ is not cuspidal. But this follows from the fact that $\Pi_{2,w}$ has non-zero Iwahori fixed vectors.

4.2. An instance of Conjecture 4.1 is provided by [**16**] Theorem 8.2. More generally, the results of [**6, 20, 23**] imply the conjecture in the case when k is a totally real or imaginary CM number field and Π_1,Π_2 are algebraic regular, by passing to the Galois side and using a density argument. In that case, note that:

– Assumption 1 on central characters is unnecessary,

– the representations $\Pi_{i,v}\otimes_{\mathbb{C}}\overline{\mathbb{Q}}_\ell$ are automatically integral for all finite v not dividing ℓ.

More precisely, assume that k is a totally real or imaginary CM number field and let Π_1, Π_2 be algebraic regular cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Assume that there exists a finite set S of places of k, containing all Archimedean places and all finite places dividing ℓ, such that, for all $v \notin S$, one has:

(1) the local components $\Pi_{1,v}$, $\Pi_{2,v}$ are unramified,

(2) the characteristic polynomials of the conjugacy classes of semisimple elements in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ associated with $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ have coefficients in $\overline{\mathbb{Z}}_\ell$ and are congruent mod $\mathfrak{m}_\ell$.

Associated with Π_i in [**6**] and [**20**], there is a continuous ℓ-adic Galois representation

$$\Sigma_i : \mathrm{Gal}(\overline{\mathbb{Q}}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

(depending on $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$) for $i = 1, 2$. For any finite place v of k not dividing ℓ, fix a decomposition subgroup Γ_v of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$. The Weil-Deligne representation associated with $\Sigma_i|_{\Gamma_v}$ is made of a smooth ℓ-adic representation $\rho_{i,v}$ together with a nilpotent operator on the space of $\rho_{i,v}$. On the other hand, the Weil-Deligne representation associated with $\Pi_{i,v} \otimes |\det|_v^{(1-n)/2}$ by the local Langlands correspondence is made of a semisimple smooth complex representation $\sigma_{i,v}$ together with a nilpotent operator on the space of $\sigma_{i,v}$. By [**23**], for any finite place v of k not dividing ℓ, one has

$$\rho_{i,v}^{\mathrm{ss}} \simeq \sigma_{i,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$$

(where $\rho_{i,v}^{\mathrm{ss}}$ stand for the semisimplification of $\rho_{i,v}$). Arguing as in [**16**] 8.2, we deduce that, for any finite place v of k not dividing ℓ, the representations $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral, their reductions mod ℓ share a generic irreducible component, which occurs with multiplicity 1.

4.3. *From now on, and until the end of this section, k is a function field of characteristic p.* We are going to prove Conjecture 4.1 in this case. We will actually prove a stronger result.

Theorem 4.5. — *Let Π_1, Π_2 be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. Let ι be a field isomorphism from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ for some prime number ℓ different from p. Suppose that there is a finite set S of places of k such that, for all $v \notin S$, one has:*

(1) *the local components $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified,*

(2) *the characteristic polynomials of their Satake parameters belong to $\overline{\mathbb{Z}}_\ell[X]$ and have the same reduction mod $\mathfrak{m}_\ell$ in $\overline{\mathbb{F}}_\ell[X]$.*

Let w be a finite place. Then:

– *the representations $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral*
– *the reductions mod ℓ of these representations have a common generic irreducible component,*
– *and such a generic component is unique and occurs with multiplicity* 1.

Remark 4.6. — Note that, by taking a bigger S, we may (and will) assume that the character ψ_v is trivial on $\mathcal{O}_v$ but not on $\mathfrak{p}_v^{-1}$ for all $v \notin S$.

First, let us prove that, under the assumptions of Theorem 4.5, for all v, the central characters of $\Pi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ take values in $\overline{\mathbb{Z}}_\ell^\times$ and are congruent mod $\mathfrak{m}_\ell$.

Lemma 4.7. — *Let χ be an automorphic character of $\mathbb{A}^\times/k^\times$ and U be a subgroup of $\mathbb{C}^\times$. Assume that there is a finite set S of places of k such that, for all $v \notin S$, the local component χ_v is unramified and takes values in U. Then, for all v, the character χ_v takes values in U.*

Proof. — If S is empty, there is nothing to prove. We thus assume that there is a place $w \in S$. Let $x \in k_w^\times$. Define an idèle $x' \in \mathbb{A}^\times$ by

$$x'_v = \begin{cases} x & \text{if } v = w, \\ 1 & \text{otherwise.} \end{cases}$$

The weak approximation theorem implies that there is a $y \in k^\times$ such that $y \in \mathrm{Ker}(\chi_v)$ if $v \in S$ and $v \neq w$, and $yx \in \mathrm{Ker}(\chi_w)$. We have

$$\chi_w(x) = \chi(x') = \chi(yx') = \chi_w(xy) \cdot \prod_{\substack{v \in S \\ v \neq w}} \chi_v(y) \cdot \prod_{v \notin S} \chi_v(y).$$

Thanks to the conditions given by the weak approximation theorem, this is equal to the product of $\chi_v(y)$ for all $v \notin S$. (Note that this is a product of finitely many terms, since y is a unit in the ring of integers of k_v for almost all $v \notin S$.) The result follows from the fact that, for such v, one has $\chi_v(y) \in U$. □

Proposition 4.8. — *Let χ_1 and χ_2 be automorphic characters of $\mathbb{A}^\times/k^\times$, and fix a field isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. Assume there is a finite set S of places of k such that, for all $v \notin S$:*

(1) *the characters $\chi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\chi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are unramified and take values in $\overline{\mathbb{Z}}_\ell^\times$,*

(2) *the reductions mod ℓ of these characters are equal.*

Then, for all places v, the characters $\chi_{1,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\chi_{2,v} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ take values in $\overline{\mathbb{Z}}_\ell^\times$ and are congruent mod $\mathfrak{m}_\ell$.

