

UNITARY DUAL OF p -ADIC SPLIT SO_{2n+1} AND Sp_{2n} : THE GOOD PARITY CASE (AND SLIGHTLY BEYOND)

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ABSTRACT. Let F be a p -adic field, and let G be either the split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or the symplectic group $\mathrm{Sp}_{2n}(F)$, with $n \geq 1$. We prove that a smooth irreducible representation of good parity of G is unitary if and only if it is of Arthur type. Combined with the algorithms of [6, 22] for detecting Arthur type representations, our result leads to an explicit algorithm for checking the unitarity of any given irreducible representation of good parity. Finally, we determine the set of unitary representations that may appear as local components of the discrete automorphic spectrum.

1. INTRODUCTION

The classification of irreducible unitary representations of classical groups over local fields stands as a central problem in the representation theory of reductive groups. This program traces its origins to I. Gel'fand's work in the 1940s and 1950s on harmonic analysis and representations of locally compact groups. For general linear groups, the classification was completed by D. Vogan [46] for real groups and by M. Tadić [40] for p -adic groups in the 1980s. For inner forms of general linear p -adic groups, the classification was achieved through the combined work of many mathematicians, see in particular [41, 38, 11, 13, 12]. In the case of classical groups over a p -adic field, only partial results are known: namely, for generic and unramified representations [27, 34], and for groups of rank less than or equal to three [45]. For recent progress in the case of real groups, see [1]. The unitary dual carries a natural topology, and one of the main challenges for classical groups is the large number of isolated representations (see [44]).

Let F be a non-archimedean local field. For $\mathrm{GL}_n(F)$ and its inner forms, Tadić showed that an irreducible representation is unitary if and only if it is parabolically induced from certain representations called essentially *Speh representations*. Note that unitary Speh representations arise as local components of discrete automorphic representations. A simplification of Tadić's proof was achieved in [25], where one of the key steps was to avoid the use of a theorem of J. Bernstein [15] that played a central role in the original proof. Bernstein's method relies on special properties of the mirabolic subgroup of $\mathrm{GL}_n(F)$ that do not extend to other groups, and thus the extension to inner forms of $\mathrm{GL}_n(F)$ required significantly heavier machinery, just to mention some [18, 14, 19, 35, 36, 37, 39]. In contrast, the approach in [25] relies only on elementary combinatorics and a careful analysis of Jacquet modules, what are now called ρ -derivatives (see Section 2.1), to control the irreducibility of the socle of parabolically induced representations.

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One of the most remarkable achievements in recent number theory is J. Arthur's classification [2] of square-integrable automorphic representations of quasi-split symplectic and orthogonal groups. This breakthrough, accomplished through the collective effort of many mathematicians, relies on the twisted trace formula and an intricate inductive procedure, known as endoscopy. Arthur introduced what are now called A -parameters, which are modifications of L -parameters designed to capture the non-tempered behavior of local components of discrete automorphic representations, and in particular, to account for the failure of the naive Ramanujan conjecture beyond GL_N . In this paper, these local components are referred to as representations of Arthur type. They are unitary representations ([2, Theorem 2.2.1]) and they generalize the unitary Speh representations discussed above to the setting of classical groups.

Representations of Arthur type were classified by C. Mœglin [29, 30, 31, 32, 33] by purely local methods (see also [47]). Let G be a symplectic or split special odd orthogonal group over a non-archimedean local field F of characteristic zero. Given an A -parameter ψ for G , one can decompose it as

$$\psi = \psi_{\mathrm{bad}}^- \oplus \psi_{\mathrm{good}} \oplus \psi_{\mathrm{bad}}^+,$$

where ψ_{good} is a sum of irreducible self-dual representations of the same type as ψ , and $\psi_{\mathrm{bad}}^- = (\psi_{\mathrm{bad}}^+)^{\vee}$ is a sum of irreducible representations of a different type (see Section 2.4 for more details). The parameter ψ is said to be of *good parity* if $\psi_{\mathrm{bad}}^- = 0$. Similarly, one can define this notion for irreducible representations of $G(F)$.

Mœglin [29] showed that the representations associated to ψ are irreducible parabolic inductions of the representations attached to ψ_{good} , together with products of unitary Speh representations corresponding to ψ_{bad}^- . This reduces the classification of Arthur type representations to the case of parameters of good parity. In [30, 31, 32, 33], she ultimately classified these representations using a combination of techniques, including the analysis of reducibility in parabolic induction, the use of Jacquet functors, and what she called the partial Aubert involution.

Mœglin's classification was later simplified by the first-named author of this paper [4], who used ρ -derivatives to reduce the construction of representations of Arthur type of good parity to those with non-negative discrete diagonal restriction (which are easy to determine). This naturally led to the question of whether the idea in [25] could be adapted to classify unitary representations of classical groups. In this paper, we carry out this program for unitary representations of good parity of $G(F)$, where G is a symplectic or split special odd orthogonal group over a non-archimedean local field F of characteristic zero. Our main theorem is as follows:

Theorem 1.1. *Let π be an irreducible representation of $G(F)$ of good parity. Then π is unitary if and only if it is of Arthur type.*

In fact, we prove a stronger result that will be useful for the eventual classification of the full unitary dual. Namely, we show that the good-parity part of any irreducible unitary representation is always of Arthur type. See Theorem 3.1 for the details.

Our result proves a conjecture of Tadić, refined in [21, Conjecture 1.2], and establishes a deep connection between unitary representations and local components of discrete automorphic representations. In these works, it was also conjectured (see [44, Conjecture 1.1]) that all isolated unitary representations should have good parity. We therefore view our theorem as a meaningful step toward the full classification of the unitary dual of classical groups. Moreover, since algorithms were developed in [6, 22] to determine whether a given irreducible representation of good parity is of Arthur type, our result shows that unitarity can be checked effectively in this case.

We now outline the main steps behind the proof of Theorem 1.1. One direction of the theorem follows from Arthur's work: every representation of Arthur type of good parity is known to be unitary ([2, Theorem 2.2.1]). For the converse, we begin by introducing some notation. Let G_n denote either the split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or the symplectic group $\mathrm{Sp}_{2n}(F)$ of rank n . If π (resp. τ_i) is a smooth representation of G_{n_0} (resp. $\mathrm{GL}_{d_i}(F)$), with $d_1 + \cdots + d_r + n_0 = n$, it is customary to denote by

$$\tau_r \times \cdots \times \tau_1 \rtimes \pi$$

the normalized parabolically induced representation from the standard parabolic subgroup P of G_n with Levi subgroup isomorphic to $\mathrm{GL}_{d_r}(F) \times \cdots \times \mathrm{GL}_{d_1}(F) \times G_{n_0}$, induced from $\tau_r \boxtimes \cdots \boxtimes \tau_1 \boxtimes \pi$.

Let now π be an irreducible unitary representation of good parity. A key input we use is that for any irreducible unitary representation σ of a general linear group, the parabolically induced representation

$$\sigma \rtimes \pi$$

is also unitary and admissible, hence semisimple. By choosing σ appropriately and using the explicit nature of the classification in [4, 6], we are able to deduce that π must be of Arthur type.

Even though the underlying idea is conceptually simple, its implementation is technically involved. For any irreducible representation π of G_n (of good parity), we define its *SZ-decomposition* (see Section 3.2): this consists of a natural sequence of irreducible representations τ_i of $\mathrm{GL}_{d_i}(F)$ (for $1 \leq i \leq r$), each induced from k_i copies of a segment-type representation, and a representation π_0 of G_{n_0} , with $d_1 + \cdots + d_r + n_0 = n$, such that π is the socle of the parabolically induced representation

$$\tau_r \times \cdots \times \tau_1 \rtimes \pi_0.$$

The representation π_0 is characterized by a specific constraint on its ρ -derivatives. Roughly speaking, it has almost no nonzero derivatives. In Theorem 3.13, we prove that any irreducible representation π_0 with this property is of Arthur type.

Let π_i denote the (irreducible) socle of $\tau_i \times \cdots \times \tau_1 \rtimes \pi_0$. We then prove the following:

Theorem 1.2 (Theorem 3.3). *Let π be an irreducible representation of G_n of good parity. If π is unitary and π_{i-1} is of Arthur type, then π_i is also of Arthur type. In particular, if π is unitary and π_0 is of Arthur type, then π is of Arthur type.*

As a result, Theorem 1.2 implies Theorem 1.1.

Finally, we give some details about the proof of Theorem 1.2. Given τ_i , we construct S , a product of k_i copies of a unitary Speh representation, such that the socle of $\tau_i \times S$ is a product of k_i copies of essentially Speh representations. We show that the unitarity of π implies that

$$S \rtimes \pi_i \quad \text{and} \quad \text{soc}(\tau_i \times S) \rtimes \pi_{i-1}$$

share a common irreducible subrepresentation. This imposes a strong constraint, which we exploit in the following step.

Let S' be a product of k_i copies of “long” essentially Speh representations such that the socle of $S' \times \tau_i \times S$ is itself a product of k_i copies of essentially Speh representations. We prove that the S -derivative (see Section 3.4) of

$$\text{soc}(S' \times \tau_i \times S) \rtimes \pi_{i-1}$$

contains the socle of $S' \rtimes \pi_i$ (Lemma 3.7). By choosing S' appropriately, and using Mœglin’s construction, we obtain some information about the Langlands data for $\text{soc}(S' \rtimes \pi_i)$. Using it together with the algorithm in [6], we deduce that π_i is of Arthur type. See Section 3.6.

In the final two sections of this article, we explore what lies beyond the good parity case. The examples of generic and unramified unitary representations suggest that the methods developed here are insufficient to fully address the general case, and that additional analytic tools will likely be necessary (see Section 5 for further discussion). However, for global applications, a particularly interesting set that extends slightly beyond the good parity case is that of representations attached to parameters which might be localizations of global A -parameters. It is well known that not all such representations are unitary (see [21, Example 5.1(1)]). In Theorem 4.1, we determine which of them are unitary, confirming [21, Conjecture 5.9].

Let us now say a few words about our restriction to symplectic and split odd orthogonal groups. We focus on these cases because the classification in [4] is currently limited to them. However, with the extension of the results in [3] to all quasi-split classical groups (see [8, Appendix C]), we expect that our methods can be adapted to this broader setting. In particular, we conjecture that all unitary representations of good parity are of Arthur type for all quasi-split classical groups. It would also be interesting to investigate whether an analogue of our result holds for real groups.

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2. PRELIMINARIES

Throughout this paper, F denotes a non-archimedean local field of characteristic zero. In this section, we recall some basic results from representation theory of p -adic groups.

2.1. Derivatives and socles. Fix $G \in \{\mathrm{GL}_n(F), \mathrm{SO}_{2n+1}(F), \mathrm{Sp}_{2n}(F)\}$, assumed to be split over F . Let $\mathrm{Rep}(G)$ denote the category of smooth representations of G of finite length. A representation $\Pi \in \mathrm{Rep}(G)$ is called *unitary* if it admits a positive-definite G -invariant hermitian form. Note that every unitary representation is semisimple since it is of finite length and hence admissible.

We write $\mathrm{Irr}(G)$ for the set of irreducible admissible representations of G , and denote by $\mathrm{Irr}_{\mathrm{unit}}(G)$ and $\mathrm{Irr}_{\mathrm{temp}}(G)$ the subsets consisting of unitary and tempered representations, respectively. Thus, we have the chain of inclusions:

$$\mathrm{Irr}(G) \supset \mathrm{Irr}_{\mathrm{unit}}(G) \supset \mathrm{Irr}_{\mathrm{temp}}(G).$$

For $\Pi \in \mathrm{Rep}(G)$, its semisimplification is denoted by $[\Pi]$. This is viewed as an element of the Grothendieck group

$$\mathcal{R}(G) = \bigoplus_{\pi \in \mathrm{Irr}(G)} \mathbb{Z}[\pi],$$

and we often identify $\pi \in \mathrm{Irr}(G)$ with its image $[\pi]$ in $\mathcal{R}(G)$. For $\Pi_1, \Pi_2 \in \mathrm{Rep}(G)$, we write

$$[\Pi_1] \geq [\Pi_2]$$

if there exists $\Pi \in \mathrm{Rep}(G)$ such that $[\Pi] = [\Pi_1] - [\Pi_2] \in \mathcal{R}(G)$.

The *socle* of Π , denoted by $\mathrm{soc}(\Pi)$, is its maximal semisimple subrepresentation. We say that Π is *socle irreducible* (SI) if $\mathrm{soc}(\Pi)$ is irreducible and appears in $[\Pi]$ with multiplicity one.

For a standard parabolic subgroup P of G , we denote the normalized Jacquet module of Π along P by $\mathrm{Jac}_P(\Pi)$. When P is clear from context, we simply write $\mathrm{Jac}(\Pi)$.

Given $\tau \in \mathrm{Rep}(\mathrm{GL}_d(F))$ and a character χ of $F^\times$, we denote by $\tau\chi$ the twist $\tau \otimes \chi \circ \det$. Let $\mathcal{C}^{\mathrm{GL}}$ denote the set of equivalence classes of supercuspidal representations of $\mathrm{GL}_d(F)$ for some $d \geq 0$. For $\rho \in \mathcal{C}^{\mathrm{GL}}$, its *exponent* is the unique real number $e(\rho)$ such that $\rho| \cdot |^{-e(\rho)}$ is unitary.

Let $d \leq n$ be two positive integers. Let $G \in \{\mathrm{GL}_n(F), \mathrm{SO}_{2n+1}(F), \mathrm{Sp}_{2n}(F)\}$, and let $P = P_d = MN$ be the standard maximal parabolic subgroup of G whose Levi component is $M \cong \mathrm{GL}_d(F) \times G_0$, where G_0 is a classical group of the same type as G . For $\tau \in \mathrm{Rep}(\mathrm{GL}_d(F))$ and $\pi_0 \in \mathrm{Rep}(G_0)$, the normalized parabolic induction $\mathrm{Ind}_P^G(\tau \otimes \pi_0)$ is denoted by

$$\tau \rtimes \pi_0.$$

When $G = \mathrm{GL}_n(F)$, this is also written as $\tau \times \pi_0$.

Fix now ρ a supercuspidal representation of $\mathrm{GL}_d(F)$. Given $\Pi \in \mathrm{Rep}(G)$, suppose that we have a decomposition

$$[\mathrm{Jac}(\Pi)] = \sum_{\tau \in \mathrm{Irr}(\mathrm{GL}_d(F))} [\tau] \otimes [\Pi_\tau]$$

in the Grothendieck group $\mathcal{R}(\mathrm{GL}_d(F)) \otimes \mathcal{R}(G_0)$. Then the ρ -*derivative* of Π is defined by

$$D_\rho(\Pi) = [\Pi_\rho].$$

If $n < d$, we simply set $D_\rho(\Pi) = 0$.

If $x - y \in \mathbb{Z}$, we define

$$D_{\rho|\cdot|^x, \dots, \rho|\cdot|^y}$$

to be the composition $D_{\rho|\cdot|y} \circ \cdots \circ D_{\rho|\cdot|x}$, where the exponents in the subscript are ordered with a fixed sign $\epsilon \in \{\pm 1\}$, depending on the sign of $x - y$.

Moreover, for $k \geq 0$, we define the k -th ρ -derivative by

$$D_\rho^{(k)}(\Pi) = \frac{1}{k!} \underbrace{D_\rho \circ \cdots \circ D_\rho}_{k \text{ times}}(\Pi),$$

which is well-defined as an element of an appropriate Grothendieck group.

2.2. Representations of the Weil–Deligne–Arthur group. For $G = \mathrm{GL}_n(F)$, $\mathrm{SO}_{2n+1}(F)$, or $\mathrm{Sp}_{2n}(F)$, the *Langlands dual group* $\widehat{G}$ is defined by

$$\widehat{G} = \begin{cases} \mathrm{GL}_n(\mathbb{C}) & \text{if } G = \mathrm{GL}_n(F), \\ \mathrm{Sp}_{2n}(\mathbb{C}) & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \mathrm{SO}_{2n+1}(\mathbb{C}) & \text{if } G = \mathrm{Sp}_{2n}(F). \end{cases}$$

Let W_F denote the Weil group of F . A *representation* of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ is a homomorphism

$$\psi: W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$$

such that $\psi(W_F)$ consists of semisimple elements and the restriction $\psi|_{\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})}$ is algebraic.

Denote by $\Psi^+(G)$ the set of equivalence classes of such homomorphisms with image in $\widehat{G}$. We say that:

- $\psi \in \Psi^+(G)$ is an *A-parameter* for G if $\psi(W_F)$ is bounded;
- $\psi \in \Psi^+(G)$ is an *L-parameter* for G if it is trivial on the second copy of $\mathrm{SL}_2(\mathbb{C})$.

We write $\Psi(G)$ (resp. $\Phi(G)$) for the subset of $\Psi^+(G)$ consisting of *A-parameters* (resp. *L-parameters*). Through the embedding

$$W_F \times \mathrm{SL}_2(\mathbb{C}) \hookrightarrow W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}), \quad (w, g) \mapsto (w, g, \mathbf{1}),$$

we identify each $\phi \in \Phi(G)$ with a homomorphism $\phi: W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \widehat{G}$.

We define the set of *tempered L-parameters* for G as

$$\Phi_{\mathrm{temp}}(G) = \Phi(G) \cap \Psi(G).$$

In other words, $\phi \in \Phi(G)$ is tempered if and only if $\phi(W_F)$ is bounded.

By the local Langlands correspondence, established by Harris–Taylor [20] and Henniart [23], we can view each $\rho \in \mathcal{C}^{\mathrm{GL}}$ as an irreducible representation of W_F . It has bounded image if and only if ρ is unitary, i.e., $e(\rho) = 0$.

Let S_a denote the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a . Then any representation ψ of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ can be decomposed as

$$\psi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$$

for some $\rho_1, \dots, \rho_r \in \mathcal{C}^{\mathrm{GL}}$.

2.3. Representations of $\mathrm{GL}_n(F)$. A *segment* is a set of the form

$$[x, y]_\rho = \{\rho \cdot |^x, \rho \cdot |^{x-1}, \dots, \rho \cdot |^y\},$$

where $\rho \in \mathcal{C}^{\mathrm{GL}}$ and $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}_{\geq 0}$. One can associate to it two irreducible representations $\Delta_\rho[x, y]$ and $Z_\rho[y, x]$ uniquely determined by

$$\Delta_\rho[x, y] \hookrightarrow \rho \cdot |^x \times \rho \cdot |^{x-1} \times \dots \times \rho \cdot |^y \twoheadrightarrow Z_\rho[y, x].$$

For $-A \leq B \leq A$, the unique irreducible subrepresentation of

$$\Delta_\rho[B, -A] \times \Delta_\rho[B+1, -A+1] \times \dots \times \Delta_\rho[A, -B]$$

is denoted by

$$\mathrm{Sp}(\rho, a, b) = \left(\begin{array}{ccc} B & \dots & A \\ \vdots & \ddots & \vdots \\ -A & \dots & -B \end{array} \right)_\rho,$$

where we set $a = A + B + 1$ and $b = A - B + 1$. We call it a *Speh representation*. It is unitary if and only if ρ is unitary.

By the local Langlands correspondence, there is a canonical bijection

$$\Phi(\mathrm{GL}_n(F)) \xrightarrow{1:1} \mathrm{Irr}(\mathrm{GL}_n(F)), \quad \phi \mapsto \tau_\phi.$$

If $\phi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i}$ with $e(\rho_1) \leq \dots \leq e(\rho_r)$, then τ_ϕ is the unique irreducible subrepresentation of

$$\Delta_{\rho_1}[x_1, -x_1] \times \dots \times \Delta_{\rho_r}[x_r, -x_r],$$

where $x_i = \frac{a_i-1}{2}$. In this case, we write

$$\tau_\phi = L(\Delta_{\rho_1}[x_1, -x_1], \dots, \Delta_{\rho_r}[x_r, -x_r])$$

or $\tau_\phi = L(\mathfrak{m})$, where

$$\mathfrak{m} = [x_1, -x_1]_{\rho_1} + \dots + [x_r, -x_r]_{\rho_r}$$

is the associated multisegment. Here, a *multisegment* is a multiset of segments, viewed as a finite formal sum.

In particular, for $\psi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi(\mathrm{GL}_n(F))$, the representation $\tau_\psi = \tau_{\phi_\psi}$ associated to the L -parameter

$$\phi_\psi: W_F \times \mathrm{SL}_2(\mathbb{C}) \ni (w, g) \mapsto \psi \left(w, g, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right)$$

is the irreducible parabolic induction of unitary Speh representations:

$$\tau_\psi = \bigtimes_{i=1}^r \mathrm{Sp}(\rho_i, a_i, b_i).$$

2.4. Representations of classical groups. In this subsection, we let G be either $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. We regard $\psi \in \Psi^+(G)$ as a self-dual representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ of symplectic type or of orthogonal type. We decompose

$$\psi = \psi_{\mathrm{good}} \oplus \psi_{\mathrm{bad}}$$

such that for each irreducible summand $\rho \boxtimes S_a \boxtimes S_b$ of ψ , it is a summand of ψ_{good} if and only if $e(\rho) \in (1/2)\mathbb{Z}$ and $\rho| \cdot |^{-e(\rho)} \boxtimes S_{a+2|e(\rho)|} \boxtimes S_b$ is self-dual of the same type as ψ . We call ψ of *good parity* (resp. of *bad parity*) if $\psi = \psi_{\mathrm{good}}$ (resp. $\psi = \psi_{\mathrm{bad}}$).

For $\psi \in \Psi(G)$, if $\psi_{\mathrm{good}} = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$, we set

$$A_\psi = \bigoplus_{i=1}^r \mathbb{Z}/2\mathbb{Z} e(\rho_i, a_i, b_i).$$

Namely, it is a free $\mathbb{Z}/2\mathbb{Z}$ -module with a canonical basis $\{e(\rho_i, a_i, b_i)\}_{i=1, \dots, r}$. Its quotient by the subgroup generated by

- $e(\rho_i, a_i, b_i) + e(\rho_j, a_j, b_j)$ such that $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \cong \rho_j \boxtimes S_{a_j} \boxtimes S_{b_j}$; and
- $z_\psi = \sum_{i=1}^r e(\rho_i, a_i, b_i)$

is denoted by $\mathcal{A}_\psi$. We often regard the Pontryagin dual $\widehat{\mathcal{A}}_\psi$ of $\mathcal{A}_\psi$ as a subgroup of the Pontryagin dual $\widehat{A}_\psi$ of A_ψ . For $\varepsilon \in \widehat{A}_\psi$, we set $\varepsilon(\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}) = \varepsilon(e(\rho_i, a_i, b_i))$. If $\psi = \phi$ is a tempered L -parameter, we write $e(\rho_i, a_i) = e(\rho_i, a_i, 1)$ and $\varepsilon(\rho_i \boxtimes S_{a_i}) = \varepsilon(e(\rho_i, a_i))$.

By Arthur's endoscopic classification ([2, Theorem 2.2.1]), for $\psi \in \Psi(G)$, there is a multi-set Π_ψ over $\mathrm{Irr}_{\mathrm{unit}}(G)$, called the *A-packet* associated to ψ , together with a map

$$\Pi_\psi \rightarrow \widehat{\mathcal{A}}_\psi, \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

characterized by certain endoscopic character identities. We call an irreducible representation π of G of *Arthur type* if there is $\psi \in \Psi(G)$ such that $\pi \in \Pi_\psi$. In [29, 30, 31, 32, 33], Mœglin explicitly constructed the *A-packet* Π_ψ , and in particular, she showed in [33] that Π_ψ is multiplicity free, i.e., is a subset of $\mathrm{Irr}_{\mathrm{unit}}(G)$. For her construction, see also [47] or Section 3.5 below. Moreover, as in [2, Theorem 2.2.1], if $\psi = \phi$ is a tempered L -parameter, then Π_ϕ is a subset of $\mathrm{Irr}_{\mathrm{temp}}(G)$, the map $\Pi_\phi \rightarrow \widehat{\mathcal{A}}_\phi$ is bijective, and

$$\mathrm{Irr}_{\mathrm{temp}}(G) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G)} \Pi_\phi.$$

For $\varepsilon \in \widehat{\mathcal{A}}_\phi$, the corresponding element in Π_ϕ is denoted by $\pi(\phi, \varepsilon)$.

By the Langlands classification, one can extend this classification of $\mathrm{Irr}_{\mathrm{temp}}(G)$ to $\mathrm{Irr}(G)$. For $\phi \in \Phi(G)$, if we write

$$\phi = \left(\bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \right) \oplus \phi_0 \oplus \left(\bigoplus_{i=1}^r \rho_i^\vee \boxtimes S_{a_i} \right)$$

with $e(\rho_1) \leq \dots \leq e(\rho_r) < 0$ and $\phi_0 \in \Phi_{\mathrm{temp}}(G_0)$, for $\varepsilon \in \widehat{\mathcal{A}}_\phi = \widehat{\mathcal{A}}_{\phi_0}$, we define $\pi(\phi, \varepsilon)$ as the unique irreducible subrepresentation of

$$\Delta_{\rho_1}[x_1, -x_1] \times \dots \times \Delta_{\rho_r}[x_r, -x_r] \rtimes \pi(\phi_0, \varepsilon)$$

with $x_i = \frac{a_i - 1}{2}$. We also write it as

$$\pi(\phi, \varepsilon) = L(\mathbf{m}; \pi(\phi_0, \varepsilon)) = L(\Delta_{\rho_1}[x_1, -x_1], \dots, \Delta_{\rho_r}[x_r, -x_r]; \pi(\phi_0, \varepsilon)),$$

where $\mathbf{m} = [x_1, -x_1]_{\rho_1} + \cdots + [x_r, -x_r]_{\rho_r}$. Then Π_ϕ is a subset of $\mathrm{Irr}(G)$, the map $\Pi_\phi \rightarrow \widehat{\mathcal{A}}_\phi$ given by $\pi(\phi, \varepsilon) \mapsto \varepsilon$ is bijective, and

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi.$$

We call Π_ϕ the *L-packet* associated to ϕ .

Let $\phi \in \Phi(G)$. As we have seen above, we can decompose it as $\phi = \phi_{\text{good}} \oplus \phi_{\text{bad}}$ with ϕ_{good} (resp. ϕ_{bad}) of good (resp. bad) parity. If $\pi = \pi(\phi, \varepsilon)$, we set $\pi_{\text{good}} = \pi(\phi_{\text{good}}, \varepsilon)$. We say that π is *of good parity* if $\pi = \pi_{\text{good}}$, or equivalently, $\phi = \phi_{\text{good}}$.

2.5. Geometric Lemma. We recall Geometric Lemma and Tadić's formula in this subsection.

Let G be either $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. Fix an F -rational Borel subgroup $B = TU$ of G . Let W be the Weyl group of G . For two standard parabolic subgroup $P_1 = M_1N_1$ and $P_2 = M_2N_2$, set

$$W^{M_1, M_2} = \{w \in W \mid w(M_1 \cap B)w^{-1} \subset B, w^{-1}(M_2 \cap B)w \subset B\}.$$

For any $w \in W^{M_1, M_2}$, we define a functor

$$F_w: \mathrm{Rep}(M_1) \rightarrow \mathrm{Rep}(M_2)$$

by

$$F_w = \mathrm{Ind}_{wP_1w^{-1} \cap M_2}^{M_2} \circ \mathrm{Ad}(w) \circ \mathrm{Jac}_{w^{-1}P_2w \cap M_1}.$$

Theorem 2.1 (Geometric Lemma ([16, 2.11])). *The functor $F = \mathrm{Jac}_{P_2} \circ \mathrm{Ind}_{P_1}^G: \mathrm{Rep}(M_1) \rightarrow \mathrm{Rep}(M_2)$ is glued from functors F_w for $w \in W^{M_1, M_2}$.*

In [43], Tadić studied W^{M_1, M_2} when P_1 and P_2 are maximal, and he got the following formula after the semisimplification.

Corollary 2.2 (Tadić's formula [43, Theorems 5.4, 6.5]). *Suppose that $P = MN$ is a maximal parabolic subgroup of G with $M \cong \mathrm{GL}_m(F) \times G_0$. Then for any finite length representation of G of the form $\tau \rtimes \pi$, we have*

$$[\mathrm{Jac}_P(\tau \rtimes \pi)] = \sum_{\substack{n_1, n_2, n_3, n_4 \geq 0 \\ n_1 + n_3 + n_4 = m}} \sum_{\substack{\tau_1 \otimes \tau_2 \otimes \tau_3 \\ \tau_4 \otimes \pi_0}} (\tau_1 \times \tau_4 \times \tau_3^\vee) \otimes (\tau_2 \rtimes \pi_0),$$

where

- $\tau_1 \otimes \tau_2 \otimes \tau_3$ runs over irreducible representations appears in $[\mathrm{Jac}_R(\tau)]$ with the standard parabolic subgroup R corresponding to a partition (n_1, n_2, n_3) ;
- $\tau_4 \otimes \pi_0$ runs over irreducible representations appears in $[\mathrm{Jac}'_{P'}(\pi_0)]$ with the standard parabolic subgroup P' whose Levi is of the form $\mathrm{GL}_{n_4}(F) \times G'_0$.

Here, when $n_i = 0$, we understand that $\tau_i = \mathbf{1}_{\mathrm{GL}_0(F)}$.

2.6. Methods for proving (non-)unitarity. Recall that an irreducible representation π of a group G is *hermitian* if $\bar{\pi} \cong \pi^\vee$, where $\bar{\pi}$ denotes the complex conjugate of π . In other words, π is hermitian if and only if it admits a non-degenerate G -invariant hermitian form. Note that if π is unitary, then it is also hermitian.

Now we list some basic methods for proving or disproving unitarity.

Proposition 2.3. *Fix a standard parabolic subgroup $P = MN$ of G with $M \cong \mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_k}(F) \times G_0$. Let $\tau_i \in \mathrm{Irr}(\mathrm{GL}_{m_i}(F))$, $\pi_0 \in \mathrm{Irr}(G_0)$ and set $\pi_M = \tau_1 \boxtimes \cdots \boxtimes \tau_k \boxtimes \pi_0 \in \mathrm{Irr}(M)$.*

- (1) *(Unitary induction (UI)) If π_M is unitary, then $\mathrm{Ind}_P^G(\pi_M)$ is a direct sum of irreducible unitary representations of G .*
- (2) *(Unitary reduction (UR)) If π_M is hermitian and if $\mathrm{Ind}_P^G(\pi_M)$ is irreducible and unitary, then π_M is also unitary.*
- (3) *(Complimentary series (CS)) Let $x_1(t), \dots, x_r(t): [0, 1] \rightarrow \mathbb{R}$ be continuous functions, and set*

$$\Pi_t = \tau_1 | \cdot |^{x_1(t)} \times \cdots \times \tau_r | \cdot |^{x_r(t)} \rtimes \pi_0.$$

If Π_t is irreducible and hermitian for $0 \leq t < 1$, and if $\Pi_0 = \mathrm{Ind}_P^G(\pi_M)$ is unitary, then all irreducible subquotients of Π_t are unitary for $0 \leq t \leq 1$.

- (4) *(Beyond the first reducibility point (RP1)) Suppose that*
 - $k = 1$;
 - $\tau_1 = \times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i)$ *is a product of unitary Speh representations with $\rho_i \cong \rho_i^\vee$ for $1 \leq i \leq r$;*
 - π_0 *is of Arthur type of good parity.*

Let $\psi_M \in \Psi(M)$ be an A -parameter with $\pi_M \in \Pi_{\psi_M}$, and let $R_P(w, \pi_M, \psi_M)$ be the normalized intertwining operator defined by Arthur [2, Section 2.4] with $w \in W(M, G)$. Assume further that $R_P(w, \pi_M, \psi_M)$ is not a scalar (so in particular $\mathrm{Ind}_P^G(\pi_M)$ is reducible). Then $\tau | \cdot |^s \rtimes \pi_0$ is not unitary for sufficiently small $s > 0$.

For (UI) and (UR) (resp. (CS)), see [42] or [34, Section 2] (resp. [27, Lemma 3.3]). On the other hand, (RP1) was established in the latter part of [34, Section 2], where another criterion (RP2) is also presented.

3. MAIN THEOREM

Let $n \geq 1$, and fix G to be either the split group $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. Accordingly, define

$$\mathrm{Irr}^G = \begin{cases} \bigcup_{m \geq 0} \mathrm{Irr}(\mathrm{SO}_{2m+1}(F)) & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \bigcup_{m \geq 0} \mathrm{Irr}(\mathrm{Sp}_{2m}(F)) & \text{if } G = \mathrm{Sp}_{2n}(F), \end{cases}$$

and set

$$\mathrm{Irr}^{\mathrm{GL}} = \bigcup_{m \geq 0} \mathrm{Irr}(\mathrm{GL}_m(F)).$$

3.1. Statement. Now we can state our main theorem.

Theorem 3.1. *If $\pi(\phi, \varepsilon)$ is an irreducible unitary representation of G , then $\pi(\phi_{\mathrm{good}}, \varepsilon)$ is of Arthur type.*

Note that one can determine whether $\pi(\phi_{\mathrm{good}}, \varepsilon)$ is of Arthur type using the algorithm described in [6]. Since every irreducible representation of Arthur type is known to be unitary, we obtain the following corollary.

Corollary 3.2. *Let $\pi \in \mathrm{Irr}(G)$ be of good parity. Then π is unitary if and only if π is of Arthur type.*

3.2. The SZ-Decomposition. By the Langlands classification, every $\pi(\phi, \varepsilon) \in \mathrm{Irr}^G$ can be written as

$$\mathrm{soc} \left(\tau_{\mathrm{bad}}^- \times \Delta_{\rho_1}[x_1, y_1]^{k_1} \times \cdots \times \Delta_{\rho_t}[x_t, y_t]^{k_t} \rtimes \pi_{\mathrm{temp}} \right),$$

where

- τ_{bad}^- is an irreducible representation of some general linear group whose L -parameter ϕ_{bad}^- satisfies that $\phi_{\mathrm{bad}} = \phi_{\mathrm{bad}}^- \oplus (\phi_{\mathrm{bad}}^-)^\vee$;
- $\pi_{\mathrm{temp}} \in \mathrm{Irr}^G$ is tempered;
- $\rho_i \in \mathcal{C}^{\mathrm{GL}}$ is unitary;
- $x_i + y_i < 0$ and $x_1 \leq \cdots \leq x_t$;
- If $\rho_i \cong \rho_j$, then

$$i < j \iff x_i < x_j, \text{ or } x_i = x_j \text{ and } y_i < y_j.$$

Here, $\Delta_{\rho_i}[x_i, y_i]^{k_i} = \Delta_{\rho_i}[x_i, y_i] \times \cdots \times \Delta_{\rho_i}[x_i, y_i]$ (k_i times). In particular, $\pi(\phi_{\mathrm{good}}, \varepsilon) = L(\Delta_{\rho_1}[x_1, y_1]^{k_1}, \dots, \Delta_{\rho_t}[x_t, y_t]^{k_t}; \pi_{\mathrm{temp}})$. Notice that $\tau_{\mathrm{bad}}^- \times \Delta_{\rho_i}[x_i, y_i]$ is irreducible by [48, Proposition 8.6].