Proof. — For Assertion 1 of the proposition, apply Lemma 4.7 to χ_i and $U = \iota^{-1}(A^\times)$. For Assertion 2, apply Lemma 4.7 to $\chi = \chi_1\chi_2^{-1}$ and $U = 1 + \iota^{-1}(\mathfrak{m}_\ell)$. □

Remark 4.9. — Note that Theorem 4.5 follows from [**13**] Théorème VI.9 by a global argument in the spirit of §4.2 (see also [**9**] IV.1.6).

4.4. The remainder of this section is devoted to the proof of Theorem 4.5. By Remark 4.2, we may and will assume that $w \in S$.

Let A denote the image of $\overline{\mathbb{Z}}_\ell$ by ι^{-1} and $\mathfrak{m}$ denote the image of $\mathfrak{m}_\ell$ by ι^{-1}. Thus A contains the complex pth roots of unity and the character ψ of Paragraph 3.1 takes values in $A^\times$. Notice that A and $\mathfrak{m}$ are sub-$\mathbb{Z}[\mu_p]$-modules of $\mathbb{C}$.

For any place v of k and $i \in \{1,2\}$, let $W_{i,v}$ be a function in the Whittaker model $\mathcal{W}(\Pi_{i,v}, \psi_v)$ satisfying the conditions:

– if $v \notin S$, then $W_{i,v}$ is the unique $\mathrm{GL}_n(\mathcal{O}_v)$-invariant function such that $W_{i,v}(1) = 1$ (see Paragraph 2.5),

– if $v \in S$, we fix an arbitrary A-valued function $f_v \in \mathrm{ind}_{N_v}^{P_v}(\psi_v)$ and let $W_{i,v} \in \mathcal{W}(\Pi_{i,v}, \psi_v)$ be the unique function extending f_v to G_v (see Paragraph 2.2),

– for all $v \in S$ such that $v \neq w$, we further assume that $f_v(1) = 1$.

For $i \in \{1, 2\}$, we consider the global Whittaker function

$$W_i = \bigotimes_v W_{i,v} \in \mathcal{W}(\Pi_i, \psi).$$

For $x \in P(\mathbb{A})$, we thus have

$$W_i(x) = \prod_{v \in S} f_v(x_v) \cdot \prod_{v \notin S} W_{i,v}(x_v).$$

It follows from Proposition 2.4 that W_1 and W_2 take values in A and $W_1 - W_2$ takes values in $\mathfrak{m}$ on $P(\mathbb{A})$. Let $\varphi_i \in \Pi_i$ be the automorphic form corresponding to W_i via (3.4), that is:

$$\varphi_i(g) = \sum_{\gamma \in N_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_i\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$. By Theorem 3.1, the functions φ_1 and φ_2 take values in A and $\varphi_1 - \varphi_2$ takes values in $\mathfrak{m}$ on $P(\mathbb{A})$.

Thanks to Proposition 4.8, the central characters of $\Pi_{1,v}$ and $\Pi_{2,v}$ take values in $A^\times$ and are congruent mod $\mathfrak{m}$ for all v. It follows that φ_1 and φ_2 take values in A and $\varphi_1 - \varphi_2$ takes values in $\mathfrak{m}$ on $Z(\mathbb{A})P(\mathbb{A})$, where Z is the centre of GL_n. Applying Theorem 3.7, we deduce that W_1 and W_2 take values in A and $W_1 - W_2$ takes values in $\mathfrak{m}$ on $\mathrm{GL}_n(\mathbb{A})$.

Now let us consider the place w. For $i = 1, 2$ and $g \in G_w \subseteq G(\mathbb{A})$, one has:

$$\begin{aligned} W_i(g) &= \prod_v W_{i,v}(g_v) \\ &= W_{i,w}(g) \cdot \prod_{v \neq w} W_{i,v}(1) \\ &= W_{i,w}(g). \end{aligned}$$

It follows that $W_{1,w}$ and $W_{2,w}$ take values in A, and that $W_{1,w} - W_{2,w}$ takes values in $\mathfrak{m}$ on G_w. We thus proved that, given any A-valued function $f_w \in \mathrm{ind}_{N_w}^{P_w}(\psi_w)$, the functions $W_{1,w}$ and $W_{2,w}$ extending f_w are A-valued. Proposition 2.5 thus implies that $\Pi_{1,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ and $\Pi_{2,w} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$ are integral. Now assume further that $f_w(1) = 1$. Then Theorem 4.5 follows from Proposition 2.1.

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