Write $\{i \mid x_i < 0\} = \{1, \dots, s\}$, and set $\tau_i^- = \Delta_{\rho_i}[x_i, y_i]^{k_i}$. If we set

$$\pi' = L(\Delta_{\rho_{s+1}}[x_{s+1}, y_{s+1}]^{k_{s+1}}, \dots, \Delta_{\rho_t}[x_t, y_t]^{k_t}; \pi_{\mathrm{temp}}),$$

then π is the socle of $\tau_{\mathrm{bad}}^- \times \tau_1^- \times \cdots \times \tau_s^- \rtimes \pi'$. Note that for $\rho \in \mathcal{C}^{\mathrm{GL}}$ with $\rho^\vee \cong \rho$ and $x \in \mathbb{R}$, if $D_{\rho|\cdot|^x}(\pi') \neq 0$, then $x \geq 0$. Next, let $\hat{\pi}'$ be the Aubert dual of π' (see [10]). We write

$$\hat{\pi}' = L(\Delta_{\rho'_1}[x'_1, y'_1]^{k'_1}, \dots, \Delta_{\rho'_{t'}}[x'_{t'}, y'_{t'}]^{k'_{t'}}; \pi'_{\mathrm{temp}})$$

as above. Writing $\{i \mid x'_i \leq -1\} = \{1, \dots, r\}$, we can find $\pi_0 \in \mathrm{Irr}^G$ such that

$$\hat{\pi}' = \mathrm{soc}(\Delta_{\rho'_1}[x'_1, y'_1]^{k'_1} \times \cdots \times \Delta_{\rho'_r}[x'_r, y'_r]^{k'_r} \rtimes \hat{\pi}_0).$$

Note that for $\rho \in \mathcal{C}^{\mathrm{GL}}$ with $\rho^\vee \cong \rho$ and $x \in \mathbb{R}$, if $D_{\rho|\cdot|^x}(\hat{\pi}_0) \neq 0$, then $-\frac{1}{2} \leq x \leq 0$. If we set $\tau_i^+ = Z_{\rho'_i}[-x'_i, -y'_i]^{k'_i}$, we conclude that π is the socle of

$$\tau_{\mathrm{bad}}^- \times \tau_1^- \times \cdots \times \tau_s^- \times \tau_1^+ \times \cdots \times \tau_r^+ \rtimes \pi_0.$$

Moreover, if we set

$$\begin{aligned} \pi_1 &= \mathrm{soc}(\tau_r^+ \rtimes \pi_0), \dots, \pi_r = \mathrm{soc}(\tau_1^+ \rtimes \pi_{r-1}), \\ \pi_{r+1} &= \mathrm{soc}(\tau_s^- \rtimes \pi_r), \dots, \pi_{r+s} = \mathrm{soc}(\tau_1^- \rtimes \pi_{r+s-1}), \end{aligned}$$

then we see that

- π_i is irreducible for all $0 \leq i \leq r + s$ with $\pi(\phi_{\mathrm{good}}, \varepsilon) = \pi_{r+s}$;
- π_0 satisfies that

$$D_{\rho|\cdot|^x}(\pi_0) \neq 0 \implies x \in \{0, \frac{1}{2}\}$$

for any $\rho \in \mathcal{C}^{\mathrm{GL}}$ with $\rho^\vee \cong \rho$;

- for $1 \leq i \leq r$, we have $\pi_i = \mathrm{soc}(\tau_{r-i+1}^+ \rtimes \pi_{i-1})$ and

$$D_{\rho|\cdot|^x}(\pi_i) \neq 0 \implies 0 \leq x \leq -x'_{r-i+1}$$

for any $\rho \in \mathcal{C}^{\mathrm{GL}}$ with $\rho^\vee \cong \rho$;

- for $r+1 \leq i \leq r+s$, we have $\pi_i = \text{soc}(\tau_{r+s-i+1}^- \rtimes \pi_{i-1})$ and

$$D_{\rho|\cdot|^x}(\pi_i) \neq 0 \implies x \geq x_{r+s-i+1}$$

for any $\rho \in \mathcal{C}^{\text{GL}}$ with $\rho^\vee \cong \rho$.

Here, we note that $x_1 \leq \dots \leq x_s < 0$ and $x'_1 \leq \dots \leq x'_r \leq -1$. We call

$$(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

the *SZ-decomposition* of π .

The following is the inductive step of the proof of Theorem 3.1.

Theorem 3.3. *Let $\pi(\phi, \varepsilon) \in \text{Irr}^G$, and let $(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$ be its SZ-decomposition. For $0 \leq j < r+s$, if $\pi(\phi, \varepsilon)$ is unitary and π_j is of Arthur type, then π_{j+1} is of Arthur type. In particular, if $\pi(\phi, \varepsilon)$ is unitary and π_0 is of Arthur type, then $\pi(\phi_{\text{good}}, \varepsilon)$ is of Arthur type.*

Sections 3.3–3.6 are devoted to proving Theorem 3.3.

3.3. A key observation. We begin with a first observation.

Lemma 3.4. *Let $\pi(\phi, \varepsilon) \in \text{Irr}^G$, and let*

$$(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

be its SZ-decomposition. Assume that $\pi(\phi, \varepsilon)$ is unitary.

- (1) *Assume $s > 0$ and fix $r \leq j < r+s$. Write $\tau_{r+s-j}^- = \Delta_\rho[B-1, -A-1]^k$ for some $B < 1$ and $A > 0$ with $A+B \geq 0$, where $\rho \in \mathcal{C}^{\text{GL}}$ is unitary. Let S be the product of k copies of the Speh representation*

$$\begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho.$$

Then there exists an irreducible representation π'_{j+1} such that:

- $\pi'_{j+1} \hookrightarrow S \rtimes \pi_{j+1}$;
 - $\pi'_{j+1} \hookrightarrow \text{soc}(\tau_{r+s-j}^- \times S) \rtimes \pi_j$.
- (2) *Assume $r > 0$ and fix $0 \leq j < r$. Write $\tau_{r-j}^+ = Z_\rho[B+1, A+1]^k$ for some $A \geq B \geq 0$, where $\rho \in \mathcal{C}^{\text{GL}}$ is unitary. Let S be the product of k copies of the Speh representation*

$$\begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho.$$

Then there exists an irreducible representation π'_{j+1} such that:

- $\pi'_{j+1} \hookrightarrow S \rtimes \pi_{j+1}$;
- $\pi'_{j+1} \hookrightarrow \text{soc}(\tau_{r-j}^+ \times S) \rtimes \pi_j$.

Proof. We only prove (2) since the proof of (1) is similar. (In fact, the proof of (2) essentially includes the one of (1).)

We fix $0 \leq j < r$. Recall that we have inclusions

$$\begin{aligned}
\pi(\phi, \varepsilon) &\hookrightarrow \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s} \\
&\hookrightarrow \tau_1^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \\
&\vdots \\
&\hookrightarrow \tau_1^- \times \cdots \times \tau_s^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_r \\
&\hookrightarrow \tau_1^- \times \cdots \times \tau_s^- \times \tau_1^+ \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1} \\
&\vdots \\
&\hookrightarrow \tau_1^- \times \cdots \times \tau_s^- \times \tau_1^+ \times \cdots \times \tau_r^+ \times \tau_{\mathrm{bad}}^- \rtimes \pi_0.
\end{aligned}$$

Consider the unitary induction $S \rtimes \pi(\phi, \varepsilon)$. We take an irreducible subquotient π''_{r+s} of $S \rtimes \pi(\phi, \varepsilon)$ such that

$$(*) \quad [\mathrm{Jac}(\pi''_{r+s})] \geq \tau_1^- \otimes \cdots \otimes \tau_s^- \otimes \tau_1^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0.$$

Since $\pi(\phi, \varepsilon)$ is unitary, so is $S \rtimes \pi(\phi, \varepsilon)$, and hence it is semisimple. Therefore $\pi''_{r+s} \hookrightarrow S \rtimes \pi(\phi, \varepsilon)$.

Recall that $\tau_1^- \times S$ is SI by [25, Corollary 4.10]. We have

$$\begin{aligned}
[\pi''_{r+s}] &\leq [S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s}] \\
&\leq [S \times \tau_1^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}] \\
&= [\tau_1^- \times S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}] \\
&= [\mathrm{soc}(\tau_1^- \times S) \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}] + \left[\frac{\tau_1^- \times S}{\mathrm{soc}(\tau_1^- \times S)} \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \right].
\end{aligned}$$

Note that

$$\left[\mathrm{Jac} \left(\frac{\tau_1^- \times S}{\mathrm{soc}(\tau_1^- \times S)} \right) \right]$$

has no irreducible subquotient of the form $\tau_1^- \otimes S'$ for any $S' \neq 0$. Recall that we denote $\tau_i^- = \Delta_{\rho_i}[x_i, y_i]^{k_i}$ and that $x_1 \leq \cdots \leq x_s$ are all negative. Since $B \geq 0$, by construction, we see that

$$\left[\mathrm{Jac} \left(\frac{\tau_1^- \times S}{\mathrm{soc}(\tau_1^- \times S)} \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \right) \right]$$

has no irreducible subquotient of the form $\tau_1^- \otimes \cdots \otimes \tau_s^- \otimes \tau_1^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \sigma$ for any $\sigma \neq 0$. Hence the composition map

$$\pi''_{r+s} \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s} \hookrightarrow S \times \tau_1^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}$$

factors through

$$\pi''_{r+s} \hookrightarrow \mathrm{soc}(\tau_1^- \times S) \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \hookrightarrow \tau_1^- \times S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}.$$

In particular, there is an irreducible subquotient π''_{r+s-1} of $S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}$ such that

$$\pi''_{r+s} \hookrightarrow \tau_1^- \rtimes \pi''_{r+s-1}.$$

By construction, the Langlands data of π''_{r+s} are obtained by adding $\tau_1^- = \Delta_{\rho_1}[x_1, y_1]^{k_1}$ to those of π''_{r+s-1} . Thus, π''_{r+s-1} is uniquely characterized by the condition

$$[\text{Jac}(\pi''_{r+s})] = \tau_1^- \otimes (m \cdot \pi''_{r+s-1}) + \sum_{\substack{\tau \in \text{Irr}(\text{GL}_{n_1}(F)) \\ \tau \not\cong \tau_1^-}} \tau \otimes \Pi_\tau,$$

for some representation Π_τ and some multiplicity $m > 0$, where n_1 is such that $\tau_1^- \in \text{Irr}(\text{GL}_{n_1}(F))$. By applying Frobenius reciprocity to the inclusion map $\pi''_{r+s} \hookrightarrow \tau_1^- \times S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}$, we have a nonzero equivariant map

$$\tau_1^- \boxtimes \pi''_{r+s-1} \rightarrow \tau_1^- \boxtimes (S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}).$$

Since π''_{r+s-1} is irreducible, we obtain an inclusion $\pi''_{r+s-1} \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}$. On the other hand, by (*), we see that

$$[\text{Jac}(\pi''_{r+s-1})] \geq \tau_2^- \otimes \cdots \otimes \tau_s^- \otimes \tau_1^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \tau_{\text{bad}}^- \otimes \pi_0.$$

Repeating this argument, for $0 \leq i \leq s$, there is an irreducible representation π''_{r+s-i} such that

- $\pi''_{r+s-i} \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-i}$; and
- $[\text{Jac}(\pi''_{r+s-i})] \geq \underbrace{\tau_{i+1}^- \otimes \cdots \otimes \tau_s^-}_{s-i} \otimes \tau_1^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \tau_{\text{bad}}^- \otimes \pi_0.$

In particular, considering $i = s$, we have an irreducible representation π''_r such that

- $\pi''_r \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_r$; and
- $[\text{Jac}(\pi''_r)] \geq \tau_1^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \tau_{\text{bad}}^- \otimes \pi_0.$

By the same argument, we have

$$\begin{aligned} [\pi''_r] &\leq [S \times \tau_{\text{bad}}^- \rtimes \pi_r] \\ &\leq [S \times \tau_1^+ \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}] \\ &= [\tau_1^+ \times S \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}] \\ &= [\text{soc}(\tau_1^+ \times S) \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}] + \left[\frac{\tau_1^+ \times S}{\text{soc}(\tau_1^+ \times S)} \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \right]. \end{aligned}$$

Recall that $\tau_i^+ = Z_{\rho'_i}[-x'_i, -y'_i]^{k'_i}$ with $-x'_1 \geq \cdots \geq -x'_{r-j} = B+1 > B$. Hence

$$\text{Jac} \left(\frac{\tau_1^+ \times S}{\text{soc}(\tau_1^+ \times S)} \rtimes \pi_{r-1} \right)$$

has no irreducible representation of the form $\tau_1^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \sigma$ for any $\sigma \neq 0$. Hence the composition map

$$\pi''_r \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_r \hookrightarrow S \times \tau_1^+ \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}$$

factors through

$$\pi''_r \hookrightarrow \text{soc}(\tau_1^+ \times S) \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \hookrightarrow \tau_1^+ \times S \times \tau_{\text{bad}}^- \rtimes \pi_{r-1},$$

and we can find an irreducible subquotient π''_{r-1} of $S \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}$ such that

$$\pi''_r \hookrightarrow \tau_1^+ \rtimes \pi''_{r-1}.$$

By applying Frobenius reciprocity to $\pi_r'' \hookrightarrow \tau_1^+ \times S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1}$, we obtain an inclusion $\pi_{r-1}'' \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1}$. Moreover, π_{r-1}'' satisfies that

$$[\mathrm{Jac}(\pi_{r-1}'')] \geq \tau_2^+ \otimes \cdots \otimes \tau_r^+ \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0,$$

This argument can be repeated for $0 \leq i \leq r-j-1$, and we obtain an irreducible representation π_{r-i}'' such that

- $\pi_{r-i}'' \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-i}$; and
- $[\mathrm{Jac}(\pi_{r-i}'')] \geq \underbrace{\tau_{i+1}^+ \otimes \cdots \otimes \tau_r^+}_{r-i} \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0$.

Moreover, by the argument for $i = r-j-1$, we see that

$$\pi_{j+1}'' \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{j+1} \hookrightarrow S \times \tau_{r-j}^+ \times \tau_{\mathrm{bad}}^- \rtimes \pi_j$$

factors through $\pi_{j+1}'' \hookrightarrow \mathrm{soc}(\tau_{r-j}^+ \times S) \times \tau_{\mathrm{bad}}^- \rtimes \pi_j$.

Finally, let $\pi'_{j+1} \in \mathrm{Irr}^G$ be such that $\pi_{j+1}'' = \mathrm{soc}(\tau_{\mathrm{bad}}^- \rtimes \pi'_{j+1})$. Applying Frobenius reciprocity again to the embeddings

$$\pi_{j+1}'' \hookrightarrow \tau_{\mathrm{bad}}^- \times S \rtimes \pi_{j+1} \quad \text{and} \quad \pi_{j+1}'' \hookrightarrow \tau_{\mathrm{bad}}^- \times \mathrm{soc}(\tau_{r-j}^+ \times S) \rtimes \pi_j,$$

we obtain inclusions

$$\pi'_{j+1} \hookrightarrow S \rtimes \pi_{j+1} \quad \text{and} \quad \pi'_{j+1} \hookrightarrow \mathrm{soc}(\tau_{r-j}^+ \times S) \rtimes \pi_j.$$

This completes the proof of Lemma 3.4. $\square$

The existence of the representation π'_{j+1} in Lemma 3.4 will be used to show that π_{j+1} is of Arthur type, assuming that π_j is of Arthur type. To simplify the notation in what follows, we denote

$$\pi_j = \sigma, \quad \pi_{j+1} = \pi, \quad \pi'_{j+1} = \pi'.$$

Theorem 3.3 is thus reduced to the following proposition.

Proposition 3.5. *Let $\sigma, \pi, \pi' \in \mathrm{Irr}^G$ be of good parity. Assume that σ is of Arthur type.*

- (1) *Fix half-integers $B < 1$ and $A > 0$ such that $A + B \geq 0$. Set $\tau = \Delta_\rho[B-1, -A-1]^k$ and let S be the product of k copies of the Speh representation*

$$\begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho,$$

where $\rho \in \mathcal{C}^{\mathrm{GL}}$ is unitary. Suppose that

- if $D_{\rho|\cdot|^x}(\sigma) \neq 0$, then $x \geq B-1$, and $D_{\rho|\cdot|^{-A-1}} \circ \cdots \circ D_{\rho|\cdot|^{B-1}}(\sigma) = 0$;
- $\pi = \mathrm{soc}(\tau \rtimes \sigma)$;
- $\pi' \hookrightarrow S \rtimes \pi$ and $\pi' \hookrightarrow \mathrm{soc}(\tau \times S) \rtimes \sigma$, i.e.,

$$S \rtimes \mathrm{soc}(\tau \rtimes \sigma) \quad \text{and} \quad \mathrm{soc}(\tau \times S) \rtimes \sigma$$

have a common irreducible subrepresentation.

Then π is also of Arthur type.

- (2) Fix half-integers $A \geq B \geq 0$. Set $\tau = Z_\rho[B+1, A+1]^k$ and let S be the product of k copies of the Speh representation

$$\begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho,$$

where $\rho \in \mathcal{C}^{\text{GL}}$ is unitary. Suppose that

- if $D_{\rho|\cdot|^x}(\sigma) \neq 0$, then $x \leq B+1$, and $D_{\rho|\cdot|^{A+1}} \circ \cdots \circ D_{\rho|\cdot|^{B+1}}(\sigma) = 0$;
- $\pi = \text{soc}(\tau \rtimes \sigma)$;
- $\pi' \hookrightarrow S \rtimes \pi$ and $\pi' \hookrightarrow \text{soc}(\tau \times S) \rtimes \sigma$, i.e.,

$$S \rtimes \text{soc}(\tau \rtimes \sigma) \quad \text{and} \quad \text{soc}(\tau \times S) \rtimes \sigma$$

have a common irreducible subrepresentation.

Then π is also of Arthur type.

The proof of Proposition 3.5 will be completed in Section 3.6. The next subsection is the critical step of the proof.

3.4. Inequalities of Jacquet modules. To attack Proposition 3.5, in this subsection, we establish some inequalities of Jacquet modules.

Define a sequence $\{x_j\}_{1 \leq j \leq ab}$ with $a = A + B + 1$ and $b = A - B + 1$ by

$$\text{Sp}(\rho, a, b) = \begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho = \begin{pmatrix} x_1 & \cdots & x_{a(b-1)+1} \\ x_2 & \cdots & x_{a(b-1)+2} \\ \vdots & \ddots & \vdots \\ x_a & \cdots & x_{ab} \end{pmatrix}_\rho.$$

Hence $-A \leq x_j \leq A$ for $1 \leq j \leq ab$. To $S = \text{Sp}(\rho, a, b)^k$, we attach an operator D_S defined by

$$D_S = D_{\rho|\cdot|^{x_{ab}}}^{(k)} \circ \cdots \circ D_{\rho|\cdot|^{x_1}}^{(k)}.$$

Remark 3.6. One might (wrongly) expect that the operator D_S distinguishes irreducible representations τ containing $S \otimes \tau'$ for some $\tau' \in \text{Irr}^G$, in the sense that $D_S(\tau) \neq 0$ if and only if $S \otimes \tau' \leq [\text{Jac}(\tau)]$ for some $\tau' \neq 0$. However, this is not the case, even when $k = 1$. We are grateful to Max Gurevich for providing us with a counterexample to this assertion.

For simplicity, let $\rho = \mathbf{1}_{\text{GL}_1(F)}$. Let $S = \text{Sp}(\rho, 4, 4)$ and consider the multisegment $\mathbf{m} = [-1, -2]_\rho + [0, 0]_\rho + [1, -1]_\rho + [2, 1]_\rho$. The representation $L(\mathbf{m}) \in \text{Irr}(\text{GL}_8(F))$ was studied by B. Leclerc [28], and is an example of what is called in [26] a non-square irreducible representation.

We claim that $D_S(L(\mathbf{m}|\cdot|^{-1} + \mathbf{m}|\cdot|)) \neq 0$. Indeed, a direct computation shows that

$$D_S(L(\mathbf{m}|\cdot|^{-1}) \times L(\mathbf{m}|\cdot|)) = \mathbb{C}^2.$$

Since S appears with multiplicity one in $L(\mathbf{m}|\cdot|^{-1}) \times L(\mathbf{m}|\cdot|)$, there must exist some other irreducible subquotient τ of this induced representation such that $D_S(\tau) \neq 0$. As the induced representation has length 4, a direct computation shows that this subquotient has to be $L(\mathbf{m}|\cdot|^{-1} + \mathbf{m}|\cdot|)$.

Using the operator D_S , we state the following lemma. This is the most technical part of the proof of Theorem 3.1. It appears naturally in Mœglin's explicit construction of A -packets (see Section 3.5).

Lemma 3.7. *Let $\sigma, \pi, \pi' \in \mathrm{Irr}^G$ be of good parity. Assume that σ is of Arthur type.*

- (1) *Suppose that we are in the situation of Proposition 3.5 (1). Fix integers $t_1, \dots, t_k > 1$, and set*

$$S' = \bigotimes_{i=1}^k \begin{pmatrix} B - t_i & \dots & B - 2 \\ \vdots & \ddots & \vdots \\ -A - t_i & \dots & -A - 2 \end{pmatrix}_\rho.$$

Then

$$D_S(\mathrm{soc}(\mathrm{soc}(S' \times \tau \times S) \rtimes \sigma)) \geq \mathrm{soc}(S' \rtimes \pi).$$

- (2) *Suppose that we are in the situation of Proposition 3.5 (2). Fix integers $t_1, \dots, t_k > 1$, and set*

$$S' = \bigotimes_{i=1}^k \begin{pmatrix} B + t_i & \dots & A + t_i \\ \vdots & \ddots & \vdots \\ B + 2 & \dots & A + 2 \end{pmatrix}_\rho.$$

Then

$$D_S(\mathrm{soc}(\mathrm{soc}(S' \times \tau \times S) \rtimes \sigma)) \geq \mathrm{soc}(S' \rtimes \pi).$$

Note that the assertions (1) and (2) in Lemma 3.7 (or Proposition 3.5) are equivalent to each other by taking Aubert duality. In the rest of this subsection, we suppose that we are in the situation of (1).

First, we show that there is $\pi_j \in \mathrm{Irr}^G$ for $0 \leq j \leq ab$ such that

- $\pi_0 = \pi'$ and $\pi_{ab} = \pi$;
- for $0 \leq j \leq ab$, we have $\pi_j \hookrightarrow L_j \rtimes \pi$, where we set

$$L_j = D_{\rho|\cdot|^{x_j}}^{(k)} \circ \dots \circ D_{\rho|\cdot|^{x_1}}^{(k)}(S) \in \mathrm{Irr}^{\mathrm{GL}};$$

- $\pi_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\pi_{j-1})$ for $1 \leq j \leq ab$.

Indeed, by induction assume the existence of π_{j-1} . Then

$$\pi_{j-1} \hookrightarrow L_{j-1} \rtimes \pi \hookrightarrow (\rho|\cdot|^{x_j})^k \times L_j \rtimes \pi.$$

By Frobenius reciprocity, we have a nonzero map

$$\mathrm{Jac}(\pi_{j-1}) \rightarrow (\rho|\cdot|^{x_j})^k \boxtimes (L_j \rtimes \pi).$$

Hence there is an irreducible subrepresentation π_j of $L_j \rtimes \pi$ such that $(\rho|\cdot|^{x_j})^k \boxtimes \pi_j$ is a subrepresentation of the image of this map. Therefore $D_{\rho|\cdot|^{x_j}}^{(k)}(\pi_{j-1}) \geq \pi_j$. In the particular case when $j = ab$, we find $\pi_{ab} \hookrightarrow \pi$, so we get $\pi_{ab} = \pi$.

Lemma 3.8. *For $0 \leq j \leq ab$, there is $\sigma_j \in \mathrm{Irr}^G$ such that $\pi_j = \mathrm{soc}(\tau \rtimes \sigma_j)$. In particular, $\sigma_{ab} = \sigma$. Moreover, $\sigma_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1})$ holds for $1 \leq j \leq ab$.*

Proof. Note that $\tau \rtimes \sigma_j$ is SI for any $\sigma_j \in \text{Irr}^G$ by [5, Proposition 3.4]. Moreover, if $\pi_j = \text{soc}(\tau \rtimes \sigma_j)$, and if we let $k_{B-1}, \dots, k_{-A-1} \geq k$ be the maximal integers such that

$$D_{\rho|\cdot|^{-A-1}}^{(k_{-A-1})} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1})}(\pi) \neq 0,$$

then since $B-1 < 0$, it is irreducible and σ_j is uniquely determined by

$$D_{\rho|\cdot|^{-A-1}}^{(k_{-A-1})} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1})}(\pi) \cong D_{\rho|\cdot|^{-A-1}}^{(k_{-A-1}-k)} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1}-k)}(\sigma_j).$$

In particular, we must have $\sigma_{ab} = \sigma$.

Write $\pi_j = L(\mathfrak{m}_j; \pi(\phi_j, \varepsilon_j))$ as in the Langlands classification. First, we show that $\mathfrak{m}_j$ does not contain $[x, y]_\rho$ such that $x < B-1$ and $y < -A$. Indeed, if there were to exist such a segment in $\mathfrak{m}_j$, then we could take $C < B-1$ such that $D_{\rho|\cdot|^{-A-1}} \circ \dots \circ D_{\rho|\cdot|^C}(\pi_j) \neq 0$. However, since

$$\pi_j \hookrightarrow L_j \rtimes \pi$$

and since the cuspidal support of $\text{Sp}(\rho, a, b)$, and hence the one of L_j , does not contain $\rho|\cdot|^{\pm(A+1)}$, we see that there is $-A-1 \leq z < B-1$ such that $D_{\rho|\cdot|^z}(\pi) \neq 0$. This contradicts that $\pi = \text{soc}(\tau \rtimes \sigma)$ and $D_{\rho|\cdot|^z}(\sigma) = 0$ for $z < B-1$. Therefore, to show the existence of $\sigma_j \in \text{Irr}^G$ such that $\pi_j = \text{soc}(\tau \rtimes \sigma_j)$, by the Langlands classification, it is enough to prove that $[\text{Jac}(\pi_j)] \geq \tau \otimes \sigma_j$ for some $\sigma_j \neq 0$. Moreover, if we were to know the existence of σ_j such that $\pi_j = \text{soc}(\tau \rtimes \sigma_j)$, then $[\text{Jac}(\pi_j)]$ would be of the form

$$[\text{Jac}(\pi_j)] \geq \tau \otimes \sigma_j + \sum_{\substack{\tau' \in \text{Irr}^{\text{GL}} \\ \tau' \not\cong \tau}} \tau' \otimes \Pi_{\tau'}.$$

By applying a Jacquet functor to the inclusion $\pi_j \hookrightarrow L_j \times \tau \rtimes \sigma$, we see that $\sigma_j \leq [L_j \rtimes \sigma]$. In particular, since

$$\pi_j = \text{soc}(\tau \rtimes \sigma_j) \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\pi_{j-1}) \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\tau \rtimes \sigma_{j-1}),$$

we would have

$$\sigma_j \leq \left(D_{\rho|\cdot|^{-A-1}}^{(k)} \circ \dots \circ D_{\rho|\cdot|^{-B-1}}^{(k)} \right) \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\tau \rtimes \sigma_{j-1}).$$

By Tadić's formula (Corollary 2.2) together with the first assumption in Proposition 3.5 (1), it would imply that $\sigma_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1})$.

Now we prove the existence of σ_j . To do this, we prepare some notations. Fix $0 \leq j \leq ab$. Let

- $d, m \geq 0$ be such that $S \in \text{Irr}(\text{GL}_d(F))$ and $L_j \in \text{Irr}(\text{GL}_m(F))$;
- G_S, G_L and G be the classical group such that $S \rtimes \pi \in \text{Rep}(G_S)$, $L_j \rtimes \pi \in \text{Rep}(G_L)$ and $L_j \times S \rtimes \pi \in \text{Rep}(G)$;
- $P = MN_P$ be the standard parabolic subgroup of G with $M \cong \text{GL}_m(F) \times G_S$;
- $\iota: \text{GL}_m(F) \times G_S \xrightarrow{\sim} M$ be a fixed isomorphism.

There is a canonical surjection $L_j \times S \rtimes \pi \twoheadrightarrow L_j \otimes (S \rtimes \pi)$ which is given by

$$L_j \times S \rtimes \pi \cong L_j \rtimes (S \rtimes \pi) \twoheadrightarrow L_j \otimes (S \rtimes \pi).$$

Here, $f \in L_j \rtimes (S \rtimes \pi)$ is regarded as a two variable function

$$f: G \times G_S \rightarrow L_j \otimes S \otimes \pi$$

such that

$$f(n \cdot \iota(a, h) \cdot g, h') = |\det(a)|^\delta L_j(a) f(g, h'h)$$

for $n \in N_P, a \in \mathrm{GL}_m(F), h, h' \in G_S$ and $g \in G$, where $\delta \in \mathbb{R}$ is independent of f . Then the isomorphism $L_j \rtimes (S \rtimes \pi) \xrightarrow{\sim} L_j \times S \rtimes \pi$ is given by $f(g, h) \mapsto f(g, \mathbf{1})$, whereas, the surjection $L_j \rtimes (S \rtimes \pi) \twoheadrightarrow L_j \otimes (S \rtimes \pi)$ is given by $f(g, h) \mapsto f(\mathbf{1}, h)$. In particular, the surjection $L_j \times S \rtimes \pi \twoheadrightarrow L_j \otimes (S \rtimes \pi)$ is described by $f(g, \mathbf{1}) \mapsto f(\iota(\mathbf{1}, h), \mathbf{1})$. This surjection factors through the M -equivariant map

$$\mathrm{Jac}_P(L_j \times S \rtimes \pi) \twoheadrightarrow L_j \otimes (S \rtimes \pi),$$

which we denote by Res_{G_S} . Similarly, we have an evaluation map $\mathrm{ev}: \mathrm{Jac}(L_j \rtimes \pi) \twoheadrightarrow L_j \otimes \pi$ at $\mathbf{1}$. This gives a surjection

$$(\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi) \twoheadrightarrow L_j \otimes (S \rtimes \pi),$$

which is also denoted by ev . By applying the contragredient and the MVW functors to the injection $\pi' \hookrightarrow S \rtimes \pi$, we obtain, by [7, Lemma 2.2], a surjection $S \rtimes \pi \twoheadrightarrow \pi'$.

Note that $S \times L_j$ is irreducible by [25, Proposition 6.6], and hence $S \times L_j \rtimes \pi \cong L_j \times S \rtimes \pi$. We have an inclusion map $S \otimes \pi_j \hookrightarrow S \otimes (L_j \rtimes \pi)$ of representations of $\mathrm{GL}_d(F) \times G_L$, which is regarded as a Levi subgroup of G . It gives an inclusion

$$\mathrm{Jac}_P(S \rtimes \pi_j) \hookrightarrow \mathrm{Jac}_P(S \rtimes (L_j \rtimes \pi)).$$

By the functoriality of the Geometric Lemma (Theorem 2.1), we have subspaces $F(S \otimes \pi_j) \subset S \rtimes \pi_j$ and $F(S \otimes (L_j \rtimes \pi)) \subset S \rtimes (L_j \rtimes \pi)$ such that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Jac}_P(F(S \otimes \pi_j)) & \hookrightarrow & \mathrm{Jac}_P(F(S \otimes (L_j \rtimes \pi))) \\ \downarrow & & \downarrow \\ (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(\pi_j) & \hookrightarrow & (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi). \end{array}$$

Incorporating the maps obtained in the previous paragraph into the picture, we obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{Jac}_P(S \rtimes \pi_j) & \hookrightarrow & \mathrm{Jac}_P(S \times L_j \rtimes \pi) & \xlongequal{\quad} & \mathrm{Jac}_P(L_j \times S \rtimes \pi) \\ \cup & & \cup & & \downarrow \mathrm{Res}_{G_S} \\ \mathrm{Jac}_P(F(S \otimes \pi_j)) & \hookrightarrow & \mathrm{Jac}_P(F(S \otimes (L_j \rtimes \pi))) & & L_j \otimes (S \rtimes \pi) \\ \downarrow & & \downarrow & \nearrow \mathrm{ev} & \downarrow \\ (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(\pi_j) & \hookrightarrow & (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi) & & L_j \otimes \pi'. \end{array}$$

We claim that this diagram is commutative. In fact, the isomorphism $S \times L_j \rtimes \pi \xrightarrow{\sim} L_j \times S \rtimes \pi$ is given by the meromorphic continuation of the Jacquet integral

$$Jf(g) = \int_{\tilde{N}_P} f(w^{-1}ug)du,$$

where $w \in G$ is a representative of a certain Weyl element, and $\tilde{N}_P$ is a quotient of N_P . In particular, the composition with Res_{G_S} is induced by $f \mapsto Jf(\iota(\mathbf{1}, h))$. On the other hand, the subspace $F(S \otimes (L_j \rtimes \pi))$ of $S \times L_j \rtimes \pi$ is taken as a subspace of f such that $Jf(\iota(a, h))$ converges absolutely for $(a, h) \in \mathrm{GL}_m(F) \times G_S$, and this is the image of f under the surjection

$F(S \otimes (L_j \rtimes \pi)) \rightarrow (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi)$. Hence the composition with ev is given by $f \mapsto Jf(\iota(\mathbf{1}, h))$. This shows the commutativity of the right square.

Note that the composition

$$(\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(\pi_j) \hookrightarrow (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi) \xrightarrow{\mathrm{ev}} L_j \otimes (S \rtimes \pi)$$

is surjective since it is induced from a surjection $\mathrm{Jac}(\pi_j) \rightarrow L_j \otimes \pi$. Hence the above commutative diagram shows that we have a surjection $\mathrm{Jac}(S \rtimes \pi_j) \rightarrow L_j \otimes \pi'$, which implies, by Frobenius reciprocity, a nonzero map

$$S \rtimes \pi_j \rightarrow L_j \rtimes \pi'.$$

Since $L_j \times \mathrm{soc}(\tau \times S)$ is irreducible by [25, Proposition 6.6], we have

$$\begin{aligned} S \rtimes \pi_j &\rightarrow L_j \rtimes \pi' \\ &\hookrightarrow L_j \times \mathrm{soc}(\tau \times S) \rtimes \sigma \\ &\cong \mathrm{soc}(\tau \times S) \times L_j \rtimes \sigma \\ &\hookrightarrow \tau \times S \times L_j \rtimes \sigma. \end{aligned}$$

This implies $[\mathrm{Jac}(S \rtimes \pi_j)] \geq \tau \otimes S \otimes L_j \otimes \sigma \neq 0$. Since $\rho|\cdot|^{-A-1} \notin \mathrm{supp}(S)$, by Tadić's formula (Corollary 2.2), there is an irreducible representation $\tau'' \otimes \sigma'_j$ with $[\mathrm{Jac}(\pi_j)] \geq \tau'' \otimes \sigma'_j$ such that

$$\underbrace{\{\rho|\cdot|^{-A-1}, \dots, \rho|\cdot|^{-A-1}\}}_k \subset \mathrm{supp}(\tau'') \subset \mathrm{supp}(\tau)$$

as multi-sets, where $\mathrm{supp}(\tau)$ is the cuspidal support of τ . By the condition on $\mathfrak{m}_j$ we have proven earlier, we must have $\tau'' = \tau$, as desired. This completes the proof of Lemma 3.8. $\square$

Now we can prove Lemma 3.7.

Proof of Lemma 3.7. As said before, it is enough to show only (1). So we suppose that we are in the situation of (1).

For $\sigma_j \in \mathrm{Irr}^G$ defined in Lemma 3.8, we claim that

$$\left(D_{\rho|\cdot|^{x_j}}^{(k)} \circ \dots \circ D_{\rho|\cdot|^{x_1}}^{(k)} \right) (\mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_0)) \geq \mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_j)$$

for $0 \leq j \leq ab$. This claim for $j = ab$ yields that

$$D_S(\mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_0)) \geq \mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_{ab}) = \mathrm{soc}(S' \rtimes \pi).$$

Since

$$\begin{aligned} \mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_0) &= \mathrm{soc}(S' \rtimes \mathrm{soc}(\tau \rtimes \sigma_0)) \\ &= \mathrm{soc}(S' \rtimes \pi') \leq \mathrm{soc}(\mathrm{soc}(S' \rtimes \tau \times S) \rtimes \sigma), \end{aligned}$$

this implies Lemma 3.7.

The claim for $j = 0$ is trivial. Fix $1 \leq j \leq ab$, and suppose that the claim holds for $j - 1$. Then,

$$\left(D_{\rho|\cdot|^{x_j}}^{(k)} \circ \dots \circ D_{\rho|\cdot|^{x_1}}^{(k)} \right) (\mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_0)) \geq D_{\rho|\cdot|^{x_j}}^{(k)} (\mathrm{soc}(\mathrm{soc}(S' \rtimes \tau) \rtimes \sigma_{j-1})).$$

Recall that $-A \leq x_j \leq A$. First, we assume $x_j \neq B$ so that $\mathrm{soc}(S' \rtimes \tau) \times (\rho| \cdot |^{x_j})^k$ is irreducible. Since

$$[\mathrm{Jac}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1}))] \geq \mathrm{soc}(S' \times \tau) \otimes (\rho| \cdot |^{x_j})^k \otimes \sigma_j,$$

there is $\tau' \in \mathrm{Irr}^{\mathrm{GL}}$ such that $[\mathrm{Jac}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1}))] \geq \tau' \otimes \sigma_j$ and $[\mathrm{Jac}(\tau')] \geq \mathrm{soc}(S' \times \tau) \otimes (\rho| \cdot |^{x_j})^k$.

We show that $\tau' = \mathrm{soc}(S' \times \tau) \times (\rho| \cdot |^{x_j})^k$. We write $\tau' = L(\mathbf{m})$ and $\mathrm{soc}(S' \times \tau) = L(\Delta_\rho[a_1, b_1]^{k_1}, \dots, \Delta_\rho[a_r, b_r]^{k_r})$ with $b_1 < \dots < b_{r-1} < b_r = -A - 1$ and $k_r = k$. By looking at the cuspidal support, we can find $a_{u,v} \geq b_u$ for $1 \leq u \leq r$ and $1 \leq v \leq k_u$ such that

$$\sum_{u=1}^r \sum_{v=1}^{k_u} [a_{u,v}, b_u]_\rho \leq \mathbf{m}.$$

Suppose that $a_{u,v} \neq a_u$ for some (u, v) . Take the minimum u with this condition. Hence $a_{i,v} = a_i$ for any $i < u$ and $1 \leq v \leq k_i$. If $a_{u,v} > a_u$, then $[\mathrm{Jac}(\tau')] \not\geq \mathrm{soc}(S' \times \tau) \otimes (\rho| \cdot |^{x_j})^k$. (See cf., [24] for $\mathrm{Jac}(\mathrm{soc}(S' \times \tau))$.) This is a contradiction. If $a_{u,v} < a_u$, then

$$\left(D_{\rho| \cdot |^{b_u}} \circ \dots \circ D_{\rho| \cdot |^{a_{u,v}}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho| \cdot |^{b_i}}^{(k_i)} \circ \dots \circ D_{\rho| \cdot |^{a_i}}^{(k_i)} \right) (\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \neq 0.$$

Since $a_u < 0$ and $D_{\rho| \cdot |^{b_i}} \circ \dots \circ D_{\rho| \cdot |^{a_i}}(\sigma_{j-1}) = 0$ for $i < u$, it implies that $D_{\rho| \cdot |^{b_u}} \circ \dots \circ D_{\rho| \cdot |^{a_{u,v}}}(\sigma_{j-1}) \neq 0$. However, since $\sigma_{j-1} \leq [L_{j-1} \rtimes \sigma]$, we can find $b_u \leq c \leq a_{u,v}$ such that $D_{\rho| \cdot |^c}(\sigma) \neq 0$. This contradicts that $D_{\rho| \cdot |^x}(\sigma) = 0$ for any $x < B - 1 = \max\{a_1, \dots, a_r\}$. Therefore, $a_{u,v} = a_u$ for any (u, v) , and hence

$$\sum_{u=1}^r k_u [a_u, b_u]_\rho \leq \mathbf{m}.$$

Comparing the cuspidal supports, we see that the difference is $k[x_j, x_j]_\rho$. Hence

$$\tau' = \mathrm{soc}(S' \times \tau) \times (\rho| \cdot |^{x_j})^k \cong (\rho| \cdot |^{x_j})^k \times \mathrm{soc}(S' \rtimes \tau).$$

In particular, we have

$$[\mathrm{Jac}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1}))] \geq (\rho| \cdot |^{x_j})^k \otimes \mathrm{soc}(S' \times \tau) \otimes \sigma_j.$$

It implies that there is $\pi'_j \in \mathrm{Irr}^G$ such that $D_{\rho| \cdot |^{x_j}}^{(k)}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \geq \pi'_j$ and $[\mathrm{Jac}(\pi'_j)] \geq \mathrm{soc}(S' \times \tau) \otimes \sigma_j$.

We show that $\pi'_j = \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_j)$. We write $\pi'_j = L(\mathbf{m}; \pi(\phi, \varepsilon))$ with $\pi(\phi, \varepsilon)$ tempered. Since $[\mathrm{Jac}(\pi'_j)] \geq \mathrm{soc}(S' \times \tau) \otimes \sigma_j$ and since $a_u < 0$, we see that

$$\sum_{u=1}^r \sum_{v=1}^{k_u} [a_{u,v}, b_u]_\rho \leq \mathbf{m}$$

for some $b_u \leq a_{u,v} \leq a_u$. Suppose that $a_{u,v} < a_u$ for some (u, v) , and we take the minimum such u . Then

$$\begin{aligned} & \left(D_{\rho| \cdot |^{b_u}} \circ \dots \circ D_{\rho| \cdot |^{a_{u,v}}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho| \cdot |^{b_i}}^{(k_i)} \circ \dots \circ D_{\rho| \cdot |^{a_i}}^{(k_i)} \right) \\ & \circ D_{\rho| \cdot |^{x_j}}^{(k)}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \neq 0. \end{aligned}$$

This implies $D_{\rho|\cdot|^c}(\sigma) \neq 0$ for some $b_u \leq c < a_u \leq B-1$, or $D_{\rho|\cdot|^{b_u}} \circ \cdots \circ D_{\rho|\cdot|^{B-1}}(\sigma) \neq 0$. This contradicts the first assumption in Proposition 3.5 (1). Therefore, $a_{u,v} = a_u$ for any (u, v) , and hence

$$\sum_{u=1}^r k_u [a_u, b_u]_\rho \leq \mathbf{m}.$$

Moreover, by the same assumption in Proposition 3.5 (1), we see that $D_{\rho|\cdot|^{-A-1}} \circ \cdots \circ D_{\rho|\cdot|^x}(\sigma_{j-1}) = 0$ for any $-A-1 \leq x \leq B-2$. This implies that $L(\mathbf{m}) = \text{soc}(\text{soc}(S' \times \tau) \times L(\mathbf{m}'))$ for some $\mathbf{m}' \leq \mathbf{m}$. Hence we can write $\pi'_j = \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma'_j)$ for some $\sigma'_j \in \text{Irr}^G$. Since $\pi'_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1}))$, we see that σ'_j is uniquely determined by the inequality

$$[\text{Jac}(\pi'_j)] \geq \Delta_\rho[a_1, b_1]^{k_1} \otimes \cdots \otimes \Delta_\rho[a_r, b_r]^{k_r} \otimes \sigma'_j.$$

Hence we have $\sigma'_j = \sigma_j$, and we conclude that $\pi'_j = \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_j)$. Therefore

$$D_{\rho|\cdot|^{x_j}}^{(k)}(\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \geq \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_j),$$

as desired. We obtain the claim for j with $x_j \neq B$.

Next, we assume that $x_j = B$. By the proof of [5, Proposition 3.4], we have

$$\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1}) = \text{soc}(S' \rtimes \text{soc}(\tau \rtimes \sigma_{j-1})) = \text{soc}(S' \rtimes \pi_{j-1}).$$

Hence by the same argument starting with

$$[\text{Jac}(\text{soc}(S' \rtimes \pi_{j-1}))] \geq S' \otimes (\rho|\cdot|^{x_j})^k \otimes \pi_j,$$

we have

$$D_{\rho|\cdot|^{x_j}}^{(k)}(\text{soc}(S' \rtimes \pi_{j-1})) \geq \text{soc}(S' \rtimes \pi_j).$$

Since $\text{soc}(S' \rtimes \pi_j) = \text{soc}(S' \rtimes \text{soc}(\tau \rtimes \sigma_j)) = \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_j)$, we obtain the claim for j with $x_j = B$. This completes the proof of Lemma 3.7. $\square$

3.5. Mœglin's construction. To show Proposition 3.5, we review Mœglin's explicit construction of A -packets. It was refined by the first author in [4], in which the following notion was introduced.

Definition 3.9. (1) An *extended segment* is a triple $([A, B]_\rho, l, \eta)$, where

- $[A, B]_\rho$ is a segment with $A, B \in (1/2)\mathbb{Z}$ (so that $A \geq B$ and $A - B \in \mathbb{Z}$);
- $l \in \mathbb{Z}$ with $0 \leq l \leq \frac{b}{2}$ with $b = A - B + 1$;
- $\eta \in \{\pm 1\}$.

(2) Two extended segments $([A, B]_\rho, l, \eta)$ and $([A', B']_{\rho'}, l', \eta')$ are said to be *equivalent* if

- $[A, B]_\rho = [A', B']_{\rho'}$;
- $l = l'$; and
- $\eta = \eta'$ if $l = l' < \frac{b}{2}$.

Two multi-sets of extended segments $\{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in I}$ and $\{([A'_i, B'_i]_{\rho'_i}, l'_i, \eta'_i)\}_{i \in I}$ with the same index set I are said to be *equivalent* if $([A_i, B_i]_{\rho_i}, l_i, \eta_i)$ and $([A'_i, B'_i]_{\rho'_i}, l'_i, \eta'_i)$ are equivalent for all $i \in I$.

(3) An *extended multi-segment* for G is an equivalence class of multi-sets of extended segments

$$\mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$$

such that

- $\rho_i \in \mathcal{C}^{\mathrm{GL}}$ with $\rho_i \cong \rho_i^\vee$;
- $A_i + B_i \geq 0$;
- $\psi_{\mathcal{E}} = \oplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$ is an A -parameter for G of good parity, where $a_i = A_i + B_i + 1$ and $b_i = A_i - B_i + 1$;
- the sign condition

$$\prod_{i \in I} (-1)^{\lfloor \frac{b_i}{2} \rfloor + l_i} \eta_i^{b_i} = 1$$

holds;

- $>$ is a total order on I , which is *admissible* in the sense that

$$\rho_i \cong \rho_j, A_i < A_j, B_i < B_j \implies i < j$$

and

$$\rho_i \cong \rho_j, B_i < B_j < 0 \implies i < j.$$

Let $\mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$ be an extended multi-segment for G . When $i < j$, we write $([A_i, B_i]_{\rho_i}, l_i, \eta_i) < ([A_j, B_j]_{\rho_j}, l_j, \eta_j)$. For simplicity, we write $I = \{1, \dots, m\}$ with the usual order. We say that $\mathcal{E}$ has a *non-negative discrete diagonal restriction (DDR)* if

$$0 \leq B_1 \leq A_1 < B_2 \leq A_2 < \dots < B_m \leq A_m.$$

In this case, we define $\pi(\mathcal{E}) \in \mathrm{Irr}(G)$ by

$$\pi(\mathcal{E}) = \mathrm{soc} \left(\bigotimes_{i=1}^m \begin{pmatrix} B_i & \dots & B_i + l_i - 1 \\ \vdots & \ddots & \vdots \\ -A_i & \dots & -(A_i - l_i + 1) \end{pmatrix}_{\rho_i} \rtimes \pi(\phi, \varepsilon) \right)$$

with

$$\phi = \bigoplus_{i=1}^m \bigoplus_{j=0}^{A_i - B_i - 2l_i} \rho_i \boxtimes S_{2(B_i + l_i + j) + 1}$$

and $\varepsilon(\rho \boxtimes S_{2(B_i + l_i + j) + 1}) = (-1)^j \eta_i$ for $1 \leq i \leq m$ and $0 \leq j \leq A_i - B_i - 2l_i$.

In the general case, given an extended multisegment $\mathcal{E}$, to define the representation $\pi(\mathcal{E})$ we proceed as follows. Take a sequence of non-negative integers $\{t_i\}_{i \in (I, >)}$ such that $\mathcal{E}_{\gg} = \{([A_i + t_i, B_i + t_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$ has a non-negative DDR, and define

$$\pi(\mathcal{E}) = \bigcirc_{1 \leq i \leq m} \left(D_{\rho_i | \cdot |^{B_i+1}, \dots, \rho_i | \cdot |^{A_i+1}} \circ \dots \circ D_{\rho_i | \cdot |^{B_i+t_i}, \dots, \rho_i | \cdot |^{A_i+t_i}} \right) (\pi(\mathcal{E}_{\gg})).$$

This definition does not depend on the choice of $\{t_i\}_{i \in (I, >)}$. Then $\pi(\mathcal{E})$ is irreducible or zero. If it is nonzero, we say that $\pi(\mathcal{E}_{\gg})$ *dominates* $\pi(\mathcal{E})$.

Theorem 3.10 ([4, Theorem 1.2]). *For $\psi \in \Psi(G)$ of good parity, we have*

$$\Pi_\psi = \{\pi(\mathcal{E}) \mid \psi_{\mathcal{E}} \cong \psi\} \setminus \{0\}.$$

3.6. Proof of Proposition 3.5. In this subsection, we prove Proposition 3.5. Note that the assertions (1) and (2) are equivalent to each other by taking Aubert duality. Hence we only prove (2).

Recall that σ is a representation of Arthur type such that $D_{\rho|\cdot|^x}(\sigma) = 0$ for $x > B + 1$, and that $D_{\rho|\cdot|^{A+1}} \circ \cdots \circ D_{\rho|\cdot|^{B+1}}(\sigma) = 0$. By [6, Theorem 4.1], there is an extended multi-segment $\mathcal{E}$ with $\sigma = \pi(\mathcal{E})$ such that if $([A', B']_{\rho}, *, *) \in \mathcal{E}$, then $B' \leq B$, or $B' = B + 1$ and $A' \leq A$.

We take integers $t_1, \dots, t_k \gg 0$, which define S' . We will give conditions for these integers later. By [5, Theorem 1.1], in the notation of Lemma 3.7 (2), any irreducible summand π'' in $\text{soc}(\text{soc}(S' \times \tau \times S) \rtimes \sigma)$ is of Arthur type, and has an extended multi-segment of the form

$$\mathcal{E}'' = \mathcal{E} \cup \bigcup_{i=1}^k \{([A, B]_{\rho}, l, \eta_i), ([A + t_i, B + t_i]_{\rho}, l, \eta'_i)\},$$

where the order $<$ on $\mathcal{E}''$ is given by

$$\begin{aligned} ([A', B']_{\rho'}, l', \eta') &< ([A, B]_{\rho}, l, \eta_1) < \cdots < ([A, B]_{\rho}, l, \eta_k) \\ &< ([A + t_1, B + t_1]_{\rho}, l, \eta'_1) < \cdots < ([A + t_k, B + t_k]_{\rho}, l, \eta'_k) \end{aligned}$$

for any $([A', B']_{\rho'}, l', \eta') \in \mathcal{E}$. Notice that this is an admissible order since if $\rho' \cong \rho$ and $B' = B + 1$, then $[A', B']_{\rho'} \subset [A, B]_{\rho}$.

We write

$$\mathcal{E} \cup \{([A, B]_{\rho}, l, \eta_i) \mid i = 1, \dots, k\} = \{([A'_j, B'_j]_{\rho'}, l'_j, \eta'_j) \mid j = 1, \dots, k'\},$$

which is independent of $t_1, \dots, t_k$. Now we choose integers $t_1, \dots, t_k \gg 0$ so that there are integers $t'_1, \dots, t'_{k'} \geq 0$ such that

$$\mathcal{E}'' \gg \{([A'_j + t'_j, B'_j + t'_j]_{\rho'}, l'_j, \eta'_j) \mid j = 1, \dots, k'\} \cup \{([A + t_i, B + t_i]_{\rho}, l, \eta'_i) \mid i = 1, \dots, k\}$$

has a non-negative DDR, i.e.,

$$\begin{aligned} 0 \leq B'_1 + t'_1 &\leq A'_1 + t'_1 < \cdots < B'_{k'} + t'_{k'} \leq A'_{k'} + t'_{k'} \\ &< B + t_1 \leq A + t_1 < \cdots < B + t_k \leq A + t_k. \end{aligned}$$

In particular, $B + t_1 > A$. Then by construction, one can write

$$\pi(\mathcal{E}'') = L(\mathbf{m}_1 + \mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)),$$

where

- $\mathbf{m}_1$ is the multi-segment given by

$$\mathbf{m}_1 = \sum_{i=1}^k \sum_{j=0}^{l-1} [B + t_i + j, -A - t_i + j]_{\rho};$$

- ϕ_1 is the representation of $W_F \times \text{SL}_2(\mathbb{C})$ given by

$$\phi_1 = \bigoplus_{i=1}^k \bigoplus_{j=0}^{A-B-2l} \rho \boxtimes S_{2(B+t_i+l+j)+1};$$

- ε satisfies that

$$\varepsilon(\rho \boxtimes S_{2(B+t_i+l+j)+1}) = (-1)^j \eta'_i$$

for $1 \leq i \leq k$ and $0 \leq j \leq A - B - 2l$;

- $\mathbf{m}_0$ is a multi-segment such that if $[x, -y]_{\rho} \in \mathbf{m}_0$, then $x < y < B + t_1$;

- ϕ_0 is a tempered representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ such that if $\rho \boxtimes S_{2d+1} \subset \phi_0$, then $d < B + t_1$.

Now, by Lemma 3.7, we take $\pi(\mathcal{E}'') \hookrightarrow \mathrm{soc}(\mathrm{soc}(S' \times \tau \times S) \rtimes \sigma)$ so that

$$D_S(\pi(\mathcal{E}'')) \geq \mathrm{soc}(S' \rtimes \pi).$$

Lemma 3.11. *We have $\mathrm{soc}(S' \rtimes \pi) = L(\mathfrak{m}_1 + \mathfrak{m}'_0; \pi(\phi_1 \oplus \phi'_0, \varepsilon'))$ for some $\mathfrak{m}'_0, \phi'_0, \varepsilon'$ with $\varepsilon'|_{\phi_1} = \varepsilon|_{\phi_1}$. In other words, $\mathfrak{m}_1$ and $(\phi_1, \varepsilon|_{\phi_1})$ remain in the Langlands data of $\mathrm{soc}(S' \rtimes \pi)$.*

Proof. We write $\mathrm{soc}(S' \rtimes \pi) = L(\mathfrak{m}'; \pi(\phi', \varepsilon'))$. For $1 \leq i \leq k$ and $0 \leq j \leq l-1$, assuming that $\mathfrak{m}'$ contains

$$\sum_{i'=i+1}^k \sum_{j'=0}^{l-1} [B + t_{i'} + j', -A - t_{i'} + j']_\rho + \sum_{j'=0}^{j-1} [B + t_i + j', -A - t_i + j']_\rho,$$

we will show $[B + t_i + j, -A - t_i + j]_\rho \in \mathfrak{m}'$.

Suppose first that for any $[x, y]_\rho \in \mathfrak{m}'$, we have $y \neq -A - t_i + j$. Then by looking at the cuspidal support, we see that $\phi' \supset \rho \boxtimes S_{2(A+t_i-j)+1}$. Note that $D_{\rho|\cdot|^x}(\mathrm{soc}(S' \rtimes \pi)) = 0$ for $B + t_i < x \leq A + t_i$ since the same condition holds for $\pi(\mathcal{E}'')$ and $t_i \gg 0$. It implies that

$$\phi' \supset \rho \boxtimes (S_{2(B+t_i+j)+1} \oplus \cdots \oplus S_{2(A+t_i-j)+1}).$$

Then for any sequence $x_1, \dots, x_m$ with $x_m = -A - t_i + j$, we have

$$D_{\rho|\cdot|^{x_m}} \circ \cdots \circ D_{\rho|\cdot|^{x_1}}(L(\mathfrak{m}'; \pi(\phi', \varepsilon'))) = 0.$$

In particular, we have $\mathrm{Jac}(L(\mathfrak{m}'; \pi(\phi', \varepsilon'))) \not\supseteq S' \otimes \pi$, which is a contradiction. Hence there is $[x, y]_\rho \in \mathfrak{m}'$ such that $y = -A - t_i + j$.

Let $[x, -A - t_i + j]_\rho \in \mathfrak{m}'$. Note that $x < A + t_i - j$. If $x > B + t_i + j$, then since $D_{\rho|\cdot|^x}(\pi(\mathcal{E}'')) = 0$ and hence $D_{\rho|\cdot|^x}(L(\mathfrak{m}'; \pi(\phi', \varepsilon'))) = 0$, we must have $[x-1, y]_\rho \in \mathfrak{m}'$ such that $y < -A - t_i + j$. However, by looking at the cuspidal support, we cannot take such y . If $x < B + t_i + j$, then $D_{\rho|\cdot|^x, \dots, \rho|\cdot|^{-A-t_i+l}}(L(\mathfrak{m}'; \pi(\phi', \varepsilon'))) \neq 0$, and hence $D_{\rho|\cdot|^x, \dots, \rho|\cdot|^{-A-t_i+l}} \circ D_S(\pi(\mathcal{E}'')) \neq 0$. Since $t_i \gg 0$, this is impossible. Therefore, we have $x = B + t_i + j$, and we conclude that $[B + t_i + j, -A - t_i + j]_\rho \in \mathfrak{m}'$.

By induction, we see that $\mathrm{soc}(S' \rtimes \pi) = \mathrm{soc}(L(\mathfrak{m}_1) \rtimes \pi_0)$ for some irreducible representation π_0 . Since

$$\begin{aligned} \mathrm{soc}(S' \rtimes \pi) &\leq D_S(\pi(\mathcal{E}'')) \\ &\leq D_S(L(\mathfrak{m}_1) \rtimes L(\mathfrak{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon))) \\ &= L(\mathfrak{m}_1) \rtimes D_S(L(\mathfrak{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon))), \end{aligned}$$

we see that $\pi_0 \leq D_S(L(\mathfrak{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)))$. Since $t_1, \dots, t_k \gg 0$, we see that $\pi_0 = L(\mathfrak{m}'_0; \pi(\phi_1 \oplus \phi'_0, \varepsilon'))$ for some $\mathfrak{m}'_0$ and (ϕ'_0, ε') . Hence $\mathrm{soc}(S' \rtimes \pi) = L(\mathfrak{m}_1 + \mathfrak{m}'_0; \pi(\phi_1 \oplus \phi'_0, \varepsilon'))$, as desired. $\square$

Recall that σ is of Arthur type.

Proposition 3.12. *Continue in the above setting. There is an extended multi-segment $\mathcal{E}_0$ such that*

- $\pi(\mathcal{E}_0) = \sigma$;
- $\mathcal{E}_0$ contains $\{([A, B]_\rho, l, \eta'_i) \mid i = 1, \dots, k\}$;
- if $([A', B'], l', \eta')$ $\in \mathcal{E}_0$ is not in $\{([A, B]_\rho, l, \eta'_i) \mid i = 1, \dots, k\}$, then $([A', B'], l', \eta') < ([A, B]_\rho, l, \eta'_i)$.

Admitting this proposition, by [6, Algorithm 3.3], we conclude that $\pi = \text{soc}(\tau \rtimes \sigma)$ is of Arthur type. In fact, π has an extended multi-segment $\mathcal{E}_1$ obtained from $\mathcal{E}_0$ by replacing $([A, B]_\rho, l, \eta'_i)$ with $([A + 1, B + 1]_\rho, l, \eta'_i)$ for $i = 1, \dots, k$. This will complete the proof of Proposition 3.5.

Proof of Proposition 3.12. First of all, by [6, Theorem 4.1], there is an extended multi-segment $\mathcal{E}$ with $\sigma = \pi(\mathcal{E})$ such that if $([A', B']_\rho, *, *) \in \mathcal{E}$, then $B' \leq B$, or $B' = B + 1$ and $A' \leq A$. We choose such an $\mathcal{E}$ so that the number

$$X = \sum_{\substack{([A', B']_\rho, l', \eta') \in \mathcal{E} \\ B' = B}} A'$$

is maximal. By changing the admissible order if necessary, we may assume that we can write

$$\mathcal{E} = \mathcal{E}_1 \cup \{([A_i, B]_\rho, l_i, \eta_i)\}_{i=1, \dots, m} \cup \{([A'_i, B + 1]_\rho, l'_i, \eta'_i)\}_{i=1, \dots, m'},$$

where

- $\{([A'_i, B + 1]_\rho, l'_i, \eta'_i)\}_{i=1, \dots, m'}$ is the multi-set contained $\mathcal{E}$ consisting of $([A', B']_\rho, *, *)$ such that $B' = B + 1$, hence $A'_i \leq A$;
- $\{([A_i, B]_\rho, l_i, \eta_i)\}_{i=1, \dots, m}$ is the multi-set contained $\mathcal{E}$ consisting of $([A', B']_\rho, *, *)$ such that $B' = B$ and $A' \geq A$;
- $\mathcal{E}_1$ is the multi-set contained $\mathcal{E}$ consisting of other $([A', B']_\rho, *, *)$;
- the admissible order is so that

$$([A', B']_{\rho'}, *, *) < ([A_i, B]_\rho, l_i, \eta_i) < ([A'_i, B + 1]_\rho, l'_i, \eta'_i)$$

for $([A', B']_{\rho'}, *, *) \in \mathcal{E}_1$, and the canonical orders on $\{1, \dots, m\}$ and $\{1, \dots, m'\}$.

Note that $[A_i, B]_\rho \supset [A'_j, B + 1]_\rho$ for any $1 \leq i \leq m$ and $1 \leq j \leq m'$. Hence by changing the admissible order and updating the notations, we can rewrite

$$\mathcal{E} = \mathcal{E}'_1 \cup \{([A, B'_i]_\rho, l'_i, \eta'_i)\}_{i=1, \dots, m'} \cup \{([A_i, B]_\rho, l_i, \eta_i)\}_{i=1, \dots, m}$$

such that

- $\{([A_i, B]_\rho, l_i, \eta_i)\}_{i=1, \dots, m}$ is the multi-set contained $\mathcal{E}$ consisting of $([A', B']_\rho, *, *)$ such that $B' = B$ and $A' \geq A$;
- $\{([A, B'_i]_\rho, l'_i, \eta'_i)\}_{i=1, \dots, m'}$ is the multi-set contained $\mathcal{E}$ consisting of $([A', B']_\rho, *, *)$ such that $B' < B$ and $A' = A$;
- $\mathcal{E}'_1$ is the multi-set contained $\mathcal{E}$ consisting of other $([A', B']_{\rho'}, *, *)$;
- the admissible order is so that

$$([A', B']_{\rho'}, *, *) < ([A, B'_i]_\rho, l'_i, \eta'_i) < ([A_i, B]_\rho, l_i, \eta_i)$$

for $([A', B']_{\rho'}, *, *) \in \mathcal{E}'_1$, and the canonical orders on $\{1, \dots, m\}$ and $\{1, \dots, m'\}$.

Note that m and m' can be equal to zero.

Let $m_0 = \#\{i = 1, \dots, m \mid A_i = A\}$ and suppose that $m_0 < k$. We may assume that $A < A_1 \leq \dots \leq A_{m-m_0}$ and $A_i = A$ for $m - m_0 + 1 \leq i \leq m$. Since the number X is maximum among $\mathcal{E}'$ such that $\pi(\mathcal{E}') = \sigma$, to hold Lemma 3.11 for $t_1, \dots, t_k \gg 0$, namely for $\mathbf{m}_1$ and $(\phi_1, \varepsilon|_{\phi_1})$ appearing in the Langlands data of $\text{soc}(S' \rtimes \pi) = \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma)$, we need to have $m > m_0$ and $m' > 0$. Moreover, by the same argument as in [4, Theorem 5.2], we see that $([A, B_{m'}]_\rho, l'_{m'}, \eta'_{m'})$ and $([A_1, B]_\rho, l_1, \eta_1)$ satisfies one of conditions in [4, Theorem

5.2]. Hence by this theorem, we can change $\mathcal{E}$ so that $\{([A, B_{m'}]_\rho, l'_{m'}, \eta'_{m'}), ([A_1, B]_\rho, l_1, \eta_1)\}$ is replaced with

$$\{([A_1, B_{m'}]_\rho, l', \eta'), ([A, B]_\rho, l, \eta)\}$$

for some (l, η) and (l', η') . Hence we can replace m_0 with $m_0 + 1$. Repeating this argument, we can achieve $\mathcal{E}_0$ as in the assertion. $\square$

This completes the proof of Proposition 3.5. Now, Theorem 3.3 follows from Lemma 3.4 and Proposition 3.5.

3.7. The initial step. Let $\pi \in \mathrm{Irr}^G$ be of good parity. In this section, we will prove that if

$$D_{\rho|\cdot|x}(\pi) \neq 0 \implies x \in \left\{0, \frac{1}{2}\right\}$$

for any $\rho^\vee \cong \rho$, then π is of Arthur type. Indeed, we will prove a slightly more general statement. To state it, we need to introduce some notation.

Let $\pi \in \mathrm{Irr}^G$ be of good parity. Write

$$\pi = L(\Delta_{\rho_1}[x_1, -y_1], \dots, \Delta_{\rho_r}[x_r, -y_r]; \pi(\phi, \varepsilon))$$

as in the Langlands classification, where

$$\phi = \bigoplus_{j=1}^s \rho'_j \boxtimes S_{2z_j+1}$$

is a tempered L -parameter such that $\rho_1, \dots, \rho_r, \rho'_1, \dots, \rho'_s \in \mathcal{C}^{\mathrm{GL}}$ are all self-dual. Fix $\rho \in \mathcal{C}^{\mathrm{GL}}$ with $\rho \cong \rho^\vee$. We set

$$\begin{aligned} L_{\rho|\cdot|x}(\pi) &= L_x(\pi) = \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho, x_i = x\}, \\ R_{\rho|\cdot|x}(\pi) &= R_x(\pi) = \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho, y_i = x\}, \\ T_{\rho|\cdot|x}(\pi) &= T_x(\pi) = \{j \in \{1, \dots, s\} \mid \rho'_j \cong \rho, z_j = x\} \end{aligned}$$

and

$$A_{\rho|\cdot|x}(\pi) = A_x(\pi) = L_x \sqcup R_x \sqcup T_x.$$

We will prove the following.

Theorem 3.13. *Notations are as above. For each self-dual $\rho \in \mathcal{C}^{\mathrm{GL}}$, denote $b_\rho \in \{0, \frac{1}{2}\}$ such that $\rho \boxtimes S_{2b_\rho+1}$ is the same type as ϕ . Assume that there are half-integers $a_\rho, a_1, \dots, a_k, b_1, \dots, b_k$ (depending on ρ) such that*

- *they are all congruent to b_ρ modulo $\mathbb{Z}$;*
- *we have*

$$b_\rho \leq a_\rho < b_k \leq a_k < b_{k-1} \leq a_{k-1} < \dots < b_1 \leq a_1;$$

- *if $D_{\rho|\cdot|x}(\pi) = 0$, then $x \in \{b_\rho, b_k, \dots, b_1\}$;*
- *if $A_x(\pi) \neq \emptyset$, then $b_\rho \leq x \leq a_\rho$ or $b_i \leq x \leq a_i$ for some $1 \leq i \leq k$;*
- *if $b_i \leq x \leq a_i$ for some $1 \leq i \leq k$, then $|A_x(\pi)| = 1$, so that $A_x(\pi)$ is equal to exactly one of $L_x(\pi)$, $T_x(\pi)$ or $R_x(\pi)$;*
- *for each $1 \leq i \leq k$,*

$$|\{x \mid b_i \leq x \leq a_i, A_x(\pi) = L_x(\pi)\}| = |\{x \mid b_i \leq x \leq a_i, A_x(\pi) = R_x(\pi)\}|.$$

Then π is of Arthur type.

When the implication $D_{\rho|\cdot|^x}(\pi) \neq 0 \implies x \in \{0, \frac{1}{2}\}$ holds for any $\rho^\vee \cong \rho$, then π satisfies the conditions of Theorem 3.13 with $k = 0$.

We prove Theorem 3.13 by induction on

$$X(\pi) = \sum_{\rho} \sum_{b_{\rho} < x \leq a_{\rho}} |A_{\rho|\cdot|^x}(\pi)|.$$

Suppose first that this number is zero. Then there are $\eta_{\rho}, \eta_i \in \{\pm 1\}$ (which depend on ρ) such that if we set

$$\mathcal{E} = \bigcup_{\rho} \left(\underbrace{\{([b_{\rho}, b_{\rho}]_{\rho}, 0, \eta_{\rho}), \dots, ([b_{\rho}, b_{\rho}]_{\rho}, 0, \eta_{\rho})\}}_{|A_{b_{\rho}}(\pi)|} \cup \bigcup_{i=1}^k \{([a_i, b_i]_{\rho}, l_i, \eta_i)\} \right)$$

with

$$l_i = |\{x \mid b_i \leq x \leq a_i, A_x(\pi) = L_x(\pi)\}| = |\{x \mid b_i \leq x \leq a_i, A_x(\pi) = R_x(\pi)\}|,$$

then we have $\pi(\mathcal{E}) = \pi$. See Section 3.5. Therefore, π is of Arthur type.

Now, we suppose that $a_{\rho} - b_{\rho} > 0$ for some $\rho \cong \rho^\vee$. We may assume that b_k is big enough so that there is an integer $t > 0$ such that

$$a_{\rho} + 1 < b_{\rho} + t \leq a_{\rho} + t < b_k - 1.$$

Here, when $k = 0$, we ignore the inequality $a_{\rho} + t < b_k - 1$. Consider

$$\pi' = \text{soc} \left(\left(\begin{pmatrix} b_{\rho} + t & \dots & a_{\rho} + t \\ \vdots & \ddots & \vdots \\ b_{\rho} + 1 & \dots & a_{\rho} + 1 \end{pmatrix}_{\rho} \right) \rtimes \pi \right)$$

which is irreducible by [5, Proposition 3.4]. By Tadić's formula (Corollary 2.2), we see that

$$D_{\rho|\cdot|^x}(\pi') \neq 0 \implies x \in \{b_{\rho}, b_{\rho} + t, b_k, \dots, b_1, -(a_{\rho} + 1)\}.$$

Lemma 3.14. *In the setting above, we have*

$$D_{\rho|\cdot|^{-(a_{\rho}+1)}}(\pi') = 0.$$

Proof. Suppose that $D_{\rho|\cdot|^{-(a_{\rho}+1)}}(\pi') \neq 0$. Then the inclusion

$$\pi' \hookrightarrow \left(\begin{pmatrix} b_{\rho} + t & \dots & a_{\rho} + t \\ \vdots & \ddots & \vdots \\ b_{\rho} + 2 & \dots & a_{\rho} + 2 \end{pmatrix}_{\rho} \right) \times Z_{\rho}[b_{\rho} + 1, a_{\rho}] \times \rho|\cdot|^{a_{\rho}+1} \rtimes \pi$$

factors through

$$\pi' \hookrightarrow \left(\begin{pmatrix} b_{\rho} + t & \dots & a_{\rho} + t \\ \vdots & \ddots & \vdots \\ b_{\rho} + 2 & \dots & a_{\rho} + 2 \end{pmatrix}_{\rho} \right) \times Z_{\rho}[b_{\rho} + 1, a_{\rho}] \rtimes \text{soc}(\rho|\cdot|^{-(a_{\rho}+1)} \rtimes \pi).$$

Hence

$$\pi = D_{\rho|\cdot|^{-(a_{\rho}+1)}} \circ D_{\rho|\cdot|^{b_{\rho}+1}, \dots, \rho|\cdot|^{a_{\rho}}} \circ \left(D_{\rho|\cdot|^{b_{\rho}+2}, \dots, \rho|\cdot|^{a_{\rho}+2}} \circ \dots \circ D_{\rho|\cdot|^{b_{\rho}+t}, \dots, \rho|\cdot|^{a_{\rho}+t}} \right) (\pi').$$

On the other hand, by definition of π' , we have

$$\pi = D_{\rho|\cdot|^{a_\rho+1}} \circ D_{\rho|\cdot|^{b_\rho+1}, \dots, \rho|\cdot|^{a_\rho}} \circ \left(D_{\rho|\cdot|^{b_\rho+2}, \dots, \rho|\cdot|^{a_\rho+2}} \circ \cdots \circ D_{\rho|\cdot|^{b_\rho+t}, \dots, \rho|\cdot|^{a_\rho+t}} \right) (\pi').$$

These equations imply that

$$\mathrm{soc}(\rho|\cdot|^{-(a_\rho+1)} \rtimes \pi) \cong \mathrm{soc}(\rho|\cdot|^{a_\rho+1} \rtimes \pi),$$

which is equivalent to saying that $\rho|\cdot|^{a_\rho+1} \rtimes \pi$ is irreducible. But since $A_{a_\rho}(\pi) \neq \emptyset$ and $A_{a_\rho+1}(\pi) = \emptyset$, this is impossible by [9, Corollary 7.2]. $\square$

Now we consider $A_x(\pi')$. Note that

$$|A_x(\pi')| = \begin{cases} |A_x(\pi)| - 1 & \text{if } b_\rho \leq x \leq a_\rho, \\ 1 & \text{if } b_\rho + t \leq x \leq a_\rho + t, \\ 1 & \text{if } b_i \leq x \leq a_i, \ 1 \leq i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $b_\rho + t \leq x \leq a_\rho + t$, exactly one of

$$A_x(\pi') = L_x(\pi'), \quad A_x(\pi') = T_x(\pi'), \quad \text{or} \quad A_x(\pi') = R_x(\pi')$$

holds. Moreover, since $D_{\rho|\cdot|^x}(\pi') = 0$ for $b_\rho + t < x \leq a_\rho + t$, by [9, Theorem 7.1], we see that

- if $x \geq b_\rho + t + 1$ and $A_x(\pi') = L_x(\pi')$, then $A_{x-1}(\pi') = L_{x-1}(\pi')$;
- if $x \leq a_\rho + t - 1$ and $A_x(\pi') = R_x(\pi')$, then $A_{x+1}(\pi') = R_{x+1}(\pi')$.

Write

$$\begin{aligned} l_+ &= |\{b_\rho + t \leq x \leq a_\rho + t \mid A_x(\pi') = L_x(\pi')\}|, \\ l_- &= |\{b_\rho + t \leq x \leq a_\rho + t \mid A_x(\pi') = R_x(\pi')\}|. \end{aligned}$$

A key lemma is as follow.

Lemma 3.15. *We have $l_+ = l_-$.*

Proof. First of all, we show that $l_+ \leq l_-$. Indeed, for $x \geq b_\rho + t$, if $A_x(\pi') = L_x(\pi')$, then $\Delta_\rho[x, -y]$ appears in the Langlands data of π' for some $y > x$. Hence $A_y(\pi') = R_y(\pi')$. The map $x \mapsto y$ induces an injection

$$\{x \geq b_\rho + t \mid A_x(\pi') = L_x(\pi')\} \hookrightarrow \{y \geq b_\rho + t \mid A_y(\pi') = R_y(\pi')\}.$$

Since

$$|\{x \mid b_i \leq x \leq a_i, A_x(\pi) = L_x(\pi)\}| = |\{y \mid b_i \leq y \leq a_i, A_x(\pi) = R_x(\pi)\}|$$

for $1 \leq i \leq k$ by assumption, we have $l_+ \leq l_-$. In particular, if $l_- = 0$, then $l_+ = 0$ as well and we get $l_+ = l_-$.

In the rest of the proof of this lemma, we assume that $l_- > 0$. In particular, $A_{a_\rho+t}(\pi') = R_{a_\rho+t}(\pi')$. It implies by [9, Corollary 7.2] that $L_{a_\rho}(\pi) = T_{a_\rho}(\pi) = \emptyset$. Since $D_{\rho|\cdot|^x}(\pi) = 0$ for any $b_\rho < x \leq a_\rho$, we see that $\Delta_\rho[b_\rho, -a_\rho]$ appears in the Langlands data of π with multiplicity exactly $k = |R_{a_\rho}(\pi)| = |A_{a_\rho}(\pi)|$. Namely, there is $\tau \in \mathrm{Irr}^G$ such that $\pi \hookrightarrow \Delta_\rho[b_\rho, -a_\rho]^k \rtimes \tau$ and such that $A_{a_\rho}(\tau) = \emptyset$.

Now, for the sake of a contradiction, we assume that $\pi' \hookrightarrow \Delta_\rho[b_\rho, -a_\rho]^k \rtimes \tau'$ for some $\tau' \in \text{Irr}^G$. Then the inclusion

$$\pi' \hookrightarrow \left(\begin{array}{ccc} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 1 & \dots & a_\rho + 1 \end{array} \right)_\rho \times \Delta_\rho[b_\rho, -a_\rho]^k \rtimes \tau$$

must factor through

$$\begin{aligned} \pi' &\hookrightarrow \text{soc} \left(\Delta_\rho[b_\rho, -a_\rho]^k \times \left(\begin{array}{ccc} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 1 & \dots & a_\rho + 1 \end{array} \right)_\rho \right) \rtimes \tau \\ &\hookrightarrow \Delta_\rho[b_\rho, -a_\rho]^k \times \left(\begin{array}{ccc} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 1 & \dots & a_\rho + 1 \end{array} \right)_\rho \rtimes \tau \\ &\hookrightarrow \Delta_\rho[b_\rho, -a_\rho]^k \times \left(\begin{array}{ccc} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 2 & \dots & a_\rho + 2 \end{array} \right)_\rho \times Z_\rho[b_\rho + 1, a_\rho] \times \rho| \cdot |^{a_\rho+1} \rtimes \tau. \end{aligned}$$

Since $A_{a_\rho}(\tau) = \emptyset$, by [9, Corollary 7.2], we see that $\rho| \cdot |^{a_\rho+1} \rtimes \tau$ is irreducible so that $\rho| \cdot |^{a_\rho+1} \rtimes \tau \cong \rho| \cdot |^{-(a_\rho+1)} \rtimes \tau$. Then the above inclusion implies that $D_{\rho| \cdot |^{-(a_\rho+1)}}(\pi') \neq 0$. This contradicts Lemma 3.14.

By the last paragraph, one can see that the Langlands data of π' contains $\Delta_\rho[x', -(a_\rho + t)]$ for some $b_\rho < x' < a_\rho + t$. Then we see that $D_{\rho| \cdot |^{x'}}(\pi') \neq 0$. Hence we conclude that $x' = b_\rho + t$, i.e., the singleton $A_{b_\rho+t}(\pi')$ is equal to $L_{b_\rho+t}(\pi')$.

If $A_y(\pi') = R_y(\pi')$ for some $b_\rho + t < y < a_\rho + t$, then the Langlands data of π' contains $\Delta_\rho[x, -y]$ for some $b_\rho \leq x < y$. We claim that $x > b_\rho + t$. For the sake of a contradiction, we assume that there is $\Delta_\rho[x, -y]$ in the Langlands data of π' such that $x \leq b_\rho + t$ and $b_\rho + t < y < a_\rho + t$. We may assume that y is maximum among this condition. Then $\Delta_\rho[x', -(y+1)]$ is in the Langlands data of π' for some $x' \geq b_\rho + t$. Since the segment $[x, -y]_\rho$ is contained in $[x', -(y+1)]_\rho$, by [9, Theorem 7.1], we have $D_{\rho| \cdot |^{y+1}}(\pi') \neq 0$, which is a contradiction. Therefore, for $b_\rho + t \leq y \leq a_\rho + t$ with $A_y(\pi') = R_y(\pi')$, if $\Delta_\rho[x, -y]$ in the Langlands data of π' , then $b_\rho + t \leq x \leq a_\rho + t$. This gives the bijection

$$\{b_\rho + t \leq y \leq a_\rho + t \mid A_y(\pi') = R_y(\pi')\} \rightarrow \{b_\rho + t \leq x \leq a_\rho + t \mid A_x(\pi') = L_x(\pi')\}$$

and we conclude that $l_+ = l_-$, as desired. $\square$

Now we set $b_{k+1} = b_\rho + t$ and $a_{k+1} = a_\rho + t$. Then π' satisfies all of the assumptions in Theorem 3.13 for $a_\rho, a_1, \dots, a_{k+1}, b_1, \dots, b_{k+1}$. Moreover, since

$$\sum_{\rho} \sum_{b_\rho < x \leq a_\rho} |A_{\rho| \cdot |^x}(\pi)| - \sum_{\rho} \sum_{b_\rho < x \leq a_\rho} |A_{\rho| \cdot |^x}(\pi')| = a_\rho - b_\rho > 0,$$

we can apply the inductive hypothesis to π' . Hence we can write $\pi' = \pi(\mathcal{E}')$. Moreover, we may assume that

$$\mathcal{E}' \ni ([a_\rho + t, b_\rho + t], l, \eta)$$

for some $\eta \in \{\pm 1\}$, where $l = l_+ = l_-$ is defined in Lemma 3.15. Then by construction, with

$$\mathcal{E} = (\mathcal{E}' \setminus \{([a_\rho + t, b_\rho + t], l, \eta)\}) \cup \{([a_\rho, b_\rho], l, \eta)\},$$

we have $\pi = \pi(\mathcal{E})$. Therefore, we conclude that π is of Arthur type. This completes the proof of Theorem 3.13.

Now Theorem 3.1 follows from Theorems 3.3 and 3.13.

4. SLIGHTLY BEYOND CASE

In this section, we treat a case slightly beyond the good parity case, which may be important for global applications.

4.1. Statement. We know that if $\psi \in \Psi(G)$, then all members in Π_ψ are unitary. However, due to the lack of a proof of the Ramanujan conjecture, if Ψ is a global A -parameter of a classical group $\mathbb{G}$ over a number field $\mathbb{F}$, and if v is a place of $\mathbb{F}$ such that $\mathbb{F}_v = F$ and $G = \mathbb{G}(\mathbb{F}_v)$, then the localization $\psi = \Psi_v$ belongs to $\Psi^+(G)$ but may not lie in $\Psi(G)$.

By the weak Ramanujan bound, which is known, it takes the form

$$\psi = \psi_0 \oplus \bigoplus_{i=1}^r (\rho_i | \cdot |^{x_i} \oplus \rho_i^\vee | \cdot |^{-x_i}) \boxtimes S_{c_i} \boxtimes S_{d_i},$$

where

- $\psi_0 = \psi_{\text{good}}$;
- $\rho_i \in \mathcal{C}^{\text{GL}}$ is unitary;
- $0 \leq x_i < \frac{1}{2}$;
- if $x_i = 0$, then $\rho_i \boxtimes S_{c_i} \boxtimes S_{d_i}$ is not self-dual of the same type as ψ .

Here, r may be zero, in which case $\psi = \psi_{\text{good}}$ is of good parity. For such a parameter ψ , its A -packet Π_ψ is defined as the set of irreducible parabolic inductions

$$\pi = \bigtimes_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) | \cdot |^{x_i} \rtimes \pi_0$$

for $\pi_0 \in \Pi_{\psi_0}$.

In this section, we prove the following theorem, which was conjectured in [21, Conjecture 5.9].

Theorem 4.1. *Let*

$$\pi = \bigtimes_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) | \cdot |^{x_i} \rtimes \pi_0$$

where π_0 is of Arthur type of good parity, ρ_i is unitary, and $0 \leq x_i < \frac{1}{2}$ for $1 \leq i \leq r$. Suppose that π is irreducible. For each $\rho \in \mathcal{C}^{\text{GL}}$ with $\rho \cong \rho^\vee$, and for any pair of positive integers (c, d) , we set

$$I(\rho, c, d) = \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho, (c_i, d_i) = (c, d)\}.$$

Then π is unitary if and only if for each ρ and any (c, d) , the following conditions hold:

- If $\rho \not\cong \rho^\vee$, then

$$\{x_i \mid i \in I(\rho, c, d), x_i \neq 0\} = \{x_i \mid i \in I(\rho^\vee, c, d), x_i \neq 0\}$$

as multisets.

- If $\rho \cong \rho^\vee$ and $|I(\rho, c, d)|$ is odd, then the unitary induction $\mathrm{Sp}(\rho, c, d) \rtimes \pi_0$ is irreducible.

For the proof, we use (UI), (UR), (CS) and (RP1) in Proposition 2.3 frequently.

4.2. Proof of unitarity. Here, we prove the “if” part of Theorem 4.1. Namely we show that if π satisfies the conditions in Theorem 4.1, then π is unitary. The proof is by induction on r .

First, suppose that $x_i = 0$ for some i . Since π is irreducible, by [17, Theorem 1.2], we see that $\text{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is also irreducible. Then since $\pi' = \times_{j \neq i} \text{Sp}(\rho_j, c_j, d_j) \cdot |^{x_j} \rtimes \pi_0$ is unitary by induction hypothesis, so is $\pi = \text{Sp}(\rho_i, c_i, d_i) \rtimes \pi'$ by (UI). In the rest of this subsection, we assume that $x_i > 0$ for any i .

Fix ρ unitary and a pair (c, d) . Suppose that $r \geq 2$ and that there are i and j with $i \neq j$ such that $\rho_i \cong \rho$, $\rho_j \cong \rho^\vee$ and $(c_i, d_i) = (c_j, d_j) = (c, d)$. We may assume that $(i, j) = (1, 2)$, $x_1 \geq x_2$ and that $x_1 = x_2$ if $\rho \not\cong \rho^\vee$. By induction hypothesis,

$$\pi' = \bigtimes_{i=3}^r \text{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$$

is unitary. Since $0 < x_2 < \frac{1}{2}$, by [40], $\text{Sp}(\rho, c, d) \cdot |^{x_2} \times \text{Sp}(\rho, c, d) \cdot |^{-x_2}$ is unitary. Hence

$$\text{Sp}(\rho, c, d) \cdot |^{x_2} \times \text{Sp}(\rho, c, d) \cdot |^{-x_2} \rtimes \pi' \cong \text{Sp}(\rho, c, d) \cdot |^{x_2} \times \text{Sp}(\rho^\vee, c, d) \cdot |^{x_2} \rtimes \pi'$$

is an irreducible unitary representation by (UI). In particular, if $\rho \not\cong \rho^\vee$, then π is unitary. If $\rho \cong \rho^\vee$ and if we set

$$\Pi_t = \text{Sp}(\rho, c, d) \cdot |^t \times \text{Sp}(\rho, c, d) \cdot |^{x_2} \rtimes \pi',$$

then for $x_2 \leq t \leq x_1$, we see that Π_t is irreducible hermitian representation since $\bar{\rho} \cong \rho^\vee \cong \rho$. Since Π_{x_2} is unitary, by (CS), we see that $\pi = \Pi_{x_1}$ is also unitary.

Hence we may assume that $\rho_i \cong \rho_i^\vee$ for any $1 \leq i \leq r$, and that $I(\rho, c, d)$ is a singleton or empty for each $\rho \cong \rho^\vee$ and (c, d) . For $(t_1, \dots, t_r)$ with $0 \leq t_i \leq x_i$, we consider

$$\Pi_{(t_1, \dots, t_r)} = \bigtimes_{i=1}^r \text{Sp}(\rho_i, c_i, d_i) \cdot |^{t_i} \rtimes \pi_0.$$

By assumption together with [17, Theorem 1.2], we see that $\Pi_{(0, \dots, 0)}$ is irreducible. In particular, $\Pi_{(t_1, \dots, t_r)}$ is an irreducible hermitian representation for any $(t_1, \dots, t_r)$ with $0 \leq t_i \leq x_i$. Since $\Pi_{(0, \dots, 0)}$ is unitary, by (CS), we see that $\pi = \Pi_{(x_1, \dots, x_r)}$ is also unitary. This completes the proof of the “if” part of Theorem 4.1.

4.3. Proof of non-unitarity. In this subsection, we will prove the “only if” part by induction on r . Suppose that π is unitary. Our goal is to show the conditions in Theorem 4.1.

First, we assume that $x_i = 0$ for some i . Put $\pi' = \times_{j \neq i} \text{Sp}(\rho_j, c_j, d_j) \cdot |^{x_j} \rtimes \pi_0$. Then $\text{Sp}(\rho_i, c_i, d_i) \otimes \pi'$ is an irreducible hermitian representation of a Levi subgroup of G such that its induction $\text{Sp}(\rho_i, c_i, d_i) \rtimes \pi'$ is unitary. Hence by (UR), we see that $\text{Sp}(\rho_i, c_i, d_i) \otimes \pi'$ is also unitary. In particular, π' is unitary. Repeating this argument, we may assume that $x_i > 0$ for $1 \leq i \leq r$.

Next, we assume that $\rho \not\cong \rho^\vee$. Since π is hermitian, we see that

$$\begin{aligned} \bar{\pi} &\cong \bigtimes_{i=1}^r \text{Sp}(\bar{\rho}_i, c_i, d_i) \cdot |^{x_i} \rtimes \bar{\pi}_0 \cong \bigtimes_{i=1}^r \text{Sp}(\rho_i^\vee, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0^\vee \\ &\cong \left(\bigtimes_{i=1}^r \text{Sp}(\rho_i, c_i, d_i) \cdot |^{-x_i} \rtimes \pi_0 \right)^\vee \cong \left(\bigtimes_{i=1}^r \text{Sp}(\rho_i^\vee, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0 \right)^\vee \end{aligned}$$

is isomorphic to $\pi^\vee$. Hence $|I(\rho, c, d)| = |I(\rho^\vee, c, d)|$ and $\{x_i \mid i \in I(\rho, c, d)\} = \{x_i \mid i \in I(\rho^\vee, c, d)\}$ as multi-sets. Moreover, if $\rho_1 \cong \rho$, $\rho_2 \cong \rho^\vee$, $(c_1, d_1) = (c_2, d_2) = (c, d)$ and $x_1 = x_2$, then since

$$\pi \cong \mathrm{Sp}(\rho, d, c) \cdot |^{x_2} \times \mathrm{Sp}(\rho, d, c) \cdot |^{-x_2} \rtimes \pi'$$

is an irreducible unitary representation with $\pi' = \times_{i=3}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$, by (UR), we see that

$$(\mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2}) \otimes \pi'$$

is also unitary. In particular, π' is unitary. Repeating this argument, we may assume that $\rho_i \cong \rho_i^\vee$ for $1 \leq i \leq r$.

Suppose that $\rho \cong \rho^\vee$. If $\rho_1 \cong \rho_2 \cong \rho$ and $(c_1, d_1) = (c_2, d_2) = (c, d)$, then by (CS), we see that

$$\mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \rtimes \pi' \cong \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2} \rtimes \pi'$$

is an irreducible unitary representation with $\pi' = \times_{i=3}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$. By (UR), we see that π' is unitary. Repeating this argument, we may assume that $I(\rho, c, d)$ is at most a singleton for each $\rho \cong \rho^\vee$ and (c, d) .

If $\mathrm{Sp}(\rho_1, c_1, d_1) \rtimes \pi_0$ is irreducible, then by [17, Theorem 1.2], we see that $\mathrm{Sp}(\rho_1, c_1, d_1) \rtimes \pi'$ is also irreducible with $\pi' = \times_{i=2}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$. By (CS) and (UR), the unitarity of π implies that π' is also unitary. Repeating this argument, we may assume that $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is reducible for $1 \leq i \leq r$.

We have proven that we may now assume the following:

- (1) $\pi = \times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$ is an irreducible unitary representation;
- (2) $\rho_i \cong \rho_i^\vee$ and $0 < x_i < \frac{1}{2}$ for $1 \leq i \leq r$;
- (3) for $i \neq j$, we have $\rho_i \not\cong \rho_j$ or $(c_i, d_i) \neq (c_j, d_j)$;
- (4) $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is reducible for $1 \leq i \leq r$.

The goal is to show that $r = 0$ in this case. For the sake of a contradiction, we assume that $r > 0$.

Proposition 4.2. *Continue in the above setting. Let $P = MN$ be the maximal parabolic subgroup of G such that $\pi_M = (\times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i)) \boxtimes \pi_0 \in \mathrm{Irr}(M)$. Then there is an A -parameter $\psi_M \in \Psi(M)$ with $\pi_M \in \Pi_{\psi_M}$ such that $R_P(w, \pi_M, \psi_M)$ is not a scalar operator, where w is the unique non-trivial Weyl element in $W(M, G)$.*

Proof. By [17, Corollary 3.33], there exist $1 \leq i \leq r$ and an irreducible summand π' of

$$\bigtimes_{\substack{1 \leq j \leq r \\ j \neq i}} \mathrm{Sp}(\rho_j, c_j, d_j) \rtimes \pi_0$$

such that $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi'$ is reducible. For simplicity, set $\rho = \rho_i$ and $(c, d) = (c_i, d_i)$. Since π' is unitary and of good parity, we can write $\pi' = \pi(\mathcal{E}')$ for some extended multi-segment $\mathcal{E}'$. In [17, Definition 3.4], Bošnjak–Stadler introduced a modification of $\mathcal{E}'$ given by

$$\mathcal{F}^{-1}: \mathcal{E}' = \bigcup_{\rho'} \{([A_i, B_i]_{\rho'}, l_i, \eta_i)\}_{i \in I_{\rho'}} \mapsto \mathcal{S}' = \bigcup_{\rho'} \{([A_i, B_i]_{\rho'}, \mu_i)\}_{i \in I_{\rho'}}$$

with

$$\mu_i = \left(\prod_{j=1}^{i-1} (-1)^{A_j - B_j} \right) \eta_i(A_i - B_i + 1 - 2l_i).$$

We may assume that $\mathcal{S}'$ is standard in the sense of [17, Definition 3.15]. Then by [17, Corollary 3.30], there are two irreducible summands π_1 and π_2 of $\mathrm{Sp}(\rho, c, d) \boxtimes \pi'$ associated to $\mathcal{S}'_1 = \mathcal{S} \cup \{([C, D]_\rho, \nu), ([C, D]_\rho, \nu)\}$ and $\mathcal{S}'_2 = \mathcal{S} \cup \{([C, D]_\rho, \nu + 2), ([C, D]_\rho, \nu + 2)\}$, respectively, for some $\nu \in \mathbb{Z}$ with $\nu \equiv d \pmod{2}$, where $C = \frac{c+d}{2} - 1$ and $D = \frac{c-d}{2}$. Note that $\mathcal{S}'_1$ and $\mathcal{S}'_2$ give the same A -parameter $\psi \in \Psi(G)$ as in [17, Definition 3.3(d)], which satisfies that $\pi_1, \pi_2 \in \Pi_\psi$. Moreover, we may assume that ψ comes from an A -parameter $\psi_M \in \Psi(M)$ so that $\pi_M \in \Pi_{\psi_M}$. That is,

$$\psi = \left(\bigoplus_{j=1}^r \rho_j \boxtimes S_{c_j} \boxtimes S_{d_j} \right)^{\oplus 2} \oplus \psi_0$$

with $\pi_0 \in \Pi_{\psi_0}$. By the description of $\mathcal{F}$ in [17, Definition 3.4] together with [4, Theorem 3.6] (corrected in [6, Appendix A]), we see that

$$\prod_{j=1}^r \frac{\langle e(\rho_j, c_j, d_j), \pi_1 \rangle_\psi}{\langle e(\rho_j, c_j, d_j), \pi_2 \rangle_\psi} = (-1)^{\frac{|\nu+2|-|\nu|}{2}} \left((-1)^{d-1} \frac{\mathrm{sgn}(\nu)}{\mathrm{sgn}(\nu+2)} \right)^d$$

with the convention $\mathrm{sgn}(0) = 1$. A case-by-case computation shows that the right-hand side is always equal to -1 .

Now by the local intertwining relation proven by Arthur [2, Theorem 2.4.1], the normalized local intertwining operator $R_P(w, \pi_M, \psi_M)$ acts on π_k by the scalar $\prod_{j=1}^r \langle e(\rho_j, c_j, d_j), \pi_k \rangle_\psi$ for $k = 1, 2$. Therefore, the eigenvalue of $R_P(w, \pi_M, \psi_M)$ on π_1 differs from the one on π_2 . $\square$

In particular, (RP1) in Proposition 2.3 implies that for sufficiently small $s > 0$, the irreducible representation $\times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^s \rtimes \pi_0$ is not unitary. However, (CS) together with the unitarity of π would imply that this representation should be unitary. This is a contradiction. This completes the proof of the “only if” part of Theorem 4.1.

5. REMARKS ON THE FURTHER BEYOND

We have described the unitary dual of split $\mathrm{SO}_{2n+1}(F)$ and $\mathrm{Sp}_{2n}(F)$ in the good parity case and a slightly beyond. In the further beyond case, several new phenomena arise, which we now explain in this section.

5.1. One-parameter complementary series. The first step beyond the good parity case is to consider the set of representations of the form

$$\Pi_s = \mathrm{Sp}(\rho, a, b) \cdot |^s \rtimes \pi_A,$$

where

- π_A is of Arthur type and of good parity;
- $\rho \in \mathcal{C}^{\mathrm{GL}}$ with $\rho^\vee \cong \rho$;
- $s > 0$.

To study this, it is important to consider the *first reducibility point*

$$\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi) = \inf \{s \geq 0 \mid \mathrm{Sp}(\rho, a, b) \cdot |^s \rtimes \pi \text{ is reducible}\}.$$

It is a half-integer and can be computed algorithmically ([5, Theorem 5.3, Corollary 5.4]). If $s \equiv \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi) \pmod{\mathbb{Z}}$, then all irreducible subquotients of Π_s are of good parity. We need to consider the other case.

Question 5.1. *Let $\pi \in \mathrm{Irr}^G$ be of good parity and of Arthur type. For $s > 0$ with $s \neq \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi) \bmod \mathbb{Z}$, if the irreducible representation $\mathrm{Sp}(\rho, a, b)| \cdot |^s \rtimes \pi$ is unitary, then must it be that $s < \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi)$?*

The converse follows from (CS). Note that if s is larger than $\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi)$ but close to it, the non-unitarity would follow if one could establish an analytic property of intertwining operators (see [34, (RP)(ii), Section 2]).

To deal with cases where s is much larger than $\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi)$, we need more explicit control over $\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi)$ than is currently available in [5]. The difficulty is that $\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi)$ really depends on the pair (a, b) , unlike in the earlier settings treated in [27] and [34].

5.2. Some example. Finally, we give some explicit example of unitary representations, beyond the good parity case. In this subsection, we set $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, and we drop ρ from the notations.

Let σ be the unique irreducible supercuspidal representation in the A -packet Π_ψ associated to $\psi = S_5 \otimes S_5 \in \Psi(\mathrm{Sp}_{24}(F))$. Then $\mathrm{FRP}(\mathrm{Sp}(2, 2), \sigma) = 4$. Fix $0 < \epsilon < \frac{1}{2}$. Since the induced representation

$$\mathrm{Sp}(2, 2)| \cdot |^{-s} \rtimes \sigma$$

is irreducible for $0 \leq s \leq 2 + \epsilon$, we see that $\mathrm{Sp}(2, 2)| \cdot |^{2+\epsilon} \rtimes \sigma$ is unitary. We consider

$$\Pi_s = \mathrm{Sp}(2, 2)| \cdot |^{-(2+\epsilon)} \times \mathrm{Sp}(2, 2)| \cdot |^{-s} \rtimes \sigma.$$

It is irreducible for $0 \leq s < \epsilon$. Therefore, all irreducible subquotients of Π_ϵ are unitary. Note that $\mathrm{Sp}(2, 2)| \cdot |^{-(2+\epsilon)} \times \mathrm{Sp}(2, 2)| \cdot |^{-\epsilon}$ contains $L| \cdot |^{-(1+\epsilon)}$ as an irreducible subquotient, where L is *Leclerc's representation* (see Remark 3.6) given by

$$L = L(\Delta[-1, -2], | \cdot |^0, \Delta[1, -1], \Delta[2, 1]).$$

Hence

$$L| \cdot |^{1+\epsilon} \rtimes \pi_A = L(\Delta[-2 - \epsilon, -3 - \epsilon], | \cdot |^{-1-\epsilon}, \Delta[-\epsilon, -2 - \epsilon], \Delta[\epsilon, \epsilon - 1]; \sigma)$$

is unitary. However, it is difficult to see from the Langlands data alone why the right-hand side is unitary. This suggests that further notions may be necessary to fully describe the unitary dual of classical groups.

Additional difficulties arise when considering the limit $\epsilon \rightarrow \frac{1}{2}$. Since all irreducible subquotients of the complementary series are unitary, we conclude that

$$\pi = L\left(\Delta\left[-\frac{5}{2}, -\frac{7}{2}\right], | \cdot |^{-\frac{3}{2}}, \Delta\left[-\frac{1}{2}, -\frac{5}{2}\right], \Delta\left[\frac{1}{2}, -\frac{1}{2}\right]; \sigma\right)$$

is also unitary.

We see from the above argument that π is not isolated in the unitary dual. However, we do not know how to prove this fact directly from the Langlands data, since the induced representation

$$\Pi = L\left(\Delta\left[-\frac{5}{2}, -\frac{7}{2}\right], | \cdot |^{-\frac{3}{2}}, \Delta\left[-\frac{1}{2}, -\frac{5}{2}\right], \Delta\left[\frac{1}{2}, -\frac{1}{2}\right]\right) \rtimes \sigma$$

is reducible. (To see that, one can check that Π is preserved under Aubert duality, whereas π is not.)

REFERENCES

- [1] J. Adams, M. van Leeuwen, P. E. Trapa and D. Vogan, *Unitary representations of real reductive groups*. *Astérisque* No. **417** (2020), x + 174 pp.
- [2] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups*. *American Mathematical Society Colloquium Publications*, **61**. American Mathematical Society, Providence, RI, 2013. xviii+590 pp.
- [3] H. Atobe, *Jacquet modules and local Langlands correspondence*. *Invent. Math.* **219** (2020), no. 3, 831–871.
- [4] H. Atobe, *Construction of local A -packets*. *J. Reine Angew. Math.* **790** (2022), 1–51.
- [5] H. Atobe, *On the socles of certain parabolically induced representations of p -adic classical groups*. *Represent. Theory* **26** (2022), 515–541.
- [6] H. Atobe, *The set of local A -packets containing a given representation*. *J. Reine Angew. Math.* **804** (2023), 263–286.
- [7] H. Atobe and W. T. Gan, *Local theta correspondence of tempered representations and Langlands parameters*. *Invent. Math.* **210** (2017), no. 2, 341–415.
- [8] H. Atobe, W. T. Gan, A. Ichino, T. Kaletha, A. Mínguez and S. W. Shin, *Local Intertwining Relations and Co-tempered A -packets of Classical Groups*. Preprint 2024, arXiv:2410.13504v1.
- [9] H. Atobe and A. Mínguez, *The explicit Zelevinsky–Aubert duality*. *Compos. Math.* **159** (2023), no. 2, 380–418.
- [10] A.-M. Aubert, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d’un groupe réductif p -adique*. *Trans. Amer. Math. Soc.* **347** (1995), no. 6, 2179–2189. Erratum: *Trans. Amer. Math. Soc.* **348** (1996), no. 11, 4687–4690.
- [11] A. I. Badulescu, *Correspondance de Jacquet–Langlands pour les corps locaux de caractéristique non nulle*. *Ann. Sci. École Norm. Sup. (4)* **35** (2002), no. 5, 695–747.
- [12] A. I. Badulescu, G. Henniart, B. Lemaire and V. Sécherre, *Sur le dual unitaire de $GL_r(D)$* . *Amer. J. Math.* **132** (2010), no. 5, 1365–1396.
- [13] A. I. Badulescu and D. A. Renard, *Sur une conjecture de Tadić*. *Glas. Mat. Ser. III* **39(59)** (2004), no. 1, 49–54.
- [14] D. Barbasch and A. Moy, *A unitarity criterion for p -adic groups*. *Invent. Math.* **98** (1989), no. 1, 19–37.
- [15] I. N. Bernstein, *P -invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-Archimedean case)*. *Lie group representations, II (College Park, Md., 1982/1983)*, 50–102, Lecture Notes in Math., **1041**, Springer, Berlin, 1984.
- [16] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p -adic groups. I*. *Ann. Sci. École Norm. Sup. (4)* **10** (1977), no. 4, 441–472.
- [17] B. Bošnjak and A. Stadler, *Irreducibility of the parabolic induction of essentially Speh representations and a representation of Arthur type over a p -adic field*. Preprint 2025, arXiv:2504.01226v2.
- [18] C. J. Bushnell and P.C. Kutzko, *The admissible dual of $GL(N)$ via compact open subgroups*. *Annals of Mathematics Studies*, **129**. Princeton University Press, Princeton, NJ, 1993. xii+313 pp.
- [19] P. Deligne, D. Kazhdan and M.-F. Vignéras, *Représentations des algèbres centrales simples p -adiques*. *Representations of reductive groups over a local field*, 33–117, Travaux en Cours, Hermann, Paris, 1984.
- [20] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*. *Annals of Mathematics Studies*, **151**. Princeton University Press, Princeton, NJ, 2001. viii+276 pp.
- [21] A. Hazeltine, D. Jiang, B. Liu, C.-H. Lo and Q. Zhang, *On representations of Arthur type and unitary dual for classical groups*. Preprint 2024, arXiv:2410.11806v1.
- [22] A. Hazeltine, B. Liu and C.-H. Lo, *On the intersection of local Arthur packets for classical groups and applications*. Preprint 2022, arXiv:2201.10539v3.
- [23] G. Henniart, *Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p -adique*. *Invent. Math.* **139** (2000), no. 2, 439–455.
- [24] A. Kret and E. Lapid, *Jacquet modules of ladder representations*. *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 21–22, 937–940.
- [25] E. Lapid and A. Mínguez, *On parabolic induction on inner forms of the general linear group over a non-archimedean local field*. *Selecta Math. (N.S.)* **22** (2016), no. 4, 2347–2400.
- [26] E. Lapid and A. Mínguez, *Geometric conditions for $\square$ -irreducibility of certain representations of the general linear group over a non-archimedean local field*. *Adv. Math.* **339** (2018), 113–190.

- [27] E. Lapid, G. Muić and M. Tadić, *On the generic unitary dual of quasisplit classical groups*. *Int. Math. Res. Not.* 2004, no. 26, 1335–1354.
- [28] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$* . *Transform. Groups* **8** (2003), no. 1, 95–104.
- [29] C. Mœglin, *Paquets d’Arthur pour les groupes classiques; point de vue combinatoire*. Preprint 2006, arXiv:math/0610189v1.
- [30] C. Mœglin, *Sur certains paquets d’Arthur et involution d’Aubert–Schneider–Stuhler généralisée*. *Represent. Theory* **10** (2006), 86–129.
- [31] C. Mœglin, *Paquets d’Arthur discrets pour un groupe classique p -adique*. *Automorphic forms and L -functions II. Local aspects*, 179–257, *Contemp. Math.*, 489, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2009.
- [32] C. Mœglin, *Holomorphie des opérateurs d’entrelacement normalisés à l’aide des paramètres d’Arthur*. *Canad. J. Math.* **62** (2010), no. 6, 1340–1386.
- [33] C. Mœglin, *Multiplicité 1 dans les paquets d’Arthur aux places p -adiques*. *On certain L -functions*, 333–374, *Clay Math. Proc.*, **13**, Amer. Math. Soc., Providence, RI, 2011.
- [34] G. Muić and M. Tadić, *Unramified unitary duals for split classical p -adic groups; the topology and isolated representations*. *On certain L -functions*, 375–438, *Clay Math. Proc.*, **13**, Amer. Math. Soc., Providence, RI, 2011.
- [35] V. Sécherre, *Représentations lisses de $\mathrm{GL}(m, D)$. I. Caractères simples*. *Bull. Soc. Math. France* **132** (2004), no. 3, 327–396.
- [36] V. Sécherre, *Représentations lisses de $\mathrm{GL}(m, D)$. II. β -extensions*. *Compos. Math.* **141** (2005), no. 6, 1531–1550.
- [37] V. Sécherre, *Représentations lisses de $\mathrm{GL}_m(D)$. III. Types simples*. *Ann. Sci. École Norm. Sup. (4)* **38** (2005), no. 6, 951–977.
- [38] V. Sécherre, *Proof of the Tadić conjecture (U0) on the unitary dual of $\mathrm{GL}_m(D)$* . *J. Reine Angew. Math.* **626** (2009), 187–203.
- [39] V. Sécherre and S. Stevens, *Représentations lisses de $\mathrm{GL}_m(D)$. IV. Représentations supercuspidales*. *J. Inst. Math. Jussieu* **7** (2008), no. 3, 527–574.
- [40] M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*. *Ann. Sci. École Norm. Sup. (4)* **19** (1986), no. 3, 335–382.
- [41] M. Tadić, *Induced representations of $\mathrm{GL}(n, A)$ for p -adic division algebras A* . *J. Reine Angew. Math.* **405** (1990), 48–77.
- [42] M. Tadić, *An external approach to unitary representations*. *Bull. Amer. Math. Soc. (N.S.)* **28** (1993), no. 2, 215–252.
- [43] M. Tadić, *Structure arising from induction and Jacquet modules of representations of classical p -adic groups*. *J. Algebra* **177** (1995), no. 1, 1–33.
- [44] M. Tadić, *On unitarizability and Arthur packets*. *Manuscripta Math.* **169** (2022), no. 3-4, 327–367.
- [45] M. Tadić, *Unitarizability in corank three for classical p -adic groups*. *Mem. Amer. Math. Soc.* **286** (2023), no. 1421, vii+120 pp.
- [46] D.A. Vogan, *The unitary dual of $\mathrm{GL}(n)$ over an Archimedean field*. *Invent. Math.* **83** (1986), no. 3, 449–505.
- [47] B. Xu, *On Mœglin’s parametrization of Arthur packets for p -adic quasisplit $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$* . *Canad. J. Math.* **69** (2017), no. 4, 890–960.
- [48] A. V. Zelevinsky, *Induced representations of reductive p -adic groups. II. On irreducible representations of $\mathrm{GL}(n)$* . *Ann. Sci. École Norm. Sup. (4)* **13** (1980), no. 2, 165–210.

